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# CHARACTERIZATIONS OF INNER PRODUCT STRUCTURES INVOLVING THE RADIUS OF THE INSCRIBED OR CIRCUMSCRIBED CIRCUMFERENCE

C. Alsina, P. Guijarro and M.S. Tomás

Abstract. We define the radius of the inscribed and circumscribed circumferences in a triangle located in a real normed space and we obtain new characterizations of inner product spaces.

## 1. On the radius of the inscribed circumference in a triangle in a normed space

In an inner product space (i.p.s.) the radius of the inscribed circumference in a triangle of sides $x, y$ and $x - y$ is given by the formula

$$\frac{\sqrt{s(s-\|x\|)(s-\|y\|)(s-\|x-y\|)}}{s} \tag{1}$$

where $s$ is the semiperimeter $s = \dfrac{(\|x\|+\|y\|+\|x-y\|)}{2}$.

Let $(E, \| \, \|)$ be a real normed space. If $x, y$ are two independent vectors in $E\backslash\{0\}$ and (see [4]) if $w(x,y) = \dfrac{\|y\|x+\|x\|y}{\|x\|+\|y\|}$ is the bisectrix of $x$ and $y$, in the triangle of sides $x, y$ and $x-y$, we can consider the bisectrices $w(x,y)$, $w(-y, x-y)$ and $w(-x, y-x)$. It is a straightforward computation to prove that these three lines intersect in a point, i.e. there exist three constants

$$\lambda = \frac{\|x\|+\|y\|}{2s}, \quad \mu = \frac{\|y\|+\|x-y\|}{2s}, \quad \gamma = \frac{\|x\|+\|x-y\|}{2s},$$

in $\mathbb{R}$ such that

$$\lambda w(x,y) = y + \mu w(-y, x-y) = x + \gamma w(-x, y-x)$$

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Now, in order to define the radius of the inscribed circumference, we need a definition of height in a real normed space. To do that, let us consider (see [5]) the functions $\rho'_{\pm} : E \times E \to \mathbb{R}$ defined by

$$\rho'_{\pm}(x, y) = \lim_{t \to 0^{\pm}} \frac{||x + ty||^2 - ||x||^2}{2t}.$$

The mappings $\rho'_{\pm}$ play a crucial role in characterizing inner product spaces. In fact, when the norm derives from an inner product $(E, \cdot)$, then $\rho'_{\pm}(x, y) = x \cdot y$, i.e., $\rho'_{\pm}$ reduce to the given inner product.

We quote here some elementary results concerning the functions $\rho'_{\pm}$ that we will use in this paper:

(i) $\rho'_{\pm}(x, x) = ||x||^2$ and $|\rho'_{\pm}(x, y)| \leq ||x|| \, ||y||$

(ii) $\rho'_{+}(\alpha x, y) = \rho'_{+}(x, \alpha y) = \alpha \rho'_{+}(x, y)$, $\alpha \geq 0$

(iii) $\rho'_{+}(\alpha x, y) = \rho'_{+}(x, \alpha y) = \alpha \rho'_{-}(x, y)$, $\alpha \leq 0$

(iv) $\rho'_{+}(x, \alpha x + y) = \rho'_{+}(x, y) + \alpha ||x||^2$

(v) $\rho'_{+}(x, y) = \rho'_{+}(y, x)$ for all $x, y$ in $E$ if and only if $E$ is an inner product space.

In [1] by using the functions $\rho'_{\pm}$ we introduce the following definition of height over the side $x - y$

$$h(x, y) = y + \frac{\rho'_{+}(y - x, y)}{||x - y||^2}(x - y) \quad \text{for all } x, y \text{ in } E \text{ linearly independent},$$

so we can define "the radius $r(x, y)$ of the inscribed circumference" as the norm of $h\left(\lambda w(x, y), \mu w(-y, x - y)\right)$, i.e.

$$r(x, y) := \frac{||x|| \, ||y||}{2s} \left\| \frac{x}{||x||} - \rho'_{-}\left(\frac{y}{||y||}, \frac{x}{||x||}\right) \frac{y}{||y||} \right\| \tag{2}$$

Observe that $r(x, y)$ is not symmetric in $x$ and $y$.

We want to find when the expressions (1) and (2) are equal.

**Theorem 1.1.** *Let $(E, || \, ||)$ be a real normed space with dim $E \geq 2$. Then $E$ is an i.p.s. if and only if for all linearly independent vectors $x, y$ in $E$*

$$r(x, y) = \frac{\sqrt{s(s - ||x||)(s - ||y||)(s - ||x - y||)}}{s}$$

*where* $s = \dfrac{||x|| + ||y|| + ||x - y||}{2}$.

**Proof.** If we assume that $r(x, y)$ defined by (2) can be obtained by means of (1), then we have:

$$r(x, y)^2 = \frac{(s - ||x||)(s - ||y||)(s - ||x - y||)}{s}$$

i.e.

$$\frac{\|x\|^2\|y\|^2}{4s^2}\left\|\frac{x}{\|x\|}-\rho'_-\left(\frac{y}{\|y\|},\frac{x}{\|x\|}\right)\frac{y}{\|y\|}\right\|^2=\frac{(s-\|x\|)(s-\|y\|)(s-\|x-y\|)}{s}$$

or equivalently

$$\left\|\|y\|x-\rho'_-(y,x)\frac{y}{\|y\|}\right\|^2=4s(s-\|x\|)(s-\|y\|)(s-\|x-y\|).$$

Substituting in this equality $y$ by $tz$, with $t>0$, dividing by $t^2$ and taking limits when $t$ tends to zero, we obtain

$$\begin{aligned}\left\|\|z\|x-\rho'_-(z,x)\frac{z}{\|z\|}\right\|^2 &= \lim_{t\to 0+}\frac{4s(s-\|x\|)(s-\|tz\|)(s-\|x-tz\|)}{t^2}\\ &= \lim_{t\to 0+}\frac{4}{t^2}\,\frac{\|x\|+t\|z\|+\|x-tz\|}{2}\cdot\frac{t\|z\|+\|x-tz\|-\|x\|}{2}\cdot\\ &\cdot\frac{\|x\|+\|x-tz\|-t\|z\|}{2}\cdot\frac{\|x\|+t\|z\|-\|x-tz\|}{2}\\ &= \lim_{t\to 0+}\frac{(\|x-tz\|+t\|z\|)^2-\|x\|^2}{2t}\,\frac{\|x\|^2-(\|x-tz\|-t\|z\|)^2}{2t}=\\ &= \lim_{t\to 0+}\left(-\frac{\|x-tz\|^2-\|x\|^2}{2(-t)}+\frac{\|z\|^2}{2}t+\|x-tz\|\,\|z\|\right)\cdot\\ &\cdot\left(\frac{\|x-tz\|^2-\|x\|^2}{2(-t)}-\frac{\|z\|^2}{2}t+\|x-tz\|\,\|z\|\right)\\ &= \left(-\rho'_-(x,z)+\|x\|\,\|z\|\right)\left(\rho'_-(x,z)+\|x\|\,\|z\|\right)\\ &= \|x\|^2\|z\|^2-\rho'_-(x,z)^2,\end{aligned}$$

and therefore

$$\left\|\|z\|^2x-\rho'_-(z,x)z\right\|^2=\|z\|^2\left(\|x\|^2\|z\|^2-\rho'_-(x,z)^2\right).$$

The substitution $x=z+y$ yields

$$\begin{aligned}\|z\|^2\left(\|z+y\|^2\|z\|^2-\rho'_-(z+y,z)^2\right) &= \left\|\|z\|^2(z+y)-\rho'_-(z,z+y)z\right\|^2\\ &= \left\|\|z\|^2z+\|z\|^2y-\|z\|^2z-\rho'_-(z,y)z\right\|^2\\ &= \left\|\|z\|^2y-\rho'_-(z,y)z\right\|^2=\\ &= \|z\|^2\left(\|y\|^2\|z\|^2-\rho'_-(y,z)^2\right),\end{aligned}$$

i.e.,

$$\|y\|^2\|z\|^2-\rho'_-(y,z)^2=\|z+y\|^2\|z\|^2-\rho'_-(z+y,z)^2.$$

Thus for all linearly independent vectors $u, v$ in $E$ with $\|u\| = \|v\| = 1$, if we substitute in the last equality $z = u - v$, $y = v$ we obtain:

$$\|v\|^2\|u-v\|^2 - \rho'_-(v, u-v)^2 = \|u\|^2\|u-v\|^2 - \rho'_-(u, u-v)^2,$$

and consequently

$$\left|\rho'_-(v, u-v)\right| = \left|\rho'_-(u, u-v)\right|$$

which in turn implies $\left|\rho'_-(v,u) - \|v\|^2\right| = \left|\|u\|^2 - \rho'_+(u,v)\right|$, i.e.,

$$\left|1 - \rho'_-(v,u)\right| = \left|1 - \rho'_+(u,v)\right|.$$

Since $\rho'_-(v,u) \le \|u\|\,\|v\| = 1$ and $\rho'_+(u,v) \le \|u\|\,\|v\| = 1$ we deduce

$$\rho'_-(v,u) = \rho'_+(u,v),$$

and interchanging the roles of $u$ and $v$:

$$\rho'_-(u,v) = \rho'_+(v,u),$$

whence $\rho'_+(v,u) = \rho'_-(u,v) \le \rho'_+(u,v)$ as well as $\rho'_+(u,v) = \rho'_-(v,u) \le \rho'_+(v,u)$, i.e., $\rho'_+(u,v) = \rho'_+(v,u)$, whenever $\|u\| = \|v\| = 1$, $u, v$ independent vectors and $E$ is an inner product space. □

Note. The value $r(x,y)$ as introduced above is not the unique possible definition of the radius of the inscribed circumference, since we can consider other heights like $h\left(\lambda w(x,y), \gamma w(-x, y-x)\right)$, $h\left(\mu w(-y, x-y), \lambda w(x,y)\right)$,
$h\left(\mu w(-y, x-y), \gamma w(-x, y-x)\right)$, $h\left(\gamma w(-x, y-x), \lambda w(x,y)\right)$ or
$h\left(\gamma w(-x, y-x), \mu w(-y, x-y)\right)$ instead of $h\left(\lambda w(x,y), \mu w(-y, x-y)\right)$.

## 2. On the radius of the circumscribed circumference in a triangle in a normed space

If $E$ is an i.p.s. and $x, y$ are two independent vectors in $E\backslash\{0\}$, in the triangle of sides $x, y$ and $x-y$, the center of the circumscribed circumference is the intersection of the perpendicular bisectors, and the radius $R$ is given by the formula

$$\frac{\|x\|\,\|y\|\,\|x-y\|}{4\sqrt{s(s-\|x\|)(s-\|y\|)(s-\|x-y\|)}}$$

where $s = \left(\|x\| + \|y\| + \|x-y\|\right)/2$ is the semiperimeter of the triangle.

In a real normed space $(E, \|\,\|)$ we can consider the perpendicular bisector of the side $x - y$ by taking

$$M(x,y) = \left\{\frac{x+y}{2} + \lambda u \,/\, \lambda \in \mathbb{R}\right\}$$

where $u$ is a vector in $E$ "orthogonal" to $x-y$.

We can assume that $u$ admits the form $u = \alpha x + \beta y$ and having in mind that in an i.p.s. $u \cdot (x-y) = 0$ is immediate to prove that

$$\alpha \left( \|x\|^2 - x \cdot y \right) = \beta \left( \|y\|^2 - x \cdot y \right).$$

Thus in the real normed space $E$, replacing the inner product $\cdot$ by $\rho'_-$, we can consider

$$u = \left( \|y\|^2 - \rho'_-(x,y) \right) x + \left( \|x\|^2 - \rho'_-(y,x) \right) y$$

and

$$M(x,y) = \left\{ \frac{x+y}{2} + \lambda \left[ (\|y\|^2 - \rho'_-(x,y))x + (\|x\|^2 - \rho'_-(y,x))y \right] \ / \ \lambda \in \mathbb{R} \right\}$$

The next step is to define the radius $R$. First in the triangle of sides $x, y$ and $x - y$ considering the corresponding three perpendicular bisectors is easy to prove that this three lines intersected in a point, i.e., there exist $\lambda, \mu, \gamma$ in $\mathbb{R}$ such that

$$\frac{x}{2} + \gamma w = \frac{y}{2} + \mu v = \frac{x+y}{2} + \lambda u$$

where $u, v$ and $w$ are respectively "orthogonal" to $x-y$, $y$ and $x$.

Then, we define $R(x,y)$ as the norm of $\frac{x}{2} + \gamma w$ and by a straightforward computation we obtain that $R(x,y)$ has the following expression

$$R(x,y) = \frac{\left\| \|y\|^2 \left( \|x\|^2 - \rho'_-(x,y) \right) x + \|x\|^2 \left( \|y\|^2 - \rho'_-(y,x) \right) y \right\|}{2\|x\|^2\|y\|^2 - \rho'_-(x,y)^2 - \rho'_-(y,x)^2}.$$

This definition is only possible if

$$\rho'_-(x,y)^2 + \rho'_-(y,x)^2 < 2\|x\|^2\|y\|^2$$

i.e., $\left|\rho'_-(x,y)\right| < \|x\|\,\|y\|$ or $\left|\rho'_-(y,x)\right| < \|x\|\,\|y\|$. For this reason we will assume that $E$ is strictly convex (in these spaces $|\rho'_-(x,y)| \neq \|x\|\,\|y\|$ for all $x$ and $y$ in $E$ linearly independent (see [7])).

**Theorem 2.1.** *Let $(E, \|\ \|)$ be a strictly convex real normed space with dim $E \geq 3$. $E$ is an i.p.s., if and only if, for all $x, y$ independent vectors in $E \backslash \{0\}$*

$$R(x,y) = \frac{\|x\|\,\|y\|\,\|x-y\|}{4\sqrt{s(s-\|x\|)(s-\|y\|)(s-\|x-y\|)}}$$

*where $s = (\|x\| + \|y\| + \|x-y\|)/2$.*

**Proof.** If we let $y = tz$ with $t > 0$ and we take limit when $t$ tends to zero we obtain

$$\frac{\|x\|^2 \left\| \|z\|^2 x - \rho'_-(z,x)z \right\|}{2\|x\|^2\|z\|^2 - \rho'_-(x,z)^2 - \rho'_-(z,x)^2} = \lim_{t\to 0+} R(x,tz) =$$

$$= \left( \lim_{t\to 0^+} \frac{\|x\|^2 t^2 \|z\|^2 \|x - tz\|^2}{[\|x-tz\|^2 - (t\|z\| - \|x\|)^2]\left[(\|x\| + t\|z\|)^2 - \|x - tz\|^2\right]} \right)^{1/2} =$$

$$= \|x\|\|z\| \left( \lim_{t\to 0^+} \frac{t\|x - tz\|}{\|x - tz\|^2 - (t\|z\| - \|x\|)^2} \right)^{1/2} \cdot$$

$$\cdot \left( \lim_{t\to 0^+} \frac{t\|x - tz\|}{(\|x\| + t\|z\|)^2 - \|x - tz\|^2} \right)^{1/2}$$

$$= \|x\|\|z\| \sqrt{\frac{\|x\|}{-2\rho'_-(x,z) + 2\|x\|\,\|z\|}} \sqrt{\frac{\|x\|}{2\|x\|\,\|z\| + 2\rho'_-(x,z)}} =$$

$$= \frac{\|z\|\,\|x\|^2}{2\sqrt{\|x\|^2\|z\|^2 - \rho'_-(x,z)^2}}$$

and therefore,

$$\left\| \frac{x}{\|x\|} - \rho'_-\left(\frac{z}{\|z\|}, \frac{x}{\|x\|}\right) \frac{z}{\|z\|} \right\| = \frac{2 - \rho'_-\left(\frac{x}{\|x\|}, \frac{z}{\|z\|}\right)^2 - \rho'_-\left(\frac{z}{\|z\|}, \frac{x}{\|x\|}\right)^2}{2\sqrt{1 - \rho'_-\left(\frac{x}{\|x\|}, \frac{z}{\|z\|}\right)^2}}.$$

If $u = \dfrac{x}{\|x\|}$, $v = \dfrac{z}{\|z\|}$, then for all unitary and independent vectors $u$ and $v$ in $E$

$$\left\| u - \rho'_-(v,u)v \right\| = \frac{2 - \rho'_-(u,v)^2 - \rho'_-(v,u)^2}{2\sqrt{1 - \rho'_-(u,v)^2}}$$

Now, if $\rho'_-(v,u) = 0$, then,

$$2\sqrt{1 - \rho'_-(u,v)^2} = 2 - \rho'_-(u,v)^2$$

and therefore $\rho'_-(u,v) = 0$ and (see [5], [6]) $E$ is an i.p.s. $\square$

Note. If we define the radius $R(x,y)$ as the norm of $\frac{y}{2} + \mu v$ or as the norm of $\frac{x+y}{2} + \lambda u$ the same expression obtained in the initial definition of $R(x,y)$ appears.

Analogous definitions of $R(x,y)$ can be given by replacing the role of $\rho'_-$ by $\rho'_+$ or by changing the order of the arguments appearing in $\rho'_-$. For example if we consider that the radius of the circumscribed circumference is given by

$$\hat{R}(x,y) = \frac{\left\| \|y\|^2 \left( \|x\|^2 - \rho'_+(x,y) \right) x + \|x\|^2 \left( \|y\|^2 - \rho'_+(y,x) \right) y \right\|}{2 \left( \|x\|^2 \|y\|^2 - \rho'_-(x,y)^2 \right)}$$

which is equal to $R(x,y)$ in an i.p.s. then, a strictly convex real normed space $E$ with $\dim E \geq 2$ is an i.p.s. if and only if

$$\hat{R}(x,y) = \frac{\|x\| \, \|y\| \, \|x-y\|}{4\sqrt{s(s-\|x\|)(s-\|y\|)(s-\|x-y\|)}}.$$

The proof is immediate using the fact that the symmetry of the second member of last expression and the symmetry of the numerator of $\hat{R}(x,y)$ imply $\rho'_-(x,y)^2 = \rho'_-(y,x)^2$ for all $x, y$ in $E$ and $E$ is an i.p.s.

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C. Alsina, P. Guijarro and M.S. Tomás
Secció de Matematiques, ETSAB
Univ. Politécnica Catalunya
Diagonal 649
08028 Barcelona, SPAIN