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A NOTE ON REGULAR POINTS FOR SOLUTIONS OF NONLINEAR ELLIPTIC SYSTEMS

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Abstract. It is shown in this paper that gradient of vector valued function $u(x)$, solution of a nonlinear elliptic system, cannot be too close to a straight line without $u(x)$ being regular.

0. - Introduction

In this paper we shall deal with points of regularity for weak solutions of nonlinear elliptic systems of the second order

$$-D_i a_i^r(x,u,Du) + a^r(x,u,Du) = -D_i f_i^r(x) + f^r(x), \quad r = 1,\dots,N, \tag{0.1}$$

in an bounded open set $\Omega \subset \mathcal{R}^n$, $n \geq 3$, with Lipschitz boundary $\partial\Omega$. Here the summation over repeated subscript is understood and $x = (x_1, \dots x_n) \in \Omega$, $u = (u_1, \dots u_N)$, $N \geq 2$, $D_i = \partial/\partial x_i, Du = (Du_1, \dots, Du_N)$. By a weak solution of (0.1) we mean a function $u \in W^{1,2}(\Omega, \mathcal{R}^N)$ (for informations see [4], [5]) such that

$$\begin{aligned} &\int_\Omega \big(a_i^r(x,u,Du) D_i\varphi^r + a^r(x,u,Du)\varphi^r\big)dx \\ &\qquad = \int_\Omega \big(f_i^r(x) D_i\varphi^r + f^r(x)\varphi^r\big)dx, \ \varphi \in C_0^\infty(\Omega, \mathcal{R}^N). \end{aligned} \tag{0.2}$$

For the sake of simplification we denote by $|\cdot|$ and $\langle .,. \rangle$ the norm and scalar product in $\mathcal{R}^n$ as well as in $\mathcal{R}^N$ and $\mathcal{R}^{nN}$. If $x \in \mathcal{R}^n$ and r is a positive real number, we set $B(x,r) = \{y \in \mathcal{R}^n : |y - x| < r\}$, i.e., the open ball in $R^n, \Omega(x,r) = B(x,r) \cap \Omega$. The meaning of $\Omega_0 \Subset \Omega$ is that the closure of Ω_0 is contained in Ω, i.e. $\overline{\Omega}_0 \subset \Omega$.

We will use the space $C_0^\infty(\Omega, R^N)$, Hölder spaces $C^{0,\alpha}(\overline{\Omega}, R^N)$, $C^{0,\alpha}(\Omega, R^N)$ and Sobolev spaces $W^{k,p}(\Omega, R^N), W_{loc}^{k,p}(\Omega, R^N), W_0^{k,p}(\Omega, R^N)$ (for detailed informations see,e.g.[4]).

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Denote by

$$f_{x_0,R} = \frac{1}{|B(x_0,R)|}\int\limits_{B(x_0,R)} f(x)dx = \fint\limits_{B(x_0,R)} f(x)dx$$

the mean value over the set $B(x_0,R)$ of the function $f \in L^1(B(x_0,R),R^N)$.

About parameters of system (0.1) we suppose:

$$a_i^r,\ a^r \in C^1(\Omega \times \mathcal{R}^N \times \mathcal{R}^{nN}). \tag{0.3}$$

For $(x,\xi,p) \in \Omega \times \mathcal{R}^N \times \mathcal{R}^{nN}$ with $|\xi| \le L,\ L>0$ is a constant

$$|a_i^r(x,\xi,p)|,\ |a^r(x,\xi,p)| \le C_1(L)(1+|p|), \tag{0.4}$$

$$\Big|\frac{\partial a_i^r(x,\xi,p)}{\partial p_j^s}\Big|,\ \Big|\frac{\partial a^r(x,\xi,p)}{\partial p_j^s}\Big| \le C_1(L), \tag{0.5}$$

$$\begin{gathered}\Big|\frac{\partial a_i^r(x,\xi,p)}{\partial \xi_k}\Big|,\ \Big|\frac{\partial a_i^r(x,\xi,p)}{\partial x_l}\Big|,\ \Big|\frac{\partial a^r(x,\xi,p)}{\partial \xi_k}\Big|,\\ \Big|\frac{\partial a^r(x,\xi,p)}{\partial x_l}\Big| \le C_1(L)(1+|p|),\end{gathered} \tag{0.6}$$

$$\frac{\partial a_i^r(x,\xi,p)}{\partial p_j^s} \longrightarrow d_{ij}^{rs}(x,\xi), \quad \text{if } |p|\to\infty, \text{ uniformly in } \Omega\times\mathcal{R}^N \tag{0.7}$$

$$f_i^r(x) \in W^{1,q}(\Omega), \quad f^r(x) \in W^{1,q/2}(\Omega),\ q>n, \tag{0.8}$$

$$\sum_{i,r}\|f_i^r(x)\|_{1,q} + \sum_r \|f^r(x)\|_{1,q} \le C_2, C_2>0 \text{ is a constant}, \tag{0.9}$$

$$\begin{gathered}\frac{\partial a_i^r(x,\xi,p)}{\partial p_j^s}\eta_i^r\eta_j^s \ge \mu(L)|\eta|^2 \quad \text{for all } \eta\in\mathcal{R}^{nN},\\ (x,\xi,p)\in\Omega\times\mathcal{R}^N\times\mathcal{R}^{nN}.\end{gathered} \tag{0.10}$$

It is known that if $u \in W^{1,2}(\Omega,\mathcal{R}^N)$ solves (0.1) in weak sense and conditions stated above are fulfiled then $u \in W^{2,2}_{loc}(\Omega,\mathcal{R}^N)$ (see e.g.[1]). Main result of this paper is the following theorem:

Theorem 0.11. *Let $M > 0$ be a constant and $u \in W^{1,2} \cap C^{0,\beta}(\Omega, \mathcal{R}^N), (0 < \beta < 1)$ be a weak solution of system (0.1) with conditions (0.3) - (0.10). There exist constants $\varepsilon_1 > 0,\ R_1 > 0$ such that if for some $x^0 \in \Omega,\ R < \min(R_1, dist(x^0, \partial\Omega)),\ \nu \in \mathcal{S}^{nN-1},\ \pi \in \mathcal{R}^{nN},\ |\pi| \le M$ we have*

$$\fint_{B(x^0,R)} \left| Du(x) - (Du)_{x^0,R} \right|^2 dx \le M^2, \tag{0.12}$$

$$\fint_{B(x^0,R)} \left| Du(x) - (Du)_{x^0,R} - \pi \right| dx - \fint_{B(x^0,R)} \left| \langle Du(x) - (Du)_{x^0,R} - \pi, \nu \rangle \right| dx < \varepsilon_1, \tag{0.13}$$

then u is regular in a neigborhoud of x^0 (there is $\delta > 0$ such that

$$u \in C^{1,\alpha}(\overline{B(x^0,\delta)}, \mathcal{R}^N), \alpha \in (0, 1 - n/q)).$$

Remark. The condition that a weak solution $u \in W^{1,2}(\Omega, \mathcal{R}^N)$ of system (0.1) is in addition from the space $C^{0,\beta}(\Omega, \mathcal{R}^N)$ be fulfiled for $n = 3$ by means of Sobolev imbedding theorem ($W^{2,2}_{loc}(\Omega, \mathcal{R}^N) \hookrightarrow C^{0,1/2}(\Omega, \mathcal{R}^N)$, see [4]). For a motivation to this result we refer to [3]. The proof of theorem 0.11 is based on some considerations of paper [2] and the fact that from (0.2) we obtain an equation in variation which has the following form (for information see e.g. [5])

$$\begin{aligned} &\int_\Omega \delta_{kl} \left[B^{rs}_{ij}(x,U) D_j U^l_s D_i \varphi^r_k + B^{rs}_j(x,U) D_j U^l_s \varphi^r_k \right] dx \\ &\qquad = \int_\Omega \left[G^{rk}_i D_i \varphi^r_k + G^{rk} \varphi^r_k \right] dx, \quad \varphi \in C^\infty_0(\Omega, \mathcal{R}^{nN}), \end{aligned} \tag{0.14}$$

where $i,j,k,l = 1,\dots,n,\ r,s = 1,\dots,N,\ U = \{U^l_s\} = \{D_l u_s\}^{s=1,\dots,N}_{l=1,\dots,n}$, $\delta_{kl}-$ Kronecker delta,

$$B^{rs}_{ij}(x,U) = \frac{\partial a^r_i}{\partial p^s_j}(x, u(x), U), \quad B^{rs}_j(x,U) = \frac{\partial a^r}{\partial p^s_j}(x, u(x), U),$$

$$G^{rk}_i(x) = D_k f^r_i - \frac{\partial a^r_i}{\partial x_k} - \frac{\partial a^r_i}{\partial \xi_s}\frac{\partial u_s}{\partial x_k}, \quad G^{rk}(x) = D_k f^r - \frac{\partial a^r}{\partial x_k} - \frac{\partial a^r}{\partial \xi_s}\frac{\partial u_s}{\partial x_k}.$$

Because the system (0.14) is quasilinear elliptic system and $U = Du$, it is sufficient to prove an assertion for quasilinear elliptic system analogous to theorem 0.11.

1. - The quasilinear case

Let us consider a quasilinear elliptic system

$$-D_i\left(A_{ij}^{rs}(x,u)D_j u^s\right) + A_j^{rs}(x,u)D_j u^s = -D_i g_i^r + g^r, \tag{1.1}$$

$x = (x_1,\dots,x_n) \in \Omega,\ \Omega \subset \mathcal{R}^n,\ n \geq 3$ is a bounded open set with Lipschitz boundary $\partial\Omega,\ u = (u^1,\dots u^N),\ N \geq 2,\ i,j = 1,\dots,n,\ r,s = 1,\dots,N.$

We suppose

$$A_{ij}^{rs}, A_j^{rs} \in C(\overline{\Omega} \times \mathcal{R}^N) \tag{1.2}$$

$$\sum_{i,j,r,s} |A_{ij}^{rs}| + \sum_{j,r,s} |A_j^{rs}| \leq L \text{ on } \Omega \times \mathcal{R}^N,\ L > 0 \text{ is a constant}, \tag{1.3}$$

$$\begin{gathered}\text{there is } \lambda > 0 \text{ such that } A_{ij}^{rs}(x,\xi)\eta_i^r\eta_j^s \geq \lambda|\eta|^2 \text{ for all } \eta \in \mathcal{R}^{nN},\\ (x,\xi) \in \overline{\Omega} \times \mathcal{R}^N\end{gathered} \tag{1.4}$$

$$A_{ij}^{rs}(x,\xi) \longrightarrow d_{ij}^{rs}(x), \quad \text{as } |\xi| \to \infty, \text{ uniformly in } \Omega, \tag{1.5}$$

$$g_i^r \in L^p(\Omega), \quad g^r \in L^{p/2}(\Omega),\ p > n. \tag{1.6}$$

By a weak solution of system (1.1) we mean a function $u \in W^{1,2}(\Omega,\mathcal{R}^N)$ such that

$$\begin{aligned}&\int_\Omega \left[A_{ij}^{rs}(x,u)D_j u^s D_i\varphi^r + A_j^{rs}(x,u)D_j u^s\varphi^r\right]dx\\ &\qquad = \int_\Omega \left[g_i^r D_i\varphi^r + g^r\varphi^r\right]dx, \quad \varphi \in C_0^\infty(\Omega,\mathcal{R}^N).\end{aligned} \tag{1.7}$$

It is matter of simple calculation to find that the type of system (0.14) is the same as the one of system (1.7) with assumptions (1.2) - (1.6). Now we may state

Theorem 1.8. *Let $\Omega' \Subset \Omega$. For every $M > 0$ there exist a constants $\varepsilon_1 > 0,\ R_1 > 0$ such that if $u \in W^{1,2}(\Omega,\mathcal{R}^N)$ is a weak solution of the system (1.1) with conditions (1.2) - (1.6) and if for some $x^0 \in \Omega',\ R < \min(R_1, dist(x^0,\partial\Omega)),\ \nu \in \mathcal{S}^{N-1},\ \pi \in \mathcal{R}^N,\ |\pi| \leq M$ we have*

$$⨍_{B(x^0,R)} \left|u(x) - (u)_{x^0,R}\right|^2 dx \leq M^2, \tag{1.9}$$

$$⨍_{B(x^0,R)} \left|u(x) - (u)_{x^0,R} - \pi\right| dx - ⨍_{B(x^0,R)} \left|\langle u(x) - (u)_{x^0,R} - \pi, \nu\rangle\right| dx < \varepsilon_1, \tag{1.10}$$

then u is regular in a neigborhoud of x^0 (there is $\delta > 0$ such that

$$u \in C^{0,\alpha}(\overline{B(x^0,\delta)}, \mathcal{R}^N), \alpha \in (0, 1 - n/p)).$$

It is clear that if theorem 1.8 will be proved then theorem 0.11 will be proved as well.

Remark. If we compare Theorem 1.8 with Theorem 3 in [3], we see the following: The assumption in Theorem 3 in [3] that for some $x_0 \in \Omega$ and R (small) $⨍_{B(x^0,R)} |u|^2 dx \leq M$ is replaced by assumptions (1.5) and $⨍_{B(x^0,R)} |u - u_{x^0,R}|^2 dx \leq M$ in Theorem 1.8. Taking into account the relation between the spaces BMO and L^∞, Theorem 1.8 may be seen as some generalization of Theorem 3 in [3].

One can say that the structural assumption (1.5) probably imply the boundedness of the solution of (1.1) and then our result is a corrolary of the result in [3]. As the following example shows, the above mentioned consideration is not true in general.

Example. [6] Let $\Omega = \{x \in \mathcal{R}^n : |x| < 1\}$ and let us consider the system

$$-D_i\left(A_{ij}^{rs}(x,u) D_j u^s\right) = 0,$$

where $A_{ij}^{rs}(x,\xi) = \delta_{ij}\delta_{rs} + \eta(|\xi|) B_{ir}(x,\xi) B_{js}(x,\xi)$, δ_{ij} -Kronecker delta, $\eta \in C^\infty([0,\infty))$, *supp* $\eta \subset [0, 1+\varepsilon]$, $\varepsilon > 0$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $[0,1]$,

$$B_{ir}(x,\xi) = c\left(\delta_{ir} + b\frac{\xi_i \xi_r |x|^{2a-2}}{1 + |\xi|^2 |x|^{2a-2}}\right),$$

$$a \in [1, \frac{n}{2}), \quad b = \frac{2n}{n-2}, \quad c^2 = \frac{a(n-a)(n-2)^2}{(n-2a)^2(n-1)^2}.$$

The coefficients of this system satisfy all assumptions (1.2)-(1.5). The function $u(x) = x/|x|^a$ is a solution of this system and u is unbounded in origin ($a = 2, 3, \ldots [n/2]$). One may see that $u \notin BMO(\Omega)$ too.

2. - The proof of Theorem 1.8

We will use the following results:

Lemma 2.1. *(see [5]) Let $g \in W^{1,2}(B(0,1))$ be a solution of the equation*

$$\int_{B(0,1)} a_{ij} D_j g D_i \varphi dx = 0, \quad \varphi \in C_0^\infty(B(0,1)) \tag{2.2}$$

in the unit ball $B(0,1)$ of $\mathcal{R}^n$, with bounded, measurable coefficients a_{ij} satisfying

$$\sum_{i,j} |a_{ij}| \leq L, \tag{2.3}$$

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \xi \in \mathcal{R}^n, \ x \in B(0,1). \tag{2.4}$$

Then there exist constants α and Q depending only on L, λ such that $g(x)$ is $\alpha-$Hölder continuous in $B(0,1/2)$ and

$$\begin{aligned} \|g\|_{C^{0,\alpha}(B(0,1/2))} = &\sup_{x\in B(0,1/2)} |g(x)| \\ &+ \sup_{x,y\in B(0,1/2),\ x\neq y} \frac{|g(x)-g(y)|}{|x-y|^\alpha} \leq Q\|g\|_{L^2(B(0,1))}. \end{aligned} \tag{2.5}$$

Using Lax-Milgram lemma we may prove

Lemma 2.6. *Let $u \in W^{1,2}(\Omega, \mathcal{R}^N)$, $x^0 \in \Omega$ and assumptions (1.2) - (1.4),(1.6) for system (1.1) be satisfied. Then there exists $0 < R_0 \leq dist(x^0, \partial\Omega)$ such that for $R \in (0, R_0]$ the linear elliptic system*

$$-D_i\left(A_{ij}^{rs}(x,u)D_j v_R^s\right) + A_j^{rs}(x,u)D_j v_R^s = -D_i g_i^r + g^r, \tag{2.7}$$

has a unique solution in $W_0^{1,2}(B(x^0,R), \mathcal{R}^N)$. Moreover

$$\fint_{B(x^0,R)} \left|v_R(x) - (v_R)_{x^0,R}\right|^2 dx \leq c_3 R^{2(1-n/p)}, \tag{2.8}$$

where $c_3 = c_3(n, N, L, \lambda, R_0, \|g_i^r\|_p, \|g^r\|_{p/2})$.

If we put $\Omega' \subset\subset \Omega$ then the above estimate will be uniform in Ω'.

The above lemma enables us to decompose the solution u of (1.1) as

$$u = v_{x^0,R} + w_{x^0,R} \text{ in } B(x^0,R). \tag{2.9}$$

If there will not be danger of misunderstanding, we will omit the subscripts x^0, R.

By classical way we may obtain for $w_{x^0,R}$ Cacciopoli's inequality:
For $x^0 \in \Omega$, $0 < \rho < R < R_0 \leq dist(x^0, \partial\Omega)$

$$\int_{B(x^0,\rho)} \left|Dw_{x^0,R}(x)\right|^2 dx \leq \frac{c_4}{(R-\rho)^2} \int_{B(x^0,R)} \left|w_{x^0,R}(x) - (w_{x^0,R})_{x^0,R}\right|^2 dx, \tag{2.10}$$

where $c_4 = c_4(n, N, L, \lambda)$.

Now we present a fundamental result concerning the partial regularity of weak solutions to the system (1.1) with assumptions (1.2)-(1.6).

Proposition 2.11. *(see [5], pp.147-149) Let $\Omega' \Subset \Omega$. There exist constants $\varepsilon_0 > 0,\ R_0 > 0$ such that if $u \in W^{1,2}(\Omega, \mathcal{R}^N)$ is a weak solution of the system (1.1) with conditions (1.2) - (1.6) and if for some $x^0 \in \Omega'$ and $R < \min(R_0, dist(x^0, \partial\Omega))$*

$$\fint_{B(x^0,R)} \left| w_R(x) - (w_R)_{x^0,R} \right|^2 dx \le \varepsilon_0^2, \tag{2.12}$$

then there exist $\delta > 0,\ \mu \in (0, 1 - n/p)$, such that $u \in C^{0,\mu}(\overline{B(x^0, \delta)}, \mathcal{R}^N)$.

Proof. The proof is easy modification those in [5], Lemma 6.2.12. Our condition (1.5) substitute the condition that $u \in L^\infty(\Omega, \mathcal{R}^N)$, that is used in the relations (6.2.16)', (6.2.17) in [5].

We remark that the constants $\varepsilon_0,\ R_0$ depend on Ω' and the parameters of system (1.1). Because using (2.8) it is matter of routine to find that

$$\begin{aligned} &\varliminf_{R \to 0+} \Big[\fint_{B(x^0,R)} \left| w_R(x) - (w_R)_{x^0,R} - \pi \right| dx \\ &\qquad - \fint_{B(x^0,R)} \left| \langle w_R(x) - (w_R)_{x^0,R} - \pi, \nu \rangle \right| dx \Big] \\ &= \varliminf_{R \to 0+} \Big[\fint_{B(x^0,R)} \left| u(x) - (u)_{x^0,R} - \pi \right| dx \\ &\qquad - \fint_{B(x^0,R)} \left| \langle u(x) - (u)_{x^0,R} - \pi, \nu \rangle \right| dx \Big] \end{aligned} \tag{2.13}$$

theorem 1.8 will be proved if we prove the following

Lemma 2.14. *Let $\Omega' \Subset \Omega$. For every $M > 0$ there exist a constants $\varepsilon_1 > 0,\ R_1 > 0$ such that if $u \in W^{1,2}(\Omega, \mathcal{R}^N)$ is a weak solution of the system (1.1) with conditions (1.2) - (1.6) and if for some $x^0 \in \Omega',\ R < \min(R_1, dist(x^0, \partial\Omega)),\ \nu \in \mathcal{S}^{N-1},\ \pi \in \mathcal{R}^N,\ |\pi| \le M$ we have*

$$\fint_{B(x^0,R)} \left| u(x) - (u)_{x^0,R} \right|^2 dx \le M^2, \tag{2.15}$$

$$\fint_{B(x^0,R)} \left| w_R(x) - (w_R)_{x^0,R} - \pi \right| dx - \fint_{B(x^0,R)} \left| \langle w_R(x) - (w_R)_{x^0,R} - \pi, \nu \rangle \right| dx \le \varepsilon_1, \tag{2.16}$$

then there exist $\delta > 0,\ \mu \in (0, 1 - n/p))$ such that $u \in C^{0,\mu}(\overline{B(x^0, \delta)}, \mathcal{R}^N)$.

Proof. Let $M > 0$ and $\Omega' \subset\subset \Omega$. We shall reduce to Proposition 2.11. For that let $\varepsilon_0 > 0, R_0 > 0$ be the constants in Proposition 2.11.

Let $\tau = \min\{1/2, (\varepsilon_0/4\sqrt{14}QM\omega_n)^{1/\alpha}\}$, where $\alpha,\ Q$ are the constant in Lemma 2.1, $\omega_n = meas(B(0,1))$. We shall prove that for $M > 0$ there exist constants ε_1 and $R_1 < R_0$ such that if u is a solution of (1.1) satisfying all conditions in Lemma 2.14, then

$$\fint_{B(x^0,\tau R)} \left|w_{\tau R}(x) - (w_{\tau R})_{x^0,\tau R}\right|^2 dx \le \varepsilon_0^2, \tag{2.17}$$

from which the conclusion follows using Proposition 2.11. Let us suppose that our assertion is false. Then it would exist

(i) sequences $\{x^k\}_1^\infty \subset \Omega',\ \{\pi_k\}_1^\infty \subset \mathcal{R}^N,\ |\pi_k| \le M,\ \{\nu_k\}_1^\infty \subset \mathcal{S}^{N-1}$,

(ii) two infinitesimal sequences $\{\varepsilon_k\}_1^\infty,\ \{R_k\}_1^\infty$,

(iii) a sequence $\{u^k\}_1^\infty$ $(u^k = w^k_{R_k} + v^k_{R_k}$ in $B(x^k, R_k))$ of solutions of the system (1.1) such that

$$\fint_{B(x^k,R_k)} \left|u^k(x) - \left(u^k\right)_{x^k,R_k}\right|^2 dx \le M^2, \tag{2.17}$$

$$\begin{aligned} &\fint_{B(x^k,R_k)} \left|w^k_{R_k}(x) - \left(w^k_{R_k}\right)_{x^k,R_k} - \pi_k\right| dx \\ -&\fint_{B(x^k,R_k)} \left|\langle w^k_{R_k}(x) - \left(w^k_{R_k}\right)_{x^k,R_k} - \pi_k, \nu_k\rangle\right| dx \le \varepsilon_k, \end{aligned} \tag{2.18}$$

but

$$\fint_{B(x^k,\tau R_k)} \left|w^k_{\tau R_k}(x) - (w^k_{\tau R_k})_{x^k,\tau R_k}\right|^2 dx > \varepsilon_0^2. \tag{2.19}$$

Put $x = x^k + R_k y,\ y \in B(0,1)$ and $h_k(y) := u^k(x^k + R_k y),\ t_k(y) := w^k_{R_k}(x^k + R_k y),\ m_k(y) := v^k_{R_k}(x^k + R_k y)$. Clearly $h_k(y) = t_k(y) + m_k(y)$. Using Lemma 2.6 we obtain from (1.1)

$$\begin{aligned} &\int_{B(0,1)} A^{rs}_{ij,k}(y, h_k(y)) D_j t^s_k(y) D_i \varphi^r(y) dy \\ +R_k &\int_{B(0,1)} A^{rs}_{j,k}(y, h_k(y)) D_j t^s_k(y) \varphi^r(y) dy = 0, \\ &\qquad \varphi \in C_0^\infty(B(0,1), \mathcal{R}^N), \end{aligned} \tag{2.20}$$

where $k = 1, 2, \ldots$, $A^{rs}_{ij,k}(y, h_k(y)) = A^{rs}_{ij}(x^k + R_k y, h_k(y))$, $A^{rs}_{j,k}(y, h_k(y)) = A^{rs}_j(x^k + R_k y, h_k(y))$. Using the transformation from above the inequalities (2.18) and (2.19) will obtain the following forms

$$\fint_{B(0,1)} \left|t_k(y) - (t_k)_{0,1} - \pi_k\right| dy - \fint_{B(0,1)} \left|\langle t_k(y) - (t_k)_{0,1} - \pi_k, \nu_k\rangle\right| dy \le \varepsilon_k, \tag{2.21}$$

where $(t_k)_{0,1} = \fint_{B(0,1)} t_k(y)dy$ and

$$\fint_{B(0,\tau)} \left|t_{k\tau}(y) - (t_{k\tau})_{0,\tau}\right|^2 dy > \varepsilon_0^2, \tag{2.22}$$

where

$$t_{k\tau}(y) = w^k_{\tau R_k}(x^k + R_k y), \quad (t_{k\tau})_{0,\tau} = \fint_{B(0,\tau)} t_{k\tau}(y)dy.$$

Let now $k \to \infty$. Passing possibly to a subsequence we may suppose that $x^k \to x^0 \in \overline{\Omega}'$, $\nu_k \to \nu \in \mathcal{S}^{N-1}$, $\pi_k \to \pi$, $|\pi| \le M$. Because we have (2.17), using Lemma 2.6 we obtain

$$\begin{aligned}
&\int_{B(0,1)} \left|t_k(y) - (t_k)_{0,1}\right|^2 dy = R_k^{-n} \int_{B(x^k,R_k)} \left|w^k_{R_k}(x) - (w^k_{R_k})_{x^k,R_k}\right|^2 dx \\
&\le 2R_k^{-n}\Big[\int_{B(x^k,R_k)} \left|u^k(x) - (u^k)_{x^k,R_k}\right|^2 dx + \int_{B(x^k,R_k)} \left|v^k_{R_k}(y) - (v^k_{R_k})_{x^k,R_k}\right|^2 dx\Big] \\
&\le \omega_n(2M^2 + c_5 R_k^{2(1-n/p)}),\ (p > n).
\end{aligned}$$

From above estimate it follows that

$$\int_{B(0,1)} \left|t_k(y) - (t_k)_{0,1}\right|^2 dy \le M_1 \tag{2.23}$$

and we may suppose that $M_1 \le 3\omega_n M^2$. The estimate (2.23) implies that (passing possibly to a subsequence) $(t_k - (t_k)_{0,1}) \rightharpoonup t$ weakly in $L^2(B(0,1), \mathcal{R}^N)$. From Caccioppoli's inequality (2.10) we see that

$$\int_{B(0,\rho)} \left|Dt_k(y)\right|^2 dy \le \frac{c_6}{(1-\rho)^2} \int_{B(0,1)} \left|t_k(y) - (t_k)_{0,1}\right|^2 dy,\ 0 < \rho < 1. \tag{2.24}$$

From the last inequality it follows that

$$(t_k - (t_k)_{0,1}) \rightharpoonup t \text{ weakly in } W^{1,2}_{loc}(B(0,1), \mathcal{R}^N),$$

$$(t_k - (t_k)_{0,1}) \to t \text{ strongly in } L^2_{loc}(B(0,1), \mathcal{R}^N).$$

Passing possibly to a subsequence we may suppose that

$$(t_k(y) - (t_k)_{0,1}) \to t(y) \text{ a.e. in } B(0,\rho),\ (0 < \rho < 1).$$

From estimate (2.8) it follows that $\|m_k\|_{L^2(B(0,1),\mathcal{R}^N)} \to 0$ as $k \to \infty$ and we may suppose (as above) $m_k(y) \to 0$ a.e. in $B(0,1)$.

In our consideration we must take into account two cases

(a) the sequence $\{(t_k)_{0,1}\}_1^\infty$ is bounded in $\mathcal{R}^N$, or

(b) $|(t_k)_{0,1}| \to \infty$ as $k \to \infty$.

(a) In this case passing possibly to a subsequence we may suppose that $(t_k)_{0,1} \to b \in \mathcal{R}^N$. Then (1.2) and the above properties imply

$$\begin{aligned} A^{rs}_{ij,k}(y, h_k(y)) &= A^{rs}_{ij}(x^k + R_k y, t_k(y) - (t_k)_{0,1} + (t_k)_{0,1} + m_k(y)) \\ &\to A^{rs}_{ij}(x^0, t(y) + b) \text{a.e. in } B(0,\rho) \text{ as } k \to \infty. \end{aligned}$$

Arguing as in [5] (chapt.6) we conclude that t satisfies

$$(2.25) \qquad \int\limits_{B(0,1)} A^{rs}_{ij}(x^0, b + t(y)) D_j t^s(y) D_i \varphi^r(y) dy = 0, \quad \varphi \in C_0^\infty(B(0,1), \mathcal{R}^N),$$

(b) In this case because (1.5) we have

$$A^{rs}_{ij,k}(y, h_k(y)) \to d^{rs}_{ij}(x^0) \text{as } k \to \infty.$$

By the same argumentation as in the case (a) we find that t satisfies

$$(2.26) \qquad \int\limits_{B(0,1)} d^{rs}_{ij}(x^0) D_j t^s(y) D_i \varphi^r(y) dy = 0, \quad \varphi \in C_0^\infty(B(0,1), \mathcal{R}^N),$$

By trivial calculation we have

$$\begin{aligned} &\fint\limits_{B(0,\tau)} \left| t_{k\tau}(y) - (t_{k\tau})_{0,\tau} \right|^2 dy \\ &= \fint\limits_{B(0,\tau)} \left| t_k(y) + m_k(y) - m_{k\tau}(y) - (t_k)_{0,\tau} - (m_k)_{0,\tau} + (m_{k\tau})_{0,\tau} \right|^2 dy \\ &= \fint\limits_{B(0,\tau)} \left| t_k(y) - (t_k)_{0,\tau} \right|^2 dy + 2 \fint\limits_{B(0,\tau)} \langle t_k(y) - (t_k)_{0,\tau}, m_k(y) - (m_k)_{0,\tau} \rangle dy \\ &\quad - 2 \fint\limits_{B(0,\tau)} \langle t_k(y) - (t_k)_{0,\tau}, m_{k\tau}(y) - (m_{k\tau})_{0,\tau} \rangle dy \\ &\quad - 2 \fint\limits_{B(0,\tau)} \langle m_k(y) - (m_k)_{0,\tau}, m_{k\tau}(y) - (m_{k\tau})_{0,\tau} \rangle dy \\ &\quad + \fint\limits_{B(0,\tau)} \left| m_k(y) - (m_k)_{0,\tau} \right|^2 dy + \fint\limits_{B(0,\tau)} \left| m_{k\tau}(y) - (m_{k\tau})_{0,\tau} \right|^2 dy \end{aligned}$$

and

$$\fint_{B(0,\tau)} \left|t_k(y) - (t_k)_{0,\tau}\right|^2 dy = \fint_{B(0,\tau)} \left|\left(t_k(y) - (t_k)_{0,1}\right) - \left(t_k(y) - (t_k)_{0,1}\right)_{0,\tau}\right|^2 dy$$
$$\to \fint_{B(0,\tau)} \left|t(y) - (t)_{0,\tau}\right|^2 dy \text{ as } k \to \infty.$$

This fact and estimations analogous to (2.8) imply

$$\fint_{B(0,\tau)} \left|t_{k\tau}(y) - (t_{k\tau})_{0,\tau}\right|^2 dy \to \fint_{B(0,\tau)} \left|t(y) - (t)_{0,\tau}\right|^2 dy, \text{ as } k \to \infty.$$

From the last information and (2.22) we have

$$\fint_{B(0,\tau)} \left|t(y) - (t)_{0,\tau}\right|^2 dy \geq \varepsilon_0^2. \tag{2.27}$$

On the other hand we have for every $0 < \rho < 1$ (using (2.21))

$$\begin{aligned} 0 &\leq \fint_{B(0,\rho)} \Big[\left|t_k(y) - (t_k)_{0,1} - \pi_k\right| - \left|\langle t_k(y) - (t_k)_{0,1} - \pi_k, \nu_k\rangle\right|\Big] dy \\ &\leq \rho^{-n} \fint_{B(0,1)} \Big[\left|t_k(y) - (t_k)_{0,1} - \pi_k\right| - \left|\langle t_k(y) - (t_k)_{0,1} - \pi_k, \nu_k\rangle\right|\Big] dy \\ &\leq \rho^{-n}\varepsilon_k \to 0, \text{ as } k \to \infty. \end{aligned}$$

and therefore

$$\fint_{B(0,\rho)} \Big[\left|t(y) - \pi\right| - \left|\langle t(y) - \pi, \nu\rangle\right|\Big] dy = 0, \quad 0 < \rho < 1, \tag{2.28}$$

so that $t(y)$ lies on a straight line

$$t(y) = \pi_1 + g(y)\nu, \tag{2.29}$$

where $\pi_1 = \pi - \langle \pi, \nu\rangle\nu$, $|\pi_1|^2 \leq 4M^2$ and $g(y) = \langle t(y), \nu\rangle$. Introducing (2.29) in (2.25), we conclude that g is a solution of the elliptic equation

$$\int_{B(0,1)} a_{ij}(y) D_j g D_i \varphi dy = 0, \quad \varphi \in C_0^\infty(B(0,1)),$$

where $a_{ij}(y) = A_{ij}^{rs}(x^0, b + \pi_1 + g(y)\nu)\nu^r\nu^s$ are bounded measurable coefficient satisfying (2.3) and (2.4). Introducing (2.29) in (2.26), we conclude that g is a solution of the elliptic equation

$$\int\limits_{B(0,1)} a_{ij} D_j g D_i \varphi dy = 0, \quad \varphi \in C_0^\infty(B(0,1)),$$

where $a_{ij} = d_{ij}^{rs}(x^0)\nu^r\nu^s$ are bounded constant coefficients with the same qualities as in previous situation.

In both cases (a) and (b) it follows from Lemma 2.1 that g is Hölder continuous in $B(0,1/2)$ and we have inequality (2.5). In particular

$$\begin{aligned}\fint\limits_{B(0,\tau)} \left|t(y) - (t)_{0,\tau}\right|^2 dy &= \fint\limits_{B(0,\tau)} \left|\pi_1 + \nu g(y) - \pi_1 - \nu(g)_{0,\tau}\right|^2 dy \\ &= \fint\limits_{B(0,\tau)} \left|g(y) - (g)_{0,\tau}\right|^2 dy \le 14 Q^2 M^2 (2\tau)^{2\alpha} \omega_n^2 \le \frac{\varepsilon_0^2}{2}\end{aligned}$$

which contradicts (2.27).

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