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# ON PACKING OF SQUARES INTO A RECTANGLE

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Abstract. It is proved in this paper that any system of squares with total area 1 may be packed into a rectangle whose area is less then 1.53.

The following problem is formulated in [7]: Determine the smallest number $S$ such that any system of squares with total area 1 may be (parallelly) packed into a rectangle of area $S$.

This problem was posed by L. Moser [4]. $S \geq \frac{1+\sqrt{2}}{2} \doteq 1.207$ follows from considering two squares of sides $x$ and $y$, where $x > y$, $x^2 + y^2 = 1$ and $x(x+y)$ is maximal. Novotný [8] proved that any system of three squares with total area 1 may be packed into a rectangle of area 1.227759 (this area is necessary for packing of three squares with sides 0.7297177, 0.5588698 and 0.3939246). The four squares with sides $x_1 = \sqrt{\frac{1}{2}}$, $x_2 = x_3 = x_4 = \sqrt{\frac{1}{6}}$ show that $S \geq \frac{2+\sqrt{3}}{3} > 1.244$.

Moon and Moser [3] found first results for the upper bound. They proved that

(1) it is possible to pack any system of squares with sides $x_1 \geq x_2 \geq x_3 \geq \cdots$ and with total area 1 into a square of side $a = x_1 + \sqrt{1 - x_1^2}$.

A consequence of this is that any system of squares with total area 1 may be packed into a square of area 2.

Meir and Moser [2] extended the result (1) and they proved that

(2) any system of squares with total area $V$ can be packed into a rectangle of size $a_1 \times a_2$ if $a_1 > x_1$, $a_2 > x_1$ and $x_1^2 + (a_1 - x_1)(a_2 - x_1) \geq V$.

Some further results for the upper bound were published by Kleitman and Krieger [1]: Any system of squares with total area 1 can be packed into a rectangle of size $\sqrt{2} \times \sqrt{\frac{4}{3}}$; its area is $\sqrt{\frac{8}{3}} \doteq 1.633$. It follows from this result that

(3) any system of squares with total area $V$ can be packed into a rectangle with sides $\sqrt{2V}$ and $\sqrt{\frac{4V}{3}}$.

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The following theorem improves the upper estimate for $S$.

**Theorem.** *Any system of squares with total area* 1 *may be packed into a rectangle whose area is less than* 1.53.

**Proof.** We denote the squares $Q_1, Q_2, Q_3, \dots$ and their sides $x_1 \geq x_2 \geq x_3 \geq \cdots$. We shall pack the squares in the dependence upon $x_1, x_2$ as it follows:

**I.** Let $x_1 \geq \sqrt{\frac{4}{7}}$. By (3) we can pack the squares $Q_2, Q_3, \dots$ into a rectangle $P$ with sides $\sqrt{2(1-x_1^2)}, \sqrt{\frac{4(1-x_1^2)}{3}}$ and the whole system can be packed into a rectangle $R$ with sides $x_1$ and $x_1 + \sqrt{2(1-x_1^2)}$ (Fig.1). The area of $R$ is less than 1.53.

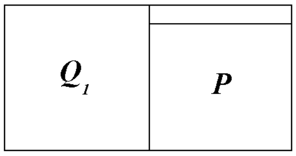


Fig. 1

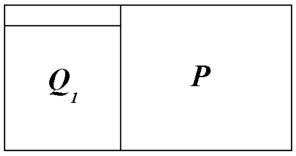


Fig. 2

**II.** Let $0.645 \leq x_1 \leq \sqrt{\frac{4}{7}}$. We pack the squares $Q_2, Q_3, \dots$ as in **I** and all squares can be packed into a rectangle $R$ with sides $\sqrt{\frac{4(1-x_1^2)}{3}}$ and $x_1 + \sqrt{2(1-x_1^2)}$ (Fig.2). The area of $R$ is less than 1.53 for every $x_1 \in \langle 0.645, \sqrt{4/7} \rangle$.

**III.** Let $x_1 \leq 0.27$. By (1) the squares can be packed into a square $R$ of side $x_1 + \sqrt{1-x_1^2}$; its area $1 + 2x_1\sqrt{1-x_1^2} < 1.53$ for every $x_1 \leq 0.27$.

It remains to investigate the domain

$$M = \{[x_1, x_2];\ 0.27 \leq x_1 \leq 0.645,\ 0 < x_2 \leq x_1\}.$$

**IV.** By (3) we can pack the squares $Q_3, Q_4, \dots$ into a rectangle $P$ with sides $\sqrt{2(1-x_1^2-x_2^2)}$ and $\sqrt{\frac{4(1-x_1^2-x_2^2)}{3}}$. All squares can be packed into a rectangle $R$ by Fig. 3 if $x_1 + x_2 \geq \sqrt{2(1-x_1^2-x_2^2)}$, i.e. $3x_1^2 + 2x_1x_2 + 3x_2^2 \geq 2$, or by Fig. 4 if $x_1 + x_2 \leq \sqrt{2(1-x_1^2-x_2^2)}$.

The area of $R$ from Fig. 3 is

$$f_1(x_1, x_2) = (x_1 + x_2)\left(x_1 + \frac{2}{\sqrt{3}}\sqrt{1-x_1^2-x_2^2}\right).$$

We have

$$\frac{\partial f_1}{\partial x_1} = \frac{1}{\sqrt{3(1-x_1^2-x_2^2)}}\left(2 - 4x_1^2 - 2x_2^2 - 2x_1x_2 + (2x_1+x_2)\sqrt{3(1-x_1^2-x_2^2)}\right).$$

If we denote $u(x_1, x_2) = 2 - 4x_1^2 - 2x_2^2 - 2x_1x_2 + (2x_1+x_2)\sqrt{3(1-x_1^2-x_2^2)}$, then evidently $\frac{\partial u}{\partial x_1} < 0, \frac{\partial u}{\partial x_2} < 0$ in $M_1$ (Fig. 11). Hence

$u(x_1,x_2) \geq u(0.645, 0.645) > 0$, $\frac{\partial f_1}{\partial x_1} > 0$, thus $f_1(x_1,x_2) \leq f_1(0.645, x_2)$ for $[x_1,x_2] \in M_1$. We verify easily that $f_1(0.645, x_2) < 1.53$ for every $x_2 \leq 0.645$.

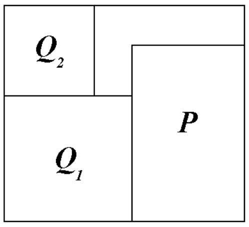



Fig. 3

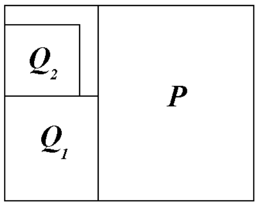



Fig. 4

The area of $R$ from Fig. 4 is

$$f_2(x_1,x_2) = \left(x_1 + \frac{2}{\sqrt{3}}\sqrt{1-x_1^2-x_2^2}\right)\sqrt{2(1-x_1^2-x_2^2)};$$

$$\frac{\partial f_2}{\partial x_1} = \frac{\sqrt{2}}{\sqrt{1-x_1^2-x_2^2}}\left(1-2x_1^2-x_2^2-\frac{4x_1}{\sqrt{3}}\sqrt{1-x_1^2-x_2^2}\right) < 0$$

for $[x_1,x_2] \in M_2$ (Fig.11). Evidently $\frac{\partial f_2}{\partial x_2} < 0$, too, and since $f_2(Z_i) < 1.53$ for $i \in \{1,2,3,4,5\}$, we have $f_2(x_1,x_2) < 1.53$ for every $[x_1,x_2] \in M_2$.

**V.** We pack the squares $Q_3, Q_4, \ldots$ as in **IV**. All squares can be packed into a rectangle $R$ by Fig. 5 if

$$x_1 + x_2 \geq \sqrt{\frac{4(1-x_1^2-x_2^2)}{3}}, \text{ i.e. } 7x_1^2 + 6x_1x_2 + 7x_2^2 \geq 4,$$

or by Fig. 6 if

$$x_1 + x_2 \leq \sqrt{\frac{4(1-x_1^2-x_2^2)}{3}}.$$

The area of $R$ from Fig. 5 is

$$f_3(x_1,x_2) = (x_1+x_2)\left(x_1 + \sqrt{2(1-x_1^2-x_2^2)}\right).$$

Since $\frac{\partial f_3}{\partial x_1} > 0$, $\frac{\partial f_3}{\partial x_2} > 0$ in $M_3$ (Fig. 11) and $f_3(Z_i) < 1.53$ for $i \in \{6,7,8,9\}$, $f_3(x_1,x_2) < 1.53$ is fulfilled for every $[x_1,x_2] \in M_3$.

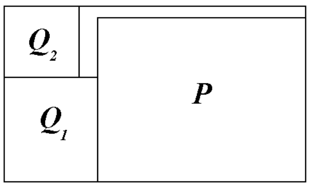



Fig. 5

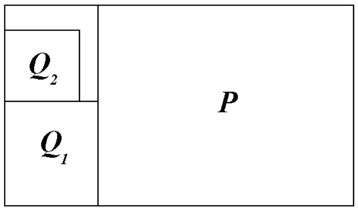



Fig. 6

The area of $R$ from Fig. 6 is

$$f_4(x_1,x_2) = \left(x_1 + \sqrt{2(1-x_1^2-x_2^2)}\right)\frac{2}{\sqrt{3}}\sqrt{1-x_1^2-x_2^2}.$$

Since $\frac{\partial f_4}{\partial x_1} < 0$, $\frac{\partial f_4}{\partial x_2} < 0$ in $M_4$ (Fig. 11) and $f_4(Z_i) < 1.53$ for $i \in \{10,11,\dots,14\}$, we have $f_4(x_1,x_2) < 1.53$ for every $[x_1,x_2] \in M_4$.

**VI.** Let $2x_2 \le x_1$. By (2) we can pack the squares $Q_4, Q_5, \dots$ into a rectangle $P$ with sides $a$ and $x_1 + x_2$ if $x_4^2 + (a - x_4)(x_1 + x_2 - x_4) = 1 - x_1^2 - x_2^2 - x_4^2$ $(\ge 1 - x_1^2 - x_2^2 - x_3^2)$. It is valid for

$$a = \frac{1 - x_1^2 - x_2^2 - 3x_4^2 + x_1x_4 + x_2x_4}{x_1 + x_2 - x_4}.$$

All squares can be packed into a rectangle $R$ by Fig. 7. Its area is

$$f_5(x_1,x_2,x_4) = (x_1+x_2)(x_1+a) = \frac{(x_1+x_2)(1 - x_2^2 - 3x_4^2 + x_1x_2 + x_2x_4)}{x_1+x_2-x_4}.$$

We have

$$\frac{\partial f_5}{\partial x_4} = \frac{(x_1+x_2)(1 + 2x_1x_2 + 3x_4^2 - 6x_1x_4 - 6x_2x_4)}{(x_1+x_2-x_4)^2}.$$

If we denote $v = 1 + 2x_1x_2 + 3x_4^2 - 6x_1x_4 - 6x_2x_4$, then $\frac{\partial v}{\partial x_4} < 0$, thus $v(x_1,x_2,x_4) \ge v(x_1,x_2,x_2) = 1 - 4x_1x_2 - 3x_2^2 > 0$ for $[x_1,x_2] \in M_5$ (Fig. 11). Hence $\frac{\partial f_5}{\partial x_4} > 0$ and

$$f_5(x_1,x_2,x_4) \le f_5(x_1,x_2,x_2) = \frac{(x_1+x_2)(1 - 3x_2^2 + x_1x_2)}{x_1} = g(x_1,x_2).$$

Since

$$\frac{\partial g}{\partial x_1} = \frac{x_2(x_1^2 + 3x_2^2 - 1)}{x_1^2} < 0 \text{ in } M_5,$$

$$g(x_1,x_2) \le g(2x_2,x_2) = \frac{3(1-x_2^2)}{2} < 1.5,$$

we have $f_5(x_1,x_2,x_4) \le g(x_1,x_2) < 1.5$ for every $[x_1,x_2] \in M_5$, $x_4 \le x_2$.

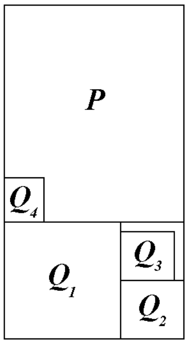



Fig. 7

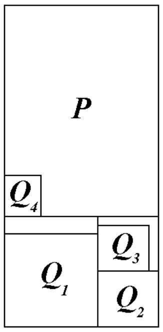



Fig. 8

**VII.** Let $2x_2 \geq x_1$. We pack $Q_4, Q_5, \dots$ as in **VI**. All squares can be packed into a rectangle $R$ by Fig. 8. The area of $R$ is

$$f_6(x_1, x_2, x_4) = (x_1 + x_2)(2x_2 + a) =$$

$$= \frac{(x_1 + x_2)(1 - x_1^2 + x_2^2 - 3x_4^2 + x_1x_4 + 2x_1x_2 - x_2x_4)}{x_1 + x_2 - x_4};$$

$$\frac{\partial f_6}{\partial x_4} = \frac{(x_1 + x_2)(1 + 2x_1x_2 + 3x_4^2 - 6x_1x_4 - 6x_2x_4)}{(x_1 + x_2 - x_4)^2} \geq$$

$$\geq \frac{(x_1 + x_2)(1 + 2x_1x_2 + 3x_2^2 - 6x_1x_2 - 6x_2^2)}{(x_1 + x_2 - x_4)^2} > 0$$

for $[x_1, x_2] \in M_6$ (Fig. 11), $x_4 \leq x_2$. Hence $f_6(x_1, x_2, x_4) \leq f_6(x_1, x_2, x_2)$. Denoting

$$h(x_1, x_2) = f_6(x_1, x_2, x_2) = \frac{(x_1 + x_2)(1 - x_1^2 - 3x_2^2 + 3x_1x_2)}{x_1},$$

we have

$$\frac{\partial h}{\partial x_1} = 2x_2 - 2x_1 + \frac{x_2(3x_2^2 - 1)}{x_1^2} < 0, \frac{\partial h}{\partial x_2} = 2x_1 + \frac{1 - 9x_2^2}{x_1} > 0$$

in $M_6$. It follows from this that $h$ is maximal in $M_6$ at some from the points $Z_{12}, Z_{16}$. But $h(Z_{12}) < 1.53, h(Z_{16}) < 1.53$.

**VIII.** By (2) we can pack $Q_4, Q_5, \dots$ into a rectangle $P$ with sides $x_1 + x_2$ and $a$ if $x_4^2 + (a - x_4)(x_1 + 2x_2 - x_4) = 1 - x_1^2 - x_2^2 - x_4^2$, i.e.

$$a = \frac{1 - x_1^2 - x_2^2 - 3x_4^2 + x_1x_4 + 2x_2x_4}{x_1 + 2x_2 - x_4}.$$

All squares can be packed into a rectangle $R$ by Fig. 9. Its area is

$$f_7(x_1, x_2, x_4) = \frac{(x_1 + 2x_2)(1 - x_2^2 - 3x_4^2 + 2x_1x_2 + 2x_2x_4)}{x_1 + 2x_2 - x_4}.$$

Since

$$\frac{\partial f_7}{\partial x_4} = \frac{(x_1 + 2x_2)(1 + 4x_1x_2 + 3x_2^2 + 3x_4^2 - 6x_1x_4 - 12x_2x_4)}{(x_1 + 2x_2 - x_4)^2} \geq$$

$$\geq \frac{(x_1 + 2x_2)(1 - 2x_1x_2 - 6x_2^2)}{(x_1 + 2x_2 - x_4)^2} > 0$$

for $[x_1, x_2] \in M_7$ (Fig. 11), $x_4 \leq x_2$, we have $f_7(x_1, x_2, x_4) \leq f_7(x_1, x_2, x_2)$. If we denote

$$k(x_1, x_2) = f_7(x_1, x_2, x_2) = \frac{(x_1 + 2x_2)(1 - 2x_2^2 + 2x_1x_2)}{x_1 + x_2},$$

then the system

$$\frac{\partial k}{\partial x_1} = \frac{x_2(6x_2^2 + 4x_1x_2 + 2x_1^2 - 1)}{(x_1 + x_2)^2} = 0,$$

$$\frac{\partial k}{\partial x_2} = \frac{x_1 + 2x_1^3 + 4x_1^2 x_2 - 10x_1 x_2^2 - 8x_2^3}{(x_1 + x_2)^2} = 0$$

has no solution in the interior of $M_7$. Therefore the function $k$ has a maximum on the boundary of $M_7$. An easy calculation shows that this maximum is at the point $Z_{15}$ (Fig. 11) and $k(Z_{15}) < 1.53$.

Further

$$\frac{\partial f_7}{\partial x_1} = \frac{2x_1^2 x_2 + 8x_1 x_2^2 - 4x_1 x_2 x_4 + 8x_2^3 - 3x_2^2 x_4 - 2x_2 x_4^2 - x_4 + 3x_4^3}{(x_1 + 2x_2 - x_4)^2} \geq$$

$$\geq \frac{9x_2^3 - x_2}{(x_1 + 2x_2 - x_4)^2} > 0$$

for $[x_1, x_2] \in M_8$ (Fig. 11), $x_4 \leq x_2$. It means that $f_7(x_1, x_2, x_4) \leq f_7(0.42, x_2, x_4)$. If we denote $\varphi(x_2, x_4) = f_7(0.42, x_2, x_4)$, then

$$\frac{\partial \varphi}{\partial x_4} = \frac{(0.42 + 2x_2)(1 + 1.68x_2 + 3x_2^2 + 3x_4^2 - 2.52x_4 - 12x_2 x_4)}{(0.42 + 2x_2 - x_4)^2}. \tag{4}$$

For $w(x_2, x_4) = 1 + 1.68x_2 + 3x_2^2 + 3x_4^2 - 2.52x_4 - 12x_2x_4$ and for $x_4 \geq 0.35$ we have $w(x_2, x_4) \leq w(x_2, 0.35) = 3x_2^2 - 2.52x_2 + 0.4855 < 0$ for all $x_2 \in \langle 0.34, 0.39\rangle$. Similarly, if $x_4 \leq 0.34$, then $w(x_2, x_4) \geq w(x_2, 0.34) = 3x_2^2 - 2.4x_2 + 0.49 > 0$ for $x_2 \in \langle 0.34, 0.39\rangle$. In consequence of this the function $\varphi$ has a maximum for $x_4 \in \langle 0.34, 0.35\rangle$. We shall estimate $\max_T \varphi(x_2, x_4)$ for $T = \langle 0.34, 0.39\rangle \times \times \langle 0.34, 0, 35\rangle$. It follows from $\frac{\partial w}{\partial x_2} < 0$, $\frac{\partial w}{\partial x_4} < 0$ that $-0.041 = w(0.39, 0.35) \leq \leq w(x_2, x_4) \leq w(0.34, 0.34) = 0.0208$. Since

$$\frac{0.42 + 2x_2}{(0.42 + 2x_2 - x_4)^2} \leq \frac{0.42 + 0.78}{(0.42 + 0.33)^2} < 2.2,$$

we have in regard of (4) $|\frac{\partial \varphi}{\partial x_4}| < 0.1$ in $T$. Further

$$\frac{\partial \varphi}{\partial x_2} = (0.148176 + 1.0584x_2 + 0.84x_2x_4 - 0.84x_4^2 - 8x_2^3 +$$

$$+ 14x_2^2 x_4 - 8x_2 x_4^2 + 6x_4^3 - 2x_4)/(0.42 + 2x_2 - x_4)^2.$$

Since the function

$$t(x_2, x_4) = 0.148176 + 1.0584x_2 + 0.84x_2x_4 - 0.84x_4^2 - 8x_2^3 + 14x_2^2x_4 - 8x_2x_4^2 + 6x_4^3 - 2x_4$$

satisfies $\frac{\partial t}{\partial x_2} > 0$, $\frac{\partial t}{\partial x_4} < 0$, we have $-0.01885 = t(0.34, 0.35) \leq t(x_2, x_4) \leq \leq t(0.39, 0.34) = 0.019828$ and because of $1/(0.42 + 2x_2 - x_4)^2 < 1.8$ we get $|\frac{\partial \varphi}{\partial x_2}| < 0.04$ in $T$.

If $U \subset T$ is a square with side of length 0.01, then for $Y_1, Y_2 \in U$ the inequality

$$|\varphi(Y_1) - \varphi(Y_2)| < 0.0014 \tag{5}$$

is satisfied. Since the function $\varphi$ gets values less than 1.527 at the points $[x_2, 0.34]$ for $x_2 \in \{0.34, 0.35, 0.36, 0.37, 0.38\}$, (5) yields $\varphi(x_2, x_4) < 1.53$ in $T$.

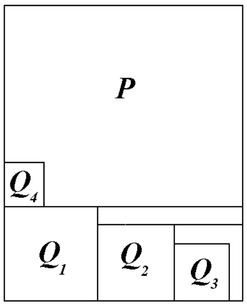



Fig. 9

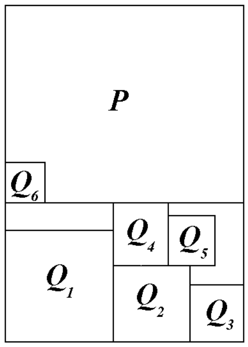



Fig. 10

**IX.** Let $[x_1, x_2] \in M_9 = \langle 0.42, 0.50\rangle \times \langle 0.29, 0.37\rangle$, $x_2 + x_4 \geq x_1$. By (2) we can pack $Q_6, Q_7, \ldots$ into a rectangle $P$ with sides $a$ and $x_1 + x_2 + x_3$ if

$$x_6^2 + (a - x_6)(x_1 + x_2 + x_3 - x_6) = 1 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_6^2,$$

i.e

$$a = \frac{1 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - 3x_6^2 + x_6(x_1 + x_2 + x_3)}{x_1 + x_2 + x_3 - x_6}.$$

All squares can be packed into a rectangle $R$ (Fig. 10) with sides $x_1 + x_2 + x_3$, $x_2 + x_4 + a$. Its area is $f_8(x_1, x_2, x_3, x_4, x_6) = (x_1 + x_2 + x_3)[(x_2 + x_4)(x_1 + x_2 + x_3 - x_6) + 1 - x_1^2 - x_2^2 - x_3^2 - x_4^2 - 3x_6^2 + x_6(x_1 + x_2 + x_3)]/(x_1 + x_2 + x_3 - x_6)$. Evidently $\frac{\partial f_8}{\partial x_4} > 0$, hence

$$f_8(x_1, x_2, x_3, x_4, x_6) \leq f_8(x_1, x_2, x_3, x_3, x_6) = m(x_1, x_2, x_3, x_6) =$$

$$= \frac{(x_1 + x_2 + x_3)(x_1x_2 + 2x_2x_3 + x_1x_3 + 1 - x_1^2 - x_3^2 - 3x_6^2 + x_1x_6)}{x_1 + x_2 + x_3 - x_6}.$$

Because of $\frac{\partial m}{\partial x_1} = [x_6(2x_1^2 + x_2^2 + 2x_3^2 + x_1x_2 + x_1x_3 - 1 + 3x_6^2 - 2x_1x_6 - x_2x_6 - x_3x_6) + (x_2 + x_3 - 2x_1)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3 - x_6)]/(x_1 + x_2 + x_3 - x_6)^2 \leq$

$$\leq \frac{x_6(2 \cdot 0.5^2 + 3 \cdot 0.37^2 + 0.37 - 1 + 3x_6^2 - 1.74x_6) - 0.1 \cdot 0.71^2}{(x_1 + x_2 + x_3 - x_6)^2} =$$

$$= \frac{3x_6^3 - 1.74x_6^2 + 0.2807x_6 - 0.05041}{(x_1 + x_2 + x_3 - x_6)^2} < 0$$

for $x_6 \leq 0.37$, $m$ has a maximum in $M_9$ for $x_1 = 0.42$. Similarly, $\frac{\partial m}{\partial x_2} = [(x_1 + x_2 + x_3)(x_1 + 2x_3)(x_1 + x_2 + x_3 - x_6) - x_6(x_1x_2 + 2x_2x_3 + x_1x_3 + 1 - x_1^2 - x_3^2 - 3x_6^2 + x_1x_6)]/(x_1 + x_2 + x_3 - x_6)^2 \geq [(0.71 + x_3)(0.42 + 2x_3) \cdot 0.71 - x_3(0.50 \cdot 0.37 + 0.74x_3 + 0.50x_3 + 1 - 0.42^2 - x_3^2 + 0.50x_3)]/(x_1 + x_2 + x_3 - x_6)^2 =$

$$= \frac{0.211722 + 0.2978x_3 - 0.32x_3^2 + x_3^3}{(x_1 + x_2 + x_3 - x_6)^2} > 0,$$

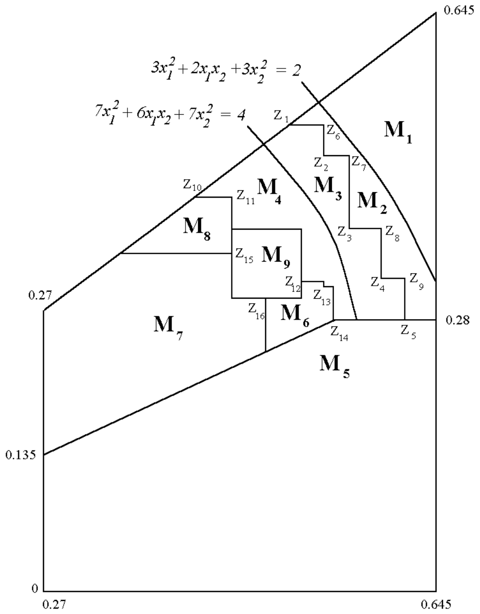
and hence $m$ has a maximum for $x_2 = 0.37$.

Further, on the assumptions $x_1 = 0.42, x_2 = 0.37$, using $x_3 \geq x_6$,

$$\frac{\partial m}{\partial x_3} \geq \frac{0.723956 - 0.3318x_3 - 1.58x_3^2 - 0.979x_6 - 1.16x_3x_6 + x_3^2x_6 + 3x_6^3 - 0.42x_6^2}{(x_1 + x_2 + x_3 - x_6)^2}.$$

Since the function $s(x_3, x_6) = 0.723956 - 0.3318x_3 - 1.58x_3^2 - 0.979x_6 - 1.16x_3x_6 + x_3^2x_6 + 3x_6^2 - 0.42x_6^2$ satisfies $\frac{\partial s}{\partial x_3} < 0$, $\frac{\partial s}{\partial x_6} < 0$, we have $s(x_3, x_6) \geq s(0.37, 0.37) > 0$, i.e. $\frac{\partial m}{\partial x_3} > 0$ and hence $m$ has a maximum for $x_3 = 0.37$. We find easily that $m$ is maximal if $x_6 = \frac{3.48 - \sqrt{6.8349}}{3}$ and that the maximal value of $f_8$ in $M_9$ is

$$f_8\left(0.42, 0.37, 0.37, 0.37, \frac{3.48 - \sqrt{6.8349}}{3}\right) < 1.53.$$




$Z_1 = [0.47, 0.47]$ $Z_2 = [0.51, 0.42]$
$Z_3 = [0.55, 0.35]$ $Z_4 = [0.59, 0.31]$
$Z_5 = [0.62, 0.28]$ $Z_6 = [0.51, 0.47]$
$Z_7 = [0.55, 0.42]$ $Z_8 = [0.59, 0.35]$
$Z_9 = [0.62, 0.31]$ $Z_{10} = [0.39, 0.39]$
$Z_{11} = [0.42, 0.37]$ $Z_{12} = [0.50, 0.32]$
$Z_{13} = [0.54, 0.31]$ $Z_{14} = [0.56, 0.28]$
$Z_{15} = [0.42, 0.34]$ $Z_{16} = [0.46, 0.29]$

Fig. 11

**X.** Let $[x_1, x_2] \in M_9, x_2 + x_4 \leq x_1$. As in **VIII**, the squares can be packed into a rectangle $R$ with area

$$f_7(x_1, x_2, x_4) = \frac{(x_1 + 2x_2)(1 - x_2^2 - 3x_4^2 + 2x_1x_2 + 2x_2x_4)}{x_1 + 2x_2 - x_4}.$$

Since

$$\frac{\partial f_7}{\partial x_1} \geq \frac{2x_1^2x_2 + 4x_1x_2^2 + 3x_2^3 - x_2}{(x_1 + 2x_2 - x_4)^2} > 0,$$

$$\frac{\partial f_7}{\partial x_4} \geq (x_1+2x_2)[1+4x_1x_2+3x_2^2+3(x_1-x_2)^2-6x_1(x_1-x_2)-$$
$$-12x_2(x_1-x_2)]/(x_1+2x_2-x_4)^2 = \frac{(x_1+2x_2)(1-3x_1^2+18x_2^2-8x_1x_2)}{(x_1+2x_2-x_4)^2} > 0$$

for $[x_1, x_2] \in M_9$, $f_7$ is maximal for $x_1 = 0.5, x_4 = 0.5 - x_2$. It is easy to show that $f_7(0.5, x_2, 0.5 - x_2) \leq f_7(0.5, 0.29, 0.21) < 1.5$ for $x_2 \in \langle 0.29, 0.37 \rangle$.

Since the domains $M_1, \ldots, M_9$ cover $M$, the proof is completed.

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