ARCHIVUM MATHEMATICUM (BRNO)
Tomus 32 (1996), 123 – 136

STANDARD HOMOGENEOUS EINSTEIN MANIFOLDS AND DIOPHANTINE EQUATIONS

Yurii G. Nikonorov and Eugene D. Rodionov

Abstract. Some new examples of standard homogeneous Einstein manifolds with semisimple transitive groups of motions and semisimple isotropy subgroups are constructed. For the construction of these examples the solutions of some systems of Diophantine equations are used.

Let g and h be the Lie algebras of the compact connected Lie groups G and H, and let g be semisimple, $g = g_1 \oplus ... \oplus g_r$, where $g_1, ..., g_r$ are simple Lie algebras. We put $B(X,Y) = -tr(adXadY)$ for all $X, Y \in g$, and we define the standard Riemannian metric ρ_B on G/H as the metric obtained from $B(X,Y)$ under the projection $\pi : G \to G/H$.

We note that in [1]-[6] a classification was given of the simply connected compact standard homogeneous Einstein manifolds $(G/H, \rho_B)$ either with a simple transitive group of motions G, or with a simple isotropy subgroup H.

Moreover, in the case of semisimple Lie groups G and H we have constructed new examples of standard homogeneous Einstein manifolds in the following way [5]

We consider the embedding

$$H = K \times L \subset (K \times ... \times K) \times L \subset K \times ... \times K \times G_0 = G ,$$

where the first embedding is of the form $diag \times id$ (K is taken t times) and the second is of the form $id \times ... \times id \times \pi$, where $\pi : K \times L \subset G_0$ is some embedding; G_0, K, L are compact connected simple Lie groups. Let g_0, k, l be the Lie algebras of the Lie groups G_0, K, L correspondingly.

Theorem A ([5],[6]). *Let $(g_0, k \oplus l)$ be a compact irreducible symmetric or compact nonsymmetric strictly isotropically-irreducible pair. Then the space $(G/H, \rho_B)$*

1991 *Mathematics Subject Classification*: 53C20, 53C25, 53C30.
Key words and phrases: Riemannian manifolds, homogeneous spaces, Einstein metrics.
Authors are partially supported by a grant from the International Science Foundation (grant # JF3100), a grant from the S.-Petersburg University (grant # 95-0-1.2-43) and a grant from the Russian Fund of Fundamental Investigations (grant # 96-01-00436).
Received March 18, 1996.

will be an Einstein manifold if and only if the Lie algebras g_0, k and l appear in the list presented in Table 1, and also in the first two cases the embeddings

$$\pi : so(n) \oplus so(m) \subset so(n+m) \ ,$$

$$\pi : sp(n) \oplus sp(m) \subset sp(n+m)$$

must be the standard embeddings whereas in the last two they must be given by

$$\pi : sp(1) \oplus sp(n) \subset so(4n) \ : \overset{1}{\circ} \otimes \overset{1}{\bullet} - \ldots - \bullet = \circ \quad (1 < n);$$

$$\pi : su(3) \oplus g_2 \subset e_6 \ : \ (\overset{1}{\circ} - \circ \otimes \overset{1}{\bullet} \equiv \circ) \oplus (\circ - \overset{2}{\circ} \otimes \bullet \equiv \circ).$$

We note that in the orthogonal and symplectic cases we have the following nontrivial solutions of Einstein equations correspondingly [6]:

$$(so) \qquad (n, m, t) = (t^2 - 4t + 6, t - 2, t) \qquad (t \in N)$$
$$(sp) \qquad (n, m, t) = (2s^2 - 1, s, 2s) \qquad (s \in N)$$

Table 1

g_0	k	l	Einstein equations
$so(n+m)$	$so(n)$	$so(m)$	$n^2 + (t-5)n + 6 - 2t =$ $= m[m + (n-2)(t-1)]$
$sp(n+m)$	$sp(n)$	$sp(m)$	$2n^2 + (5-t)n + 3 - t =$ $= 2m[(t-1)(n+1) + m]$
$so(4n)$	$sp(n)$	$sp(1)$	The metric is Einstein iff $t = 11$ and $n = 8$
e_6	g_2	$su(3)$	The metric is Einstein iff $t = 2$

In this paper we find all solutions of the above Diophantine equations. Our main result is the following one

Theorem B. *Let $(g_0, k \oplus l)$ be ether the pair $(so(n+m), so(n) \oplus so(m))$, or the pair $(sp(n+m), sp(n) \oplus sp(m))$. Then the space $(G/H, \rho_B)$ will be an Einstein manifold if and only if the triple (n, m, t) is contained in the list of Table 2.*

Table 2

$(g_0, k \oplus l)$	Einstein equations	(n, m, t)
$(so(n+m),$ $so(n) \oplus so(m))$	$n^2 + (t-5)n + 6 - 2t =$ $= m[m + (n-2)(t-1)]$	$\left(\frac{4sa^2}{c} + 2, \frac{2sa(b-as)}{c}, s+1\right)$, where c is a divisor of $4s$ $(s \in N)$, and a, b satisfy the Diophantine equation: $b^2 - (s^2+4)a^2 = c$
$(sp(n+m),$ $sp(n) \oplus sp(m))$	$2n^2 + (5-t)n + 3 - t =$ $= 2m[m + (n+1)(t-1)]$	$\left(\frac{2sa^2}{c} - 1, \frac{sa(b-as)}{c}, s+1\right)$, where c is a divisor of $2s$ $(s \in N)$, and a, b satisfy the Diophantine equation: $b^2 - (s^2+4)a^2 = -c$

It is easy to see that every natural solution of equation $b^2 - (s^2+4)a^2 = \pm c$ generates the triple (n, m, t) which consists of natural numbers. We also note that the equation $b^2 - (s^2+4)a^2 = \pm c$ have infinitely many solutions. So, for example, if we put $c = 1$ in the orthogonal case, we get the Pell equation $b^2 - (s^2+4)a^2 = 1$, which has infinitely many solutions for every $s \in N$ [7].

In this paper we also present other examples of standard homogeneous Einstein manifolds with semisimple Lie groups G and H.

Let

$$H = SO(k) \times SO(n) \times SO(m) \subset SO(k) \times [SO(n) \times ... \times SO(n)] \times SO(m) \subset$$

$$\subset SO(k+n) \times [SO(n) \times ... \times SO(n)] \times SO(n+m) = G \ ,$$

where the first embedding is of the form $id \times diag \times id$ ($SO(n)$ is taken t times) and the second is of the form $\pi_1 \times id \times ... \times id \times \pi_2$ ($SO(n)$ is taken $(t-2)$ times); $\pi_1 : SO(k) \times SO(n) \subset SO(k+n)$, $\pi_2 : SO(n) \times SO(m) \subset SO(n+m)$ are standard embeddings.

We also consider the analogous constructions for the unitary and symplectic cases:

$$H = SU(k) \times SU(n) \times SU(m) \subset SU(k+n) \times [SU(n) \times ... \times SU(n)] \times SU(n+m) = G \ ,$$

$$H = Sp(k) \times Sp(n) \times Sp(m) \subset Sp(k+n) \times [Sp(n) \times ... \times Sp(n)] \times Sp(n+m) = G \ .$$

Theorem C. *Let (g, h) be either the pair*
$(so(k+n) \oplus (t-2) \cdot so(n) \oplus so(n+m),\ so(k) \oplus so(n) \oplus so(m))$, *or the pair*
$(su(k+n) \oplus (t-2) \cdot su(n) \oplus su(n+m),\ su(k) \oplus su(n) \oplus su(m))$, *or the pair*

$(sp(k+n) \oplus (t-2)\cdot sp(n) \oplus sp(n+m),\ sp(k) \oplus sp(n) \oplus sp(m))$. Then the space $(G/H, \rho_B)$ will be an Einstein manifold if and only if the triple (n, m, t) is contained in the list of Table 3.

Table 3

cases	Einstein equations	(n, m, t)
orthogonal	$(n-2m-1)(m+n-2) =$ $= (m-1)(n-2)(t-2)$, $k=m$	$\left(\frac{y-3s}{s^2+8}+2, \frac{z+1}{4}-\frac{s(y-3s)}{4(s^2+8)}, s+1\right)$, where (y,z) is natural solution of the Diophantine equation: $y^2-(s^2+8)z^2 = 8(s^2-1)$ $(s \in N)$
unitary	$m(n^2+1)(m+n) =$ $= (m^2-1)n(2m+nt)$, k=m	There are no solutions for every $n, m, t \in N$
symplectic	$(2n-4m+1)(m+n+1) =$ $= (2m+1)(n+1)(t-2)$, k=m	$\left(\frac{y+3s}{2(s^2+8)}-1, \frac{z-1}{8}-\frac{s(y+3s)}{8(s^2+8)}, s+1\right)$, where (y,z) is natural solution of the Diophantine equation: $y^2-(s^2+8)z^2 = 8(s^2-1)$ $(s \in N)$

We note that every solution of the equation $y^2 - (s^2+8)z^2 = 8(s^2-1)$ does not generate natural solution of Einstein equations. So, for example, for $s = 1$ in orthogonal case we have solutions $n = 2m+1$, but in symplectic case we have no solutions for $s = 1$. Below we show that Einstein equation in orthogonal case has infinitely many solutions for every $s > 1$ and we find sufficient conditions which imply the existence of infinite family of solutions for the symplectic case.

1. Preliminaries

The proof of Theorems is based on a series of lemmas which will be formulated under the assumptions of the theorems.

Let g and h be the Lie algebras of the compact connected Lie groups G and H, and let g be semisimple, $g = g_1 \oplus ... \oplus g_r$, where $g_1, ..., g_r$ are simple Lie algebras. We put $B(X,Y) = -tr(adX adY)$ for all $X, Y \in g$, where $adX(Z) = [X,Z]$, and we consider the standard Riemannian metric ρ_B on G/H. It is easy to see that $B_g = B_{g_1} + ... + B_{g_r}$ and $g = h \oplus p = h \perp_B p$, where p is $ad(h)$-invariant (i.e. $[h,p] \subset p$).

We introduce some more notation: χ is the isotropy representation of the group H on $T_{\bar{e}}(G/H) = p$; then $p = p_0 \oplus p_1 \oplus \ldots \oplus p_s$, where χ acts trivially on p_0 and irreducibly on $p_1, \ldots, p_s$.

Lemma 1 [4]. *The space $(G/H, \rho_B)$ with $H \neq e$ is an Einstein manifold if and only if $p_0 = 0$ and $B_g|_h^*(\lambda_i, \lambda_i + 2\delta) = B_g|_h^*(\lambda_j, \lambda_j + 2\delta)$, where λ_i is the highest weight of the representation χ on p_i, 2δ is the sum of positive roots of the algebra h, and $B_g|_h^*$ is the scalar product on h^* induced by $B_g|_h$.*

Given a simple Lie algebra k, we consider the scalar product B'_k defined by $B_k = \alpha_k B'_k$, where α_k is the Casimir constant of the adjoint representation of the algebra k [4].

If l, k are both simple Lie algebras and $k \subset l$, then the index of k in l is the constant $[l : k]$ so that $B'_l = [l : k] \cdot B'_k$. In [8] Dynkin showed that this constant is an integer number.

Lemma 2 (corresponds to Theorem C).

(i) *Let*

$$H = SO(k) \times SO(n) \times SO(m) \subset SO(k+n) \times [SO(n) \times \ldots \times SO(n)] \times SO(n+m) = G,$$

with embeddings as in Theorem C. Then the standard metric on G/H is Einstein if and only if $(n - 2m - 1)(m + n - 2) = (m - 1)(n - 2)(t - 2)$ and $m = k$.

(ii) *Let*

$$H = SU(k) \times SU(n) \times SU(m) \subset SU(k+n) \times [SU(n) \times \ldots \times SU(n)] \times SU(n+m) = G,$$

with embeddings as in Theorem C. Then the standard metric on G/H is Einstein if and only if $m(n^2 + 1)(m + n) = (m^2 - 1)n(2m + nt)$ and $m = k$.

(iii) *Let*

$$H = Sp(k) \times Sp(n) \times Sp(m) \subset Sp(k+n) \times [Sp(n) \times \ldots \times Sp(n)] \times Sp(n+m) = G,$$

with embeddings as in Theorem C. Then the standard metric on G/H is Einstein if and only if $(2n - 4m + 1)(m + n + 1) = (2m + 1)(n + 1)(t - 2)$ and $m = k$.

Proof. For (i), if we pass to the Lie algebras, we have $\chi = \chi_1 \oplus \chi_2 \oplus \chi_3$, where $\chi_1 = id \hat{\otimes} (\oplus_{i=1}^{t-2} ad_{so(n)}) \hat{\otimes} id$, $\chi_2 = (\rho_k \hat{\otimes} \rho_n)$, $\chi_3 = (\rho_n \hat{\otimes} \rho_m)$; ρ_n — a standard representation, $\alpha_{so(n)} = 2(n - 2)$,

$$B_g|_h = 2(k+n-2) \cdot B'_{so(k)} + [2(k+n-2) + 2(t-2)(n-2) + 2(n+m-2)] \cdot B'_{so(n)} + \\ + 2(n + m - 2) B'_{so(m)} \quad .$$

Then if we use the criteria that the standard homogeneous Riemannian manifold is Einstein, we get a system of Einstein equations

$$\frac{2(n-2)}{2[2n + k + m - 4 + (n-2)(t-2)]} = \frac{k-1}{2(k+n-2)} + \\ + \frac{n-1}{2[2n + k + m - 4 + (n-2)(t-2)]} =$$

$$= \frac{n-1}{2[2n+k+m-4+(n-2)(t-2)]} + \frac{m-1}{2(n+m-2)} \quad ,$$

or equivalently $k = m$ and $\frac{n-3}{2n+2m-4+(n-2)(t-2)} = \frac{m-1}{m+n-2}$.

From this we deduce $(n-2m-1)(m+n-2) = (m-1)(n-2)(t-2)$ and $k = m$.

For (ii) we have $\chi = \chi_1 \oplus \chi_2 \oplus \chi_3$, where $\chi_1 = id\hat{\otimes}(\oplus_{i=1}^{t-2} ad_{su(n)})\hat{\otimes} id$, $\chi_2 = (\mu_k \hat{\otimes} \mu_n)$, $\chi_3 = (\mu_n \hat{\otimes} \mu_m)$; μ_n — a standard representation, $\alpha_{su(n)} = 2n$,

$$B_g|_h = 2(k+n) \cdot B'_{su(k)} + 2[2n+m+k+n(t-2)] \cdot B'_{su(n)} + 2(n+m)B'_{su(m)} \quad ,$$

and a system of Einstein equations

$$\frac{2n}{2[2n+k+m+n(t-2)]} = \frac{k^2-1}{k} \cdot \frac{1}{2(k+n)} + \frac{n^2-1}{n} \cdot \frac{1}{2[2n+k+m+n(t-2)]} =$$

$$= \frac{n^2-1}{n} \cdot \frac{1}{2[2n+k+m+n(t-2)]} + \frac{m^2-1}{m} \cdot \frac{1}{2(n+m)} \quad ,$$

or equivalently $k = m$ and $\frac{n^2+1}{n[2n+2m+n(t-2)]} = \frac{m^2-1}{m(m+n)}$, or $m(n^2+1)(m+n) = (m^2-1)n(2m+nt)$ and $k = m$.

For (iii) we have similarly, $\chi = \chi_1 \oplus \chi_2 \oplus \chi_3$, $\chi_1 = id\hat{\otimes}(\oplus_{i=1}^{t-2} ad_{sp(n)})\hat{\otimes} id$, $\chi_2 = (\nu_{2k} \hat{\otimes} \nu_{2n})$, $\chi_3 = (\nu_{2n} \hat{\otimes} \nu_{2m})$; ν_{2n} — a standard representation, $\alpha_{sp(n)} = 2(n+1)$,

$$B_g|_h = 2(k+n+1) \cdot B'_{sp(k)} + 2[2n+k+m+(n+1)(t-2)] \cdot B'_{sp(n)} + 2(n+m+1)B'_{sp(m)} \quad ,$$

and a system of Einstein equations

$$\frac{2(n+1)}{2[2n+k+m+2+(n+1)(t-2)]} = \frac{k+1/2}{2(k+n+1)} +$$

$$+ \frac{n+1/2}{2[2n+k+m+2+(n+1)(t-2)]} =$$

$$= \frac{n+1/2}{2[2n+k+m+2+(n+1)(t-2)]} + \frac{m+1/2}{2(n+m+1)} \quad ,$$

or $k = m$ and $(2n-4m+1)(m+n+1) = (2m+1)(n+1)(t-2)$. □

We shall use also some well known facts about solutions of Diophantine equations such a Pell equation and its generalizations. The equation

$$(1) \qquad x^2 - ay^2 = 1 \ ,$$

where a is natural number different from perfect squared is called Pell equation. It has infinitely many solutions into the class of natural numbers. If the pair (x_0, y_0) is minimal solution of equation (1) (i.e. $x_0 + \sqrt{a}y_0$ has minimal value among all numbers of type $x + \sqrt{a}y$, where (x, y) — arbitrary natural solution of (1) different from trivial $(1, 0)$) then general solution of Pell equation consists of pairs (x_n, y_n), where

$$x_n = \frac{1}{2}\Big((x_0 + \sqrt{a}y_0)^n + (x_0 - \sqrt{a}y_0)^n\Big) \ ,$$

$$y_n = \frac{1}{2\sqrt{a}}\Big((x_0 + \sqrt{a}y_0)^n - (x_0 - \sqrt{a}y_0)^n\Big) \ .$$

More general equation

$$x^2 - ay^2 = c \ , \tag{2}$$

where c — any integer number, has natural solution not for all value of c. Nevertheless, in the case when there is even one solution $(\tilde{x}, \tilde{y})$ of (2) this equation has infinitely many solutions of type $x = \tilde{x}x_n + a\tilde{y}y_n$, $y = \tilde{x}y_n + \tilde{y}x_n$, where (x_n, y_n) — a solution of Pell equation with the same value of a.

More precisely, it is known that all natural solutions of (2) are generated by this way from some finite set of solutions. All this results one can find, for example, in [7].

2. Proof of Theorem B

At first we consider orthogonal groups. In this case we have Diophantine equation which after change of variables $l = n + 2$, $s = t - 1$ can be reduced to the following one

$$l^2 - m^2 = sl(m - 1) \ . \tag{3}$$

Consider any natural solution of (3) when s is fixed natural number. Obviously $l^2 - 1$ is divided by $m - 1$, then $l^2 - 1 = k(m - 1)$, where k — a natural number.

By substituting this expression into (3) we obtain equation

$$k(m-1) - (m-1)(m+1) = sl(m-1) \ ,$$

which is equivalent (when $m \neq 1$) to the next one

$$k - m - 1 = sl \ .$$

Note that numbers k and $1 - m$ are precisely roots of quadratic equation

$$x^2 - (sl + 2)x + (1 - l^2) = 0 \ .$$

Really, $k + (1 - m) = sl + 2$ and $k(1 - m) = 1 - l^2$. Since this quadratic equation has integer roots, its discriminant D is perfect square of natural number, i.e.

$$D = l((s^2 + 4)l + 4s) = z^2 \ .$$

Let u be greatest common divisor of numbers l and $4s$, then $4s = cu$, $l = du$ for some natural c and d. It is necessary that $z = z_1 u$ ($z_1 \in N$) and

$$d((s^2 + 4)d + c) = z_1^2 \ .$$

Using that c and d are relatively prime we get $d = a^2$ and $(s^2+4)d+c = b^2$, where a and b — some natural numbers satisfying to the condition $ab = z_1$. From last two expressions we finally obtain

$$b^2 - (s^2 + 4)a^2 = c \ ,$$

where c is some divisor of $4s$. If numbers a and b satisfy this equation, then m and l can be easily computed in reverse order by formulas: $m = 2sa(b - as)/c$

and $l = 4sa^2/c$. It is easy to show by direct calculation that deriving numbers are solution of (3).

Proof of second part of theorem B we develope by the same scheme. After change of variables $l = n + 1$, $s = t - 1$ we obtain the following Diophantine equation

$$2l^2 - 2m^2 = sl(2m + 1) \ . \tag{4}$$

Consider any natural solution of (4) when s is fixed natural number. It is easy to see that $4l^2 - 1$ is divided by $2m + 1$, then $4l^2 - 1 = k(2m + 1)$ for any natural k.

We substitute this expression into (4) and we obtain equation

$$k(2m + 1) - (2m - 1)(2m + 1) = 2sl(2m + 1) \ ,$$

which is equivalent to the next one

$$k - 2m + 1 = 2sl \ .$$

Obviously, numbers k and $-(2m + 1)$ are precisely roots of quadratic equation

$$x^2 - (2sl - 2)x + (1 - 4l^2) = 0 \ .$$

In fact, $k - (2m + 1) = 2sl - 2$ and $-k(2m + 1) = 1 - 4l^2$. Since this quadratic equation has integer roots, its discriminant D is perfect square of natural number, i.e.

$$D = l((4s^2 + 16)l - 8s) = z^2 \ .$$

It is necessary that z is even number, i.e. $z = 2z_1$. Let u be greatest common divisor of numbers l and $2s$, then $2s = cu$, $l = du$ for some natural c and d, $z_1 = z_2 u$ ($z_2 \in N$) and

$$d((s^2 + 4)d - c) = z_2^2 \ .$$

Using that c and d are relatively prime we get $d = a^2$ and $(s^2+4)d-c = b^2$, where a and b — some natural numbers satisfying to the condition $ab = z_1$. From last two expressions we finally obtain

$$b^2 - (s^2 + 4)a^2 = -c \ ,$$

where c is some divisor of $2s$. If numbers a and b satisfy this equation then m and l can be easily computed in reverse order by formulas: $m = as(b - as)/c$ and $l = 2a^2 s/c$. It is easy to show by direct calculation that deriving numbers are solution of (4). The theorem is proved.

Remark 1. Note that equation (3) has infinitely many solutions for all natural s. Really, we can choose $c = 1$ and equation

$$b^2 - (s^2 + 4)a^2 = 1 \ ,$$

being Pell equation, has infinitely many solutions.

Equation (4) has infinitely many solutions for all even s. In this case we can choose $c = 4$ and equation

$$b^2 - (s^2 + 4)a^2 = -4$$

has one natural solution $b = s$, $a = 1$ and as follows from the theory of such equations it has infinitely many solutions.

The case when s is odd natural number require of special consideration. Let, for example, be $s = 1$. Then c is equal to 1 or to 2. The equation $b^2 - 5a^2 = -2$ has no integer solutions, but the equation $b^2 - 5a^2 = -1$ has the partial solution $b = 2$ and $a = 1$ and, as follows from the theory of such equations, it has infinitely many natural solutions.

Remark 2. It is easy to see that solutions of equation $b^2 - (s^2 + 4)a^2 = \pm c$ for different value of c can generate one and the same solution m, l of (3) or of (4). Really, all solutions which are obtained for $c = q$ consist in the set of solutions which are obtained for $c = p^2 q$.

3. Examples

Consider some examples of Theorem B when $t = 2$ ($s = 1$). We note that these examples appeared at first in paper of Mc Kenzie Y. Wang and Wolfgang Ziller [4]. In that paper they obtained only Einstein equations of the pairs of Theorem B without solutions of corresponding Diophantine equations.

(i) Let $t = 2$ and $(g_0, k \oplus l) = (so(n+m), so(n) \oplus so(m))$. Then we have $s = 1$ and the Diophantine equation

$$b^2 - 5a^2 = c \quad ,$$

where c is a divisor of 4. Using remark 2 from previous item, we can assume that $c = 4$ or $c = 2$. It is easy to see, that the equation

$$b^2 - 5a^2 = 2$$

has not natural solutions, but the equation

$$b^2 - 5a^2 = 4$$

has partial solution $b = 3$, $a = 1$, and it generates infinite family of solutions of above Diophantine equation.

(ii) Let $t = 2$ and $(g_0, k \oplus l) = (sp(n+m), sp(n) \oplus sp(m))$. Then we have $s = 1$ and the Diophantine equation

$$b^2 - 5a^2 = c \quad ,$$

where c is equal to either 1 or 2. Obviously, the equation

$$b^2 - 5a^2 = -2$$

has not natural solutions, but the equation

$$b^2 - 5a^2 = -1$$

has partial solution $b = 2$, $a = 1$, and it generates infinite family of solutions of above Diophantine equation.

Hence, in both these cases we obtain two infinite families of Einstein manifolds.

4. Proof of Theorem C

For the proof of theorem C we use Lemma 2.

As above, at first we consider orthogonal groups. In this case we have Diophantine equation, which after change of variables $l = n - 2$, $k = m - 1$, $s = t - 1$ can be reduced to the following one

$$2k^2 + skl + 3k + 1 - l^2 = 0 \ . \tag{5}$$

Fix natural number s and consider any natural solution of (5). It is easy to see, that $l^2 - 1$ is divided by k, then $l^2 - 1 = kp$, where p — a natural number.

By substituting this expression into (5) we obtain equation

$$2k + sl + 3 - p = 0 \ .$$

Note, that numbers p and $-2k$ are precisely roots of quadratic equation

$$x^2 - (sl + 3)x + (2 - 2l^2) = 0 \ .$$

Since this quadratic equation has integer roots, its discriminant D is perfect square of natural number, i.e.

$$D = s^2 l^2 + 6sl + 1 + 8l^2 = z^2 \ .$$

Natural number l is the root of quadratic equation

$$(s^2 + 8)l^2 + 6sl + 1 - z^2 = 0 \ .$$

Obviously, discriminant D_1 of last equation must be perfect square of even natural number

$$D_1 = 36s^2 - 4(s^2 + 8)(1 - z^2) = (2y)^2 \ .$$

Finally we obtain the equation

$$y^2 - (s^2 + 8)z^2 = 8(s^2 - 1) \ . \tag{6}$$

Numbers k and l can be easily computed in reverse order by formulas:
$l = (y - 3s)/(s^2 + 8)$, $k = (z - 3 - ls)/4$, where (y, z) is natural solution of the last Diophantine equation.

Note, that arbitrary solution of (6) does not generate natural l and m, we must choose only solution which satisfy to the following conditions
$y \equiv 3s(mod(s^2 + 8))$, $z \equiv 3 + sl(mod 4)$. A little below we discuss this problem.

Now we consider second part of the theorem. After change of variables $l = n+1$, $s = t - 1$ we obtain the following Diophantine equation

$$2l^2 - s(2m + 1)l - 4m^2 - m = 0 \ . \tag{7}$$

Fix natural number s and consider any natural solution of (7). Obviously, $4l^2 - 1$ is divided by $2m + 1$, i.e. $4l^2 - 1 = (2m + 1)p$, where p — a natural number. We substitute this expression into (7) and we get the equation

$$p - 2(2m - 1) - 1 = 2sl \ .$$

Note, that numbers p and $-2(2m+1)$ are precisely roots of quadratic equation

$$x^2 - (2sl - 3)x + (2 - 8l^2) = 0 \ .$$

Since this quadratic equation has integer roots, its discriminant D is perfect square of natural number, i.e.

$$D = 4s^2l^2 - 12sl - 1 + 32l^2 = z^2 \ .$$

Natural numbers l are the roots of quadratic equation

$$(4s^2 + 32)l^2 - 12sl + 1 - z^2 = 0 \ .$$

Obviously, discriminant D_1 of last equation is perfect square of some natural number, which is divided by 4, i.e.

$$D_1 = 144s^2 - 4(4s^2 + 32)(1 - z^2) = (4y)^2 \ .$$

Finally we get the equation, which is the same with (6)

$$y^2 - (s^2 + 8)z^2 = 8(s^2 - 1) \ .$$

Numbers k and l we compute in reverse order by formulas
$l = (y + 3s)/(2(s^2 + 8))$, $m = (z - 1 - 2ls)/8$, where y, z are natural solutions of the last Diophantine equation.

Note, that we must choose solutions of (6), which satisfy to the following conditions $y \equiv -3s(mod 2(s^2 + 8))$, $z \equiv 1 + 2sl(mod 8)$.

It is interesting, that the equation (7) has no solutions for some value of s (for example, for s=1, it follows from obvious fact that $4m + 2n + 1 \neq 0$ for all natural m and n) and has infinitely many solutions for some other value of s (below we consider the case $s = 4$).

At last we consider the equation

$$m(n^2 + 1)(m + n) = (m^2 - 1)n(2m + tn) \ . \tag{8}$$

Let d be greatest common divisor of numbers m and n, then $m = da$, $n = db$, a and b are relatively prime natural numbers. Equation (8) can be reduced to the following one

$$a(d^2b^2 + 1)(a + b) = (d^2a^2 - 1)b(2a + tb) \ .$$

Obviously, a^2 is divided by b, but a and b are relatively prime, then necessarily $b = 1$. We obtain the equation

$$a(d^2 + 1)(a + 1) = (d^2a^2 - 1)(2a + t) \ ,$$

where a and d are natural numbers.

If $a \geq 3$, then $d^2a^2 - 1 > d^2a + a$ ($a^2 - a > a + 1$ and $d^2(a^2 - a) > a + 1$). Since $2a + t > a + 1$, in this case there is no solutions.

If $a = 1$, then 4 is divided by $d^2 - 1$, but it is impossible for natural d.

If $s = 2$, then $10 = 10d^2 + (4d^2 - 1)t$, but it is impossible for natural d and t.

Theorem is proved.

Now we find some sufficient conditions for the existence of infinite family of solutions for Einstein equations (5) and (7).

Proposition 1. *For all natural $s > 1$ the Pell equation*

$$\tilde{y}^2 - (s^2+8)\tilde{z}^2 = 1 \tag{9}$$

has infinitely many natural solutions, which satisfy to the following conditions $\tilde{y} \equiv 1(mod 8(s^2+8))$, $\tilde{z} \equiv 0(mod\, 8)$.

Proof. Let (y_1, z_1) be arbitrary natural nontrivial solution of (9). Consider another solution (y_2, z_2), which is obtained as follows

$$y_2 + \sqrt{s^2+8}z_2 = (y_1 + \sqrt{s^2+8}z_1)^8 \ .$$

Obviously, $z_2 \equiv 0(mod 8)$, y_2 and $8(s^2+8)$ are relatively prime.

Let φ be Euler function ($\varphi(q)$ is the cardinality of natural numbers, which are less than q and relatively prime with them) and $\varphi(8(s^2+8)) = \alpha$, then by Euler theorem $y_2^\alpha \equiv 1(mod 8(s^2+8))$. We consider one more solution (y_3, z_3) of (9):

$$y_3 + \sqrt{s^2+8}z_3 = (y_2 + \sqrt{s^2+8}z_2)^\alpha \ .$$

Simple calculation shows, that $y_3 \equiv 1(mod 8(s^2+8))$, $z_3 \equiv 0(mod 8)$. Now we define a family of solutions of (9):

$$\tilde{y} + \sqrt{s^2+8}\tilde{z} = (y_3 + \sqrt{s^2+8}z_3)^m \ ,$$

where m is any natural number. All this solutions satisfy to the conditions $\tilde{y} \equiv 1(mod 8(s^2+8))$, $\tilde{z} \equiv 0(mod 8)$. □

Proposition 2. *For every $s \geq 1$ the Einstein equation, in orthogonal case, has infinitely many natural solutions.*

Proof. For $s = 1$ we have solutions $n = 2m+1$. Consider other cases. For all value of s we have partial solution of (6) $y_0 = 3s$, $z_0 = -1$. Using Proposition 1 we construct the family (y, z) of solutions of (6): $y = \tilde{y}y_0 + (s^2+8)\tilde{z}z_0 = 3s\tilde{y} - (s^2+8)\tilde{z}$, $z = \tilde{y}z_0 + \tilde{z}y_0 = -\tilde{y} + 3s\tilde{z}$. Obviously, such solution of (6) generate an integer solution of (5).

Really, $y = 3s\tilde{y} - (s^2+8)\tilde{z} \equiv 3s(mod 4(s^2+8))$, $l = (y-3s)/(s^2+8)$ is integer and $l \equiv 0(mod 4)$, $z = -\tilde{y} + 3s\tilde{z} \equiv 3(mod 4)$, m is integer too.

Now we show that obtained solutions (y, z) of (6) are natural. Since for $s > 1$, $3s > \sqrt{s^2+8}$ and $\tilde{y} - \sqrt{s^2+8}\tilde{z} = 1/(\tilde{y} + \sqrt{s^2+8}\tilde{z}) > 0$ then $3s\tilde{y} - (s^2+8)\tilde{z} > 0$, we proved that $y > 0$.

Obviously, that for $s > 1$, $\tilde{y} \neq 3s\tilde{z}$ and $9s^2 > s^2+8$. Then $(3s\tilde{z} - \tilde{y})(3s\tilde{z} + \tilde{y}) = 9s^2\tilde{z}^2 - \tilde{y}^2 > (s^2+8)\tilde{z}^2 - \tilde{y}^2 = -1$ and $3s\tilde{z} - \tilde{y} > 0$, i.e. $z > 0$.

It is sufficient to show that triples (n, m, t) which obtained from the solutions of (6) as above (see Table 3) consists of natural numbers. Since $(y-3s)(y+3s) = y^2 - 9s^2 = (s^2+8)(z^2-1) > 0$, then $y - 3s > 0$ and $n > 2$. Now suppose that

$z+1 \leq s(y-3s)/(s^2+8)$ then $z-1 < s(y+3s)/(s^2+8)$ and multiplying last two inequalities we have

$$z^2 - 1 < \frac{s^2(y^2-9s^2)}{(s^2+8)^2} < \frac{y^2-9s^2}{s^2+8} ,$$

and we obtain the contradiction with the equation (6). Therefore,
$z+1 > s(y-3s)/(s^2+8)$ and $m > 0$

Since $t = s+1 > 2$, then we really found infinitely many solutions of Einstein equation in orthogonal case. □

Proposition 3. *If the equation (6) has the natural solution* (y_0, z_0), *which satisfies to the condition* $y_0 \equiv -3s(mod 8(s^2+8))$, $z_0 \equiv 1(mod 8)$, *then the equation (7) has infinitely many natural solutions.*

Proof. Using Proposition 1 we construct the family (y, z) of solutions of (6): $y = \tilde{y}y_0 + (s^2+8)\tilde{z}z_0$, $z = \tilde{y}z_0 + \tilde{z}y_0$. Obviously, such solution of (6) generates the solution of (7). Really, $y = \tilde{y}y_0 + (s^2+8)\tilde{z}z_0 \equiv -3s(mod 8(s^2+8))$,
$z = \tilde{y}z_0 + \tilde{z}y_0 \equiv 1(mod 8)$. It is easy to see that every such solution generates the solution of (7) and corresponding triple (n, m, t) from Table 3 consists of natural numbers.

When s is even or moreover $s \equiv 0(mod 4)$ obvious changes into the proof show that sufficiently to find one partial solution of (6) with property
$y_0 \equiv -3s(mod 4(s^2+8))$, $z_0 \equiv 1(mod 8)$ or $y_0 \equiv 3s(mod 2(s^2+8))$, $z_0 \equiv 1(mod 8)$ correspondingly, and then (7) has infinitely many solutions.

For example, consider the case $s = 4$, then $s^2+8 = 24$ and (6) has the form

$$y^2 - 24z^2 = 120 .$$

This equation has partial solution $(y_0, z_0) = (84, 17)$, $y_0 \equiv -12(mod 48)$, $z_0 \equiv 1(mod 8)$. Then the equation (7) has infinitely many natural solutions for $s = 4$.

Note that for $s \equiv 0(mod 4)$ from Proposition 3 it follows that the existence of one natural solution of (7) implies the existence of family of natural solutions for the corresponding equation.

References

[1] Cartan, É., *Geometry of Lie groups and symmetric spaces*, IL, Moscow, 1949 (Russian).

[2] Manturov, O. V., *Homogeneous Riemannian manifolds with irreducible isotropy group*, Trudy Sem. Vektor. Tensor. Anal. Vyp. 13 (1966), 68-145 (Russian).

[3] Joseph A. Wolf, *The geometry and structure of isotropy irreducible homogeneous spaces*, Acta Math., **120** (1968), 59-148.

[4] McKenzie, Y. W., Ziller, W., *On normal homogeneous Einstein manifolds*, Ann. Sci. Ecole Norm. Sup. (4) 18 (1985), 563-633.

[5] Rodionov, E. D., *Standard homogeneous Einstein manifolds,* Russian Acad. Sci. Dokl. Math. **47** (1993), no.1, 37-40.
[6] Rodionov, E. D., *Homogeneous Riemannian manifolds with Einstein metrics,* Doctor dissertation in Mathematics, Institute of Mathematics, Novosibirsk, 1994.
[7] Ireland, K., Rosen, M., *A Classical Introduction to Modern Number Theory,* Berlin: Springer-Verlag, 1993.
[8] Dynkin, E. B., *Semi-simple Subalgebras of Semi-simple Lie Algebras,* Transl. Amer. Math. Soc., Series 2, **6** (1957), 111-244.

Yurii G. Nikonorov and Eugene D. Rodionov
Dimitrov Street, 66
Altai State University
656 099 Barnaul, RUSSIA