ARCHIVUM MATHEMATICUM (BRNO)
Tomus 32 (1996), 137 – 145

LOCAL ISOMETRY CLASSES OF RIEMANNIAN 3-MANIFOLDS WITH CONSTANT RICCI EIGENVALUES $\rho_1 = \rho_2 \neq \rho_3 > 0$

Oldřich Kowalski and Masami Sekizawa

Abstract. We prove that the all local isometry classes as in the title can be parametrized by two arbitrary functions of one variable. This result is extended also for a special case of nonconstant Ricci eigenvalues.

1. Introduction

In [3], the classification of Riemannian 3-manifolds (M, g) with constant Ricci eigenvalues $\rho_1 = \rho_2 \neq \rho_3 \neq 0$ was done. For $\rho_3 < 0$, Theorem 6.2 from [3] says that the local isometry classes of such Riemannian manifolds are parametrized by two arbitrary functions of one variable. In [4], the same result was extended to the case when $\rho_1 = \rho_2$ is not constant on the whole (M, g) but only along the trajectories of the one-dimensional eigenspaces of the Ricci eigenvalue ρ_3 (which are called "principal geodesics" in the sequel), and where the scalar curvature is prescribed as a function with the same property (see [4], Theorem 5.5).

In the case $\rho_3 > 0$ the situation is more complicated. In [3], Theorem 8.1, the author only proved the *existence* of a family of solutions whose local isometry classes depend on two arbitrary functions of one variable. Recently, this result was improved by P.Bueken [1] who showed that, also for $\rho_3 > 0$, the local isometry classes of *all* solutions depend on two arbitrary functions of one variable. The proof in [1] based on the concept of "shear tensor" is short and elegant; yet it does not seem to be completely rigorous. (In fact, P.Bueken is using the method from the article [5] by D.McManus devoted to the same topic, correcting at the same time an essential error in this last paper.)

The aim of this paper is to give a rigorous proof of this fact. Moreover, the method of the shear tensor does not extend immediately to the case of a non-constant eigenvalue $\rho_1 = \rho_2$, whereas our method does.

1991 *Mathematics Subject Classification*: Primary 53C25, Secondary 53B20.
Key words and phrases: Riemannian manifolds, principal Ricci curvatures, curvature homogeneous spaces.
Received April 3, 1996.
This research was supported by the grant GAČR 201/96/0227 .

2. The general solution of the problem

First of all, we summarize from [3] the basic information known about the "elliptic case" $\rho_1 = \rho_2 \neq \rho_3 > 0$.

Let λ be a positive constant, and let φ_1, φ_2, φ_3, φ_4 be arbitrary functions of two variables w and x. Then we define functions a_0, a_1, a_2, a_3, b_1, b_2, b_3, φ_0 and h by

(2.1)
$$a_1 = \frac{\varphi_2(\lambda^2 - (\varphi_4)^2) - 2\lambda\varphi_1\varphi_4}{\lambda(\varphi_0)^2}, \quad a_2 = \frac{\varphi_1((\varphi_4)^2 - \lambda^2) - 2\lambda\varphi_2\varphi_4}{\lambda(\varphi_0)^2},$$

(2.2)
$$a_3 = \frac{\varphi_3(\lambda^2 + (\varphi_4)^2)}{\lambda(\varphi_0)^2},$$

(2.3)
$$b_1 = \frac{\lambda\varphi_1 + \varphi_2\varphi_4}{\lambda\varphi_0}, \quad b_2 = \frac{\lambda\varphi_2 - \varphi_1\varphi_4}{\lambda\varphi_0}, \quad b_3 = -\frac{\varphi_3\varphi_4}{\lambda\varphi_0},$$

(2.4)
$$a_0\varphi_0 = -\frac{\lambda^2 + (\varphi_4)^2}{\lambda}, \quad \varphi_0 = \sqrt{(\varphi_3)^2 - (\varphi_1)^2 - (\varphi_2)^2},$$

(2.5)
$$h = -\frac{2\lambda\varphi_3}{\varphi_0}.$$

(Here the obvious condition $\varphi_0 \neq 0$ must be satisfied, which is the only limitation for the choice of the functions $\varphi_1, \dots, \varphi_4$.) Further, let A, f and C be functions of three variables w, x and y defined by

$$\left\{\begin{aligned} &A^2 = a_1\cos(2\lambda y) + a_2\sin(2\lambda y) + a_3,\\ &AC = b_1\cos(2\lambda y) + b_2\sin(2\lambda y) + b_3,\\ &f^2 + C^2 = \frac{1}{\lambda}[\varphi_1\sin(2\lambda y) - \varphi_2\cos(2\lambda y) + \varphi_3],\\ &Af = 1, \quad f > 0, \end{aligned}\right. \tag{2.6}$$

and let H be a function of w and x such that $H'_x = h$ (where H'_x indicates the partial derivative of H with respect to the suffix x). One can check easily that the conditions (2.6) are compatible and determine A, f and C in a unique way. Finally, let $g = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2$ be a Riemannian metric defined on a domain U in $\mathbb{R}^3(w, x, y)$ by

$$\omega^1 = f\,dw, \quad \omega^2 = A\,dx + C\,dw, \quad \omega^3 = dy + H\,dw, \tag{2.7}$$

which is an orthonormal coframe. Then the Ricci tensor of the metric g has constant eigenvalues $\rho_1 = \rho_2 \neq \rho_3 = 2\lambda^2$ if and only if the following system of

PDE is satisfied:

$$\varphi'_{1x} - 2b_1\lambda^2 H + \frac{\lambda}{a_0}a'_{1w} - \frac{\varphi_4}{a_0}a'_{2w} = 0, \tag{2.8}$$

$$\varphi'_{2x} - 2b_2\lambda^2 H + \frac{\varphi_4}{a_0}a'_{1w} + \frac{\lambda}{a_0}a'_{2w} = 0, \tag{2.9}$$

$$H'_x = -\frac{2\lambda\varphi_3}{\varphi_0}, \tag{2.10}$$

$$(A\alpha)'_w + R'_x = -\rho_1, \tag{2.11}$$

where

$$R = ff'_x - C\alpha + H\beta, \tag{2.12}$$

$$\begin{cases} \alpha = A'_w - C'_x - HA'_y, \\ \beta = \dfrac{1}{2}(H'_x + AC'_y - CA'_y). \end{cases} \tag{2.13}$$

Moreover, *all* Riemannian metrics with the prescribed constant Ricci eigenvalues $\rho_1 = \rho_2 \neq \rho_3 = 2\lambda^2$ can be locally constructed in the previous way (see [3] for more details).

Let us mention that the new variable y in (2.6) and (2.7) measures the arc-length along the principal geodesics, and the principal geodesics satisfy the equation $\omega^1 = \omega^2 = 0$.

Calculations using formulas (7.24)–(7.26) in [3] for $A\alpha$ and for R, respectively, show that (2.11) evaluated at $y = 0$ takes on the form

$$\varphi''_{3xx} - \varphi''_{2xx} = F(\varphi_i, H, \varphi'_{iw}, \varphi'_{ix}, H'_w, \varphi''_{iwx}, \varphi''_{iww}), \tag{2.14}$$

where F is a real analytic function of its variables. The equation (2.14) is then equivalent to (2.11) modulo (2.8) and (2.9).

Using the same procedure as in [2], proof of Proposition 8.1, one can introduce new variables $\tilde{x} = \tilde{x}(w,x)$, $\tilde{w} = \tilde{w}(w,x)$ and $\tilde{y} = y + \psi(w,x)$ such that all previous formulas are satisfied in these new variables and, moreover, $b_1 = b_2$. The last condition implies, due to (2.3), (2.1) and (2.2),

$$\varphi_4 = \frac{\lambda(\varphi_2 - \varphi_1)}{\varphi_1 + \varphi_2}, \tag{2.15}$$

$$a_1 = 2\varphi_1\lambda\Omega, \quad a_2 = -2\varphi_2\lambda\Omega, \quad a_3 = 2\varphi_3\lambda\Omega, \tag{2.16}$$

where

$$\Omega = \frac{(\varphi_1)^2 + (\varphi_2)^2}{(\varphi_0)^2(\varphi_1 + \varphi_2)^2} > 0 \tag{2.17}$$

and

$$b_1 = b_2 = \frac{(\varphi_1)^2 + (\varphi_2)^2}{\varphi_0(\varphi_1+\varphi_2)}, \quad b_3 = \frac{\varphi_3(\varphi_1-\varphi_2)}{\varphi_0(\varphi_1+\varphi_2)}. \tag{2.18}$$

Now consider the PDE system formed by the equations (2.8)–(2.10) and (2.14). We can apply to this system an easy modification of the Cauchy-Kowalewski Theorem (see *e.g.* [2], Section 9). It follows that *the general solution of the problem depends* formally *on five arbitrary functions of the variable* w, namely on $\varphi_1(w,x_0)$, $\varphi_2(w,x_0)$, $\varphi_3(w,x_0)$, $\varphi'_{3x}(w,x_0)$ and $H(w,x_0)$ around a point $(w_0,x_0)\in\mathbb{R}^2(w,x)$. (We notice here that φ_4 is given in terms of φ_1 and φ_2 by (2.15) and that we assume the generic case $\varphi_1+\varphi_2\neq 0$.)

Remark. In Proposition 7.1 of [3], the formulas corresponding to (2.1) contain a misprint (missing λ's).

3. The geometric existence theorem for the elliptic case

We give in this section the complete solution of the isometry problem in the elliptic case.

Theorem 1. *The local isometry classes of all Riemannian 3-manifolds with constant Ricci eigenvalues $\rho_1 = \rho_2 \neq \rho_3 > 0$ are parametrized by two arbitrary functions of one variable.*

Proof. Suppose that (M,g) and $(\bar M,\bar g)$ are two Riemannian manifolds with the same constant Ricci eigenvalues $\rho_1=\rho_2\neq\rho_3 = 2\lambda^2$, and let the local expression for (M,g) be given by (2.7), (2.6) and (2.16)–(2.18). Suppose that the local expression for $(\bar M,\bar g)$ is given by the analogous formulas. The only basic functions are φ_i, H and $\bar\varphi_i$, $\bar H$, respectively, $i=1,2,3$.

Let $F : U \longrightarrow \bar U$ be a local isometry between (M,g) and $(\bar M,\bar g)$ with the coordinate expression

$$\bar w = \bar w(w,x,y), \quad \bar x = \bar x(w,x,y), \quad \bar y = \bar y(w,x,y). \tag{3.1}$$

In the same way as in [3], using the geometrical meaning of the orthonormal coframe (2.7), we obtain (denoting the induced forms $F^*\bar\omega^i$ simply by $\bar\omega^i$, $i = 1,2,3$)

$$\begin{cases} \bar\omega^1 = \cos\varphi\,\omega^1 + \varepsilon\sin\varphi\,\omega^2, \\ \bar\omega^2 = -\sin\varphi\,\omega^1 + \varepsilon\cos\varphi\,\omega^2, \qquad (\varepsilon,\ \varepsilon' = \pm 1), \\ \bar\omega^3 = \varepsilon'\,\omega^3, \end{cases} \tag{3.2}$$

i.e.,

$$(\bar\omega^1)^2 + (\bar\omega^2)^2 = (\omega^1)^2 + (\omega^2)^2, \quad \bar\omega^3 = \varepsilon'\,\omega^3. \tag{3.3}$$

We can assume $\varepsilon' = 1$, the opposite case is treated similarly. We obtain from (3.2) and $(2.7)_3$

$$\bar w = \bar w(w,x), \quad \bar x = \bar x(w,x), \tag{3.4}$$

$$\bar{y} = y + \phi(w, x), \quad d\phi = -\bar{H} d\bar{w} + H dw. \tag{3.5}$$

We shall now substitute into the first equation of (3.3). We get first

$$\begin{aligned}(\omega^1)^2 + (\omega^2)^2 = \frac{1}{\lambda}&[\varphi_1 \sin(2\lambda y) - \varphi_2 \cos(2\lambda y) + \varphi_3] dw^2 \\ &+ 2[b_1 \cos(2\lambda y) + b_2 \sin(2\lambda y) + b_3] dw dx \\ &+ [a_1 \cos(2\lambda y) + a_2 \sin(2\lambda y) + a_3] dx^2,\end{aligned} \tag{3.6}$$

$$\begin{aligned}(\bar{\omega}^1)^2 + (\bar{\omega}^2)^2 = \frac{1}{\lambda}&[\bar{\varphi}_1 \sin(2\lambda \bar{y}) - \bar{\varphi}_2 \cos(2\lambda \bar{y}) + \bar{\varphi}_3] d\bar{w}^2 \\ &+ 2[\bar{b}_1 \cos(2\lambda \bar{y}) + \bar{b}_2 \sin(2\lambda \bar{y}) + \bar{b}_3] d\bar{w} d\bar{x} \\ &+ [\bar{a}_1 \cos(2\lambda \bar{y}) + \bar{a}_2 \sin(2\lambda \bar{y}) + \bar{a}_3] d\bar{x}^2.\end{aligned} \tag{3.7}$$

In (3.6) we put $y = \bar{y} - \phi$ and use the standard trigonometric formulas for developing the sine and cosine of a difference of arguments; in (3.7) we substitute

$$d\bar{w} = \bar{w}'_w dw + \bar{w}'_x dx, \quad d\bar{x} = \bar{x}'_w dw + \bar{x}'_x dx. \tag{3.8}$$

Then the equality of the right-hand sides of (3.6) and (3.7) means the equalities between three pairs of quadratic forms in dw and dx which are coefficients of 1, $\sin(2\lambda\bar{y})$ and $\cos(2\lambda\bar{y})$, respectively. For each pair of quadratic forms we compare the coefficients of dx^2, $dxdw$ and dw^2, respectively. As a result, we obtain the following system of nine PDE for the functions $\bar{x} = \bar{x}(w, x)$ and $\bar{w} = \bar{w}(w, x)$:

$$\begin{cases} \bar{a}_3 P^2 + 2\bar{b}_3 PR + \frac{1}{\lambda}\bar{\varphi}_3 R^2 = a_3, \\ \bar{a}_3 Q^2 + 2\bar{b}_3 QS + \frac{1}{\lambda}\bar{\varphi}_3 S^2 = \frac{1}{\lambda}\varphi_3, \\ \bar{a}_3 PQ + \bar{b}_3(PS + QR) + \frac{1}{\lambda}\bar{\varphi}_3 RS = b_3, \end{cases} \tag{3.9}$$

$$\begin{cases} \bar{a}_2 P^2 + 2\bar{b}_2 PR + \frac{1}{\lambda}\bar{\varphi}_1 R^2 = a_1 \sin(2\lambda\phi) + a_2 \cos(2\lambda\phi), \\ \bar{a}_2 Q^2 + 2\bar{b}_2 QS + \frac{1}{\lambda}\bar{\varphi}_1 S^2 = \frac{1}{\lambda}[\varphi_1 \cos(2\lambda\phi) - \varphi_2 \sin(2\lambda\phi)], \\ \bar{a}_2 PQ + \bar{b}_2(PS + QR) + \frac{1}{\lambda}\bar{\varphi}_1 RS = b_1 \sin(2\lambda\phi) + b_2 \cos(2\lambda\phi), \end{cases} \tag{3.10}$$

$$\begin{cases} \bar{a}_1 P^2 + 2\bar{b}_1 PR - \frac{1}{\lambda}\bar{\varphi}_2 R^2 = a_1 \cos(2\lambda\phi) - a_2 \sin(2\lambda\phi), \\ \bar{a}_1 Q^2 + 2\bar{b}_1 QS - \frac{1}{\lambda}\bar{\varphi}_2 S^2 = -\frac{1}{\lambda}[\varphi_1 \sin(2\lambda\phi) + \varphi_2 \cos(2\lambda\phi)], \\ \bar{a}_1 PQ + \bar{b}_1(PS + QR) - \frac{1}{\lambda}\bar{\varphi}_2 RS = b_1 \cos(2\lambda\phi) - b_2 \sin(2\lambda\phi), \end{cases} \tag{3.11}$$

where we use the notation

$$P = \bar{x}'_x, \quad Q = \bar{x}'_w, \quad R = \bar{w}'_x, \quad S = \bar{w}'_w. \tag{3.12}$$

A lengthy but routine calculation shows that, under a necessary and sufficient condition

$$\left(\frac{\bar{\varphi}_3}{\bar{\varphi}_0}\right)^2 = \left(\frac{\varphi_3}{\varphi_0}\right)^2, \tag{3.13}$$

the square of each last equation of (3.9), (3.10) and (3.11) is equivalent to the product of the first two equations. Hence, all the last equations $(3.9)_3$, $(3.10)_3$ and $(3.11)_3$ are, *up to the sign*, consequences of the remaining equations, and of the condition (3.13), which can be also written in the form

$$\frac{(\bar{\varphi}_1)^2 + (\bar{\varphi}_2)^2}{(\bar{\varphi}_0)^2} = \frac{(\varphi_1)^2 + (\varphi_2)^2}{(\varphi_0)^2}. \tag{3.14}$$

(Let us mention that (3.13) together with (2.5) means that h^2 is a Riemannian invariant, which can be seen more directly from [3] or [4].)

In what follows, we can assume, without the loss of generality,

$$\varphi_0, \ \bar{\varphi}_0, \ \varphi_3, \ \bar{\varphi}_3 > 0. \tag{3.15}$$

Now, the remaining equations $(3.9)_{1,2}$, $(3.10)_{1,2}$ and $(3.11)_{1,2}$ can be solved with respect to the unknowns P^2, R^2 and PR; or Q^2, S^2 and QS, respectively, by the Cramer's rule. The determinant of each system is (due to (3.13) and (3.14))

$$\mathcal{D} = \begin{vmatrix} \bar{a}_3 & 2\bar{b}_3 & \frac{1}{\lambda}\bar{\varphi}_3 \\ \bar{a}_2 & 2\bar{b}_2 & \frac{1}{\lambda}\bar{\varphi}_1 \\ \bar{a}_1 & 2\bar{b}_1 & -\frac{1}{\lambda}\bar{\varphi}_2 \end{vmatrix} = -\frac{4((\varphi_1)^2 + (\varphi_2)^2)\varphi_3}{(\varphi_0)^3} \tag{3.16}$$

and hence $\mathcal{D}$ depends only on φ_1, φ_2, φ_3 and is strictly negative. Further, we get

$$\begin{cases} P^2 = \dfrac{2}{\lambda\bar{\varphi}_0(-\mathcal{D})}[((\bar{\varphi}_1)^2 + (\bar{\varphi}_2)^2)a_3 + \bar{\varphi}_3(\bar{\varphi}_2 L_2 - \bar{\varphi}_1 L_1)], \\ R^2 = \dfrac{4\lambda\bar{\Omega}}{\bar{\varphi}_0(-\mathcal{D})}[((\bar{\varphi}_1)^2 + (\bar{\varphi}_2)^2)a_3 + \bar{\varphi}_3(\bar{\varphi}_2 L_1 - \bar{\varphi}_1 L_2)], \\ PR = \dfrac{2\bar{\Omega}(\bar{\varphi}_1 + \bar{\varphi}_2)}{-\mathcal{D}}[(\bar{\varphi}_2 - \bar{\varphi}_1)a_3 + \bar{\varphi}_3(L_1 + L_2)], \end{cases} \tag{3.17}$$

where

$$\begin{cases} L_1 = a_1\sin(2\lambda\phi) + a_2\cos(2\lambda\phi), \\ L_2 = a_1\cos(2\lambda\phi) - a_2\sin(2\lambda\phi). \end{cases} \tag{3.18}$$

A routine computation using (3.13) or (3.14) implies that the equalities (3.17) are compatible in the sense that the product of the right-hand sides of $(3.17)_1$ and

$(3.17)_2$ is equal to the square of the right-hand side of $(3.17)_3$. We also want to show that the right-hand sides of $(3.17)_1$ and $(3.17)_2$ are always non-negative. For the first equation, the computation proceeds as follows:

$$\begin{aligned}
&((\bar\varphi_1)^2+(\bar\varphi_2)^2)a_3+\bar\varphi_3(\bar\varphi_2L_2-\bar\varphi_1L_1)\\
&=((\bar\varphi_1)^2+(\bar\varphi_2)^2)a_3-\bar\varphi_3(\bar\varphi_2a_2-\bar\varphi_1a_1)\sin(2\lambda\phi)+\bar\varphi_3(\bar\varphi_2a_1-\bar\varphi_1a_2)\cos(2\lambda\phi)\\
&=(2\lambda\Omega)[((\bar\varphi_1)^2+(\bar\varphi_2)^2)\varphi_3+\bar\varphi_3(\bar\varphi_2\varphi_2-\bar\varphi_1\varphi_1)\sin(2\lambda\phi)\\
&\quad+\bar\varphi_3(\bar\varphi_2\varphi_1+\bar\varphi_1\varphi_2)\cos(2\lambda\phi)].
\end{aligned}$$

Now, it can be seen easily that the function $p(t)=A\cos t+B\sin t$ with constant A, B and variable t satisfies $|p(t)|\le\sqrt{A^2+B^2}$. Hence we get

$$\begin{aligned}
&|\bar\varphi_3(\bar\varphi_2\varphi_2-\bar\varphi_1\varphi_1)\sin(2\lambda\phi)+\bar\varphi_3(\bar\varphi_2\varphi_1+\bar\varphi_1\varphi_2)\cos(2\lambda\phi)|\\
&\le\bar\varphi_3\sqrt{((\varphi_1)^2+(\varphi_2)^2)((\bar\varphi_1)^2+(\bar\varphi_2)^2)}
\end{aligned}$$

and, on the other hand, using the identity $((\bar\varphi_1)^2+(\bar\varphi_2)^2)(\varphi_3)^2=((\varphi_1)^2+(\varphi_2)^2))(\bar\varphi_3)^2$, we get

$$\bar\varphi_3\sqrt{((\varphi_1)^2+(\varphi_2)^2)((\bar\varphi_1)^2+(\bar\varphi_2)^2)}=((\bar\varphi_1)^2+(\bar\varphi_2)^2)\varphi_3.$$

Hence it follows that the right-hand side of $(3.17)_1$ is non-negative. The calculation for $(3.17)_2$ is similar.

Further, we get by the Cramer's rule

$$(3.19)\qquad\begin{cases}
Q^2=\dfrac{2}{\lambda^2\bar\varphi_0(-\mathcal{D})}\left[((\bar\varphi_1)^2+(\bar\varphi_2)^2)\varphi_3-\bar\varphi_3(\bar\varphi_1R_1+\bar\varphi_2R_2)\right],\\
S^2=\dfrac{4\bar\Omega}{\bar\varphi_0(-\mathcal{D})}\left[((\bar\varphi_1)^2+(\bar\varphi_2)^2)\varphi_3+\bar\varphi_3(\bar\varphi_2R_1+\bar\varphi_1R_2)\right],\\
QS=\dfrac{2\bar\Omega(\bar\varphi_1+\bar\varphi_2)}{\lambda(-\mathcal{D})}\left[(\bar\varphi_2-\bar\varphi_1)\varphi_3+\bar\varphi_3(R_1-R_2)\right],
\end{cases}$$

where

$$(3.20)\qquad\begin{cases}
R_1=\varphi_1\cos(2\lambda\phi)-\varphi_2\sin(2\lambda\phi),\\
R_2=\varphi_1\sin(2\lambda\phi)+\varphi_2\cos(2\lambda\phi).
\end{cases}$$

The equalities (3.19) are again compatible and the right-hand sides of $(3.19)_1$ and $(3.19)_2$ are always non-negative.

Now, in the *generic* case, we can assume that the right-hand sides of $(3.17)_1$ and $(3.19)_2$ are strictly positive. Then we can define P and S as the corresponding positive square roots and calculate R and Q from $(3.17)_3$ and $(3.19)_3$, respectively.

Notice that (3.13) and (3.14) imply (together with (2.4))

$$
\tag{3.21}
\begin{cases}
\bar{\varphi}_3 = \sqrt{\dfrac{(\bar{\varphi}_1)^2 + (\bar{\varphi}_2)^2}{(\varphi_1)^2 + (\varphi_2)^2}}\,\varphi_3, \\[2ex]
\bar{\varphi}_0 = \sqrt{\dfrac{(\bar{\varphi}_1)^2 + (\bar{\varphi}_2)^2}{(\varphi_1)^2 + (\varphi_2)^2}}\,\sqrt{(\varphi_3)^2 - (\varphi_1)^2 - (\varphi_2)^2},
\end{cases}
$$

that is, $\bar{\varphi}_3$ and $\bar{\varphi}_0$ can be expressed through $\bar{\varphi}_1$, $\bar{\varphi}_2$ and φ_1, φ_2, φ_3. Hence P, Q, R and S can be expressed, in a *unique way*, as real analytic functions of $\bar{\varphi}_1$, $\bar{\varphi}_2$, ϕ, φ_1, φ_2 and φ_3. (These functions will be denoted later by the same letters.)

Now, if we substitute these functions for P, Q, R and S into the last equations $(3.9)_3$, $(3.10)_3$ and $(3.11)_3$, these equations will be satisfied up to sign. But if φ_1, φ_2 and φ_3 are "generic", and if we put $\bar{w} = w$, $\bar{x} = x$ and $\bar{\varphi}_i = \varphi_i$ for $i = 1, 2, 3$, we get $P = S = 1$ and $Q = R = 0$. Thus, if some of the equations $(3.9)_3$-$(3.11)_3$ is, after our general substitution, satisfied with the opposite sign, we come to a contradiction with this special case.

We conclude that all equations (3.9), (3.10) and (3.11) are, in general, consequence of our functional expressions for P, Q, R and S. Now, due to the meaning of these quantities in (3.12), we still have to satisfy the integrability conditions

$$
\tag{3.22}
P'_w = Q'_x, \quad R'_w = S'_x.
$$

We see easily that these two conditions can be written down in the form resolved with respect to $\bar{\varphi}'_{1x}$ and $\bar{\varphi}'_{2x}$, respectively. We add the last partial differential equation

$$
\tag{3.23}
\phi'_x = (\phi'_w - H)\frac{R}{S},
$$

which is enforced by the second equality of (3.5).

For φ_1, φ_2, φ_3 and H given, we obtain a system of three PDE of first order for the unknown functions $\bar{\varphi}_1$, $\bar{\varphi}_2$ and ϕ in the form where the Cauchy-Kowalewski theorem can be applied. The general solution depends on three arbitrary functions of the variable w. Taking any particular solution $(\bar{\varphi}_1, \bar{\varphi}_2, \phi)$, the functions $\bar{w} = \bar{w}(w, x)$ and $\bar{x} = \bar{x}(w, x)$ are determined by the given P, Q, R and S up to an additive constant, and $\bar{y} = y + \phi(w, x)$ is also determined. Now, denoting $\tilde{H} = -\dfrac{\phi'_x}{R}$, we get from (3.23) that $\tilde{H} = \dfrac{H - \phi'_w}{S}$. Hence we obtain

$$
\tag{3.24}
\phi'_x = -\tilde{H}\bar{w}'_x, \quad \phi'_w = H - \tilde{H}\bar{w}'_w,
$$

which is equivalent to

$$
\tag{3.25}
d\phi = -\tilde{H}\,d\bar{w} + H\,dw.
$$

According to (3.5), we see $\tilde{H} = \bar{H}$. This shows that $\bar{A}$, $\bar{f}$, $\bar{C}$ and $\bar{H}$ can be determined from A, f, C and H using three arbitrary functions of the variable w. In other words, *each local isometry class of our family of metrics depends* (geometrically) *on three arbitrary functions of one variable* w, say $\bar{\varphi}_1(w, x_0)$, $\bar{\varphi}_2(w, x_0)$ and $\bar{H}(w, x_0)$. Combining this with the existence result (five arbitrary functions of w!) in the previous section we have proved Theorem 1. □

Further, if we combine the above technique with the computations in [4], we obtain the following improvements of Theorem 7.2 and Theorem 7.3 from [4]: let us denote by $\mathcal{M}$ the class of all 3-dimensional Riemannian manifolds (M, g) whose Ricci eigenvalues ρ_1, ρ_2 and ρ_3 satisfy the following conditions:

a) $\rho_3 > 0$ is a prescribed constant,
b) $\rho_1 = \rho_2$ $(\neq \rho_3)$ is a function which is constant along each principal geodesics (and so is the scalar curvature of (M, g)).

We have

Theorem 2. *The local isometry classes of $\mathcal{M}$ are parametrized by one arbitrary function of two variables modulo one arbitrary function of one variable.*

Theorem 3. *The local isometry classes of all $(M, g) \in \mathcal{M}$ with prescribed scalar curvature are parametrized by two arbitrary functions of one variable.*

Obviously, if the scalar curvature is prescribed as a constant, we obtain Theorem 1 as a special case.

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Oldřich Kowalski
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
186 00 Praha 8, CZECH REPUBLIC
e-mail: mu@karlin.mff.cuni.cz

Masami Sekizawa
Tokyo Gakugei University
Koganei-shi Nukuikita-machi 4-1-1,
Tokyo 184, JAPAN
e-mail: sekizawa@u-gakugei.ac.jp