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MAPPINGS RELATED TO CONFLUENCE

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Abstract. Necessary and sufficient conditions are found in the paper for a mapping between continua to be monotone, confluent, semi-confluent, joining, weakly confluent and pseudo-confluent. Three lists of these conditions are presented. Two are formulated in terms of components and of quasi-components, respectively, of connected closed subsets of the range space, while the third one in terms of connectedness between subsets of the domain space. Some basic relations concerning these concepts are studied.

A mapping (i.e. a continuous function) $f : X \to Y$ between metric continua X and Y is called:

– *confluent* provided that for each subcontinuum Q of Y each component of the inverse image $f^{-1}(Q)$ is mapped onto Q under f;

– *semi-confluent* provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$;

– *joining* provided that for each subcontinuum Q of Y and for every two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ we have $f(C_1) \cap f(C_2) \neq \emptyset$;

– *weakly confluent* provided that for each subcontinuum Q of Y there is a component of the inverse image $f^{-1}(Q)$ which is mapped onto Q under f;

– *pseudo-confluent* provided that for each irreducible subcontinuum Q of Y there is a component of the inverse image $f^{-1}(Q)$ which is mapped onto Q under f.

The above definitions of the discussed classes of mappings are formulated in terms of continua, and are applicable rather to mappings between continua (compare [L2] and [L3], for example) than to mappings between arbitrary topological spaces, because studying continua requires a combination of two properties: compactness and connectedness. So, it is hard to work with these concepts when mappings between topological spaces (without any additional assumptions) are studied. Therefore, it is natural to ask about a possibility to reformulate these

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concepts in such a way that the new definitions could be applied to mappings between topological spaces and that they would coincide with old ones in the case when mappings between continua are under consideration.

There were some efforts in this direction, made first for the most important from among the above recalled classes of mappings, viz. the class on confluent ones, in [L1]. Namely a modification of the definition of the concept has been proposed there which was expressed in terms of connectedness between sets ([L1], p. 223). The same idea was later used by Lelek and Tymchatyn in [LT] for some related classes of mappings, and it was applied to study mappings of hereditarily normal spaces and mappings onto locally connected ones (compare Chapters 2 and 3 of [LT]). The obtained results have led to new theorems in continuum theory (Chapters 4 and 5 of [LT]). Also the paper [G] by Grispolakis contains extensions of the concepts of confluent, weakly confluent and pseudo-confluent mappings to the case when both domain and range are general topological spaces.

Let $f : X \to Y$ be a mapping and let $Q \subset Y$ be a connected closed subset of Y. In [LT] two kinds of such extensions are introduced. One (definitions (c), (w), and (p), p. 1336 of [LT]) is formulated in terms of connectivity of $f^{-1}(Q)$ between some of its subsets; the other (definitions (c′), (w′) and (p′), p. 1337 of [LT]) uses condition related to quasi-components of $f^{-1}(Q)$. The paper [G] is a large study of several concepts of (perfect) mappings between topological spaces related to confluence (confluent, weakly confluent and pseudo-confluent), but notions introduced and discussed there are formulated in a different manner then the corresponding concepts considered previously in [LT]). Roughly speaking, conditions concerning quasi-components of $f^{-1}(Q)$ are considered if the set $Q \subseteq Y$ is assumed to be merely either nonempty (H-confluent, H-weakly confluent and H-pseudo-confluent), or connected (h-confluent, h-weakly confluent and h-pseudo-confluent), without being closed ([G], p. 113). Neither [LT] nor [G] contains any similar treatment of semi-confluent and of joining mappings. The class of semi-confluent mappings is intermediate between classes on confluent and of weakly confluent mappings (when considered as defined on continua), and the class of joining mappings is larger than the class of semi-confluent ones (see [M] , 3.3, 3.4 and Theorem 3.8, p.13).

The aim of the present paper is to extend the known definitions of these two classes of mappings, viz. semi-confluent and joining ones, to the general case of mappings between arbitrary topological spaces. Three kinds of such new definitions are presented in the paper. The first two are formulated in terms of components and of quasi-components of connected closed subsets of the range space, respectively, while the third one is formulated in terms of connectedness between subsets of the domain space. We also present some basic relations concerning these concepts.

Reflections about semi-confluent and joining mappings lead us to another set of definitions, expressed in a similar way as previous ones in [LT] for confluent, weakly confluent and pseudo-confluent mappings. Since notation used in [LT] and [G] are different, we do not follow any of them, and we label the discussed conditions with (C), (SC), (J), (WC) and (PC) when referred to confluent, semi-confluent, joining, weakly confluent and pseudo-confluent mappings, respectively.

At the end of the paper we discuss interrelations between various definitions of monotone mappings, either known in the literature of formulated along the ideas presented for other classes of mappings.

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1. Preliminaries.

A topological space X is said to be *connected between two its subsets A and B* provided there is no closed and open subset in X that contains A and is disjoint with B (see [K], p. 142). In other words, a space X is connected between A and B provided that there are no subsets A' and B' of X, both closed and open simultaneously, such that

$$A \subset A', \quad B \subset B' \quad A' \cap B' = \emptyset \quad \text{and} \quad X = A' \cup B' .$$

Clearly, connectedness of a space X between points is an equivalence relation on X. The equivalence classes of this relation are called quasi-components of the space, i.e., a *quasi-component* of a space X containing a point $p \in X$ is the set of all points $x \in X$ such that the space X is connected between $\{p\}$ and $\{x\}$. In other words, a quasi-component of a space X containing a point $p \in X$ is the intersection of all closed and open subsets of X containing p. The reader is referred to [K], §46, IV and V, p. 142-151 for a detailed information concerning further properties of these concepts. In particular, the following statements are known (compare [K], §46, IV, Theorems 1a – 1d, p. 143, and V, Theorem 1, p. 148).

1.1. Statement. *If a topological space is connected between its subsets A and B, then $A \neq \emptyset \neq B$, and if $A \subset A_1$ and $B \subset B_1$, then the space is connected between A_1 and B_1.*

1.2. Statement. *A topological space is connected if and only if for each two points a and b the space is connected between $\{a\}$ and $\{b\}$.*

1.3. Statement. *The component containing a point p of a topological space is contained in the quasi-component of the space containing p.*

1.4. Statement. *Let subsets A and B of a topological space X be given. If there exists a quasi-component of X that intersects both A and B, then X is connected between A and B.*

Proof. Let C stand for the quasi-component of X such that

$$A \cap C \neq \emptyset \neq B \cap C .$$

Take $a \in A \cap C$ and $b \in B \cap C$. Since a and b are in C, the space is connected between $\{a\}$ and $\{b\}$ by the definition, whence X is connected between A and B by Statement 1.1. □

1.5. Remark. Connectedness of a space X between its subsets A and B does not imply the existence of a quasi-component C of X that intersects both A and B, even if X is a compact metric space. In fact, let

$$H = \{0\} \cup \{1/n : n \in \mathbb{N}\},$$

put $X = H \times [0,1]$ (with the usual topology inherited from the plane), and take $A = \{(0,0)\}$ and $B = (H\backslash\{0\}) \times (0,1]$. Then the space X is compact and metric, and it is connected between A and B, while the component $\{0\} \times [0,1]$ of X contains A and is disjoint with B.

1.6. Remark. Note that in the example of Remark 1.5 the space X, being connected between the sets A and B, is not connected between any pair of singletons $\{a\}$ and $\{b\}$ such that $a \in A$ and $b \in B$. This holds because the set B is not closed.

The above remark is related to a property of topological spaces, which is due to S. Mazurkiewicz (see e.g. [K], footnote ([1]) on p. 168).

(M) If a space X is connected between its closed subsets A and B, then there are points $a \in A$ and $b \in B$ such that X is connected between $\{a\}$ and $\{b\}$.

The following results are known (see [K], §47, II, Theorem 1, p. 168, and Theorem 2, p. 169).

1.7. Statement. *Every compact Hausdorff space X has property* (M).

1.8. Statement. *If a topological space has property* (M)*, in particular if it is a compact Hausdorff space, then their quasi-components coincide with the components.*

As a consequence of Statements 1.4 and 1.8 we get the next one (compare [K], §47, Theorem 3, p. 170).

1.9. Statement. *A space with property* (M) *(in particular a compact Hausdorff space) is connected between its closed subsets A and B if and only if there exists a quasi-component (i.e. a component) C of the space such that*

$$A \cap C \neq \emptyset \neq B \cap C.$$

Below we extend the concepts of mappings named in the beginning of the paper by considering mappings between topological spaces X and Y, replacing a subcontinuum of Y by a connected closed subset of Y, and replacing components by quasi-components. Therefore, the extended concepts coincide with the original ones if mappings between continua are under consideration. More precisely, we formulate conditions (C1), (SC1), (J1), (WC1) and (PC1), as well as conditions (C2), (SC2), (J2), (WC2) and (PC2), related to mappings between topological spaces, which are equivalent to the condition that the mapping is confluent, semi-confluent, joining and pseudo-confluent correspondingly, when applied to mappings between *continua* (i.e. compact connected Hausdorff spaces).

2. Confluent mappings.

Confluence of a mapping $f : X \to Y$ between topological spaces X and Y can be defined by the following conditions.

(C1) For each connected closed nonempty subset Q of Y each component of the inverse image $f^{-1}(Q)$ is mapped onto Q under f.

(C2) For each connected closed nonempty subset Q of Y each quasi-component of the inverse image $f^{-1}(Q)$ is mapped onto Q under f.

(C3) For each connected closed nonempty subset Q of Y and points $x \in f^{-1}(Q)$ and $y \in Q$ the set $f^{-1}(Q)$ is connected between $\{x\}$ and $f^{-1}(y)$.

2.0. Remark. Note that condition (C2) is just (c′) of [LT], p. 1337 and ($\dot{c}$) of [G], p. 113, as well as (C3) coincides with (c) of [LT], p. 1336.

To formulate relations between the three conditions let us recall that if $f : X \to Y$ is a mapping between topological spaces X and Y, then a subset A of X is said to be an *inverse set* provided $A = f^{-1}(f(A))$. Obviously for each subset Q of Y the set $f^{-1}(Q)$ is an inverse set (compare [W], p. 137).

2.1. Theorem. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) (C1) *implies* (C2).

(b) (C2) *implies* (C3).

(c) *If both X and Y are Hausdorff spaces and if each inverse closed subset of the space X has property* (M), *then* (C3) *implies* (C1).

Proof. (a). Apply Statement 1.3.

(b). Assume (C2) and let Q, x and y be as in (C3). Take the quasi-component C of $f^{-1}(Q)$ that contains the point x. Then $f(C) = Q$ by (C2), whence $C \cap f^{-1}(y) \neq \emptyset$. Thus $f^{-1}(Q)$ is connected between $\{x\}$ and $f^{-1}(y)$ according to Statement 1.4, so (C3) holds.

(c). Assume (C3). Let a nonempty subset Q of Y be connected and closed, and let C be a component of $f^{-1}(Q)$. Obviously $f(C) \subset Q$. To show that $Q \subset f(C)$ take a point $y \in Q$, and choose a point $x \in C$. By (C3) the set $f^{-1}(Q)$ is connected between $\{x\}$ and $f^{-1}(y)$. By continuity of f the set $f^{-1}(Q)$ is a closed subset of X, and therefore it has property (M) by assumption. By Statement 1.8 we infer that C is a quasi-component of $f^{-1}(Q)$. Since $f^{-1}(y)$ is closed, property (M) implies the existence of a point x' in $f^{-1}(y)$ such that $f^{-1}(Q)$ is connected between $\{x\}$ and $\{x'\}$. According to the definition of a quasi-component we have $x' \in C$, whence $y = f(x') \in f(C)$, and therefore $Q \subset f(C)$ as needed. □

2.2. Corollary. *For mappings $f : X \to Y$ between compact Hausdorff spaces X and Y conditions* (C1), (C2) *and* (C3) *are equivalent.*

2.3. Corollary. *For a mapping $f : X \to Y$ between continua X and Y each of conditions* (C1), (C2) *and* (C3) *is equivalent to confluence of f.*

Assumptions in Theorem 2.1 (c) and in Corollary 2.2 are essential. This will be shown in Example 3.6 and Remark 3.7.

The following result is known (see [LT], Corollary 1.4, p. 1337).

2.4. Theorem. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y such that $f^{-1}(y)$ is compact for each $y \in Y$. Then conditions* (C2) *and* (C3) *are equivalent.*

2.5. Question. Does (C2) imply (C1) for surjective mappings $f : X \to Y$ with compact point inverses between topological spaces X and Y?

3. Semi-confluent mappings.

Now we are going to discuss some conditions related to semi-confluence of a mapping $f : X \to Y$. Let us consider the following three conditions.

(SC1) For each connected closed nonempty subset Q of Y and for every two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

(SC2) For each connected closed nonempty subset Q of Y and for every two quasi-components C_1 and C_2 of the inverse image $f^{-1}(Q)$ we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

(SC3) For each connected closed nonempty subset Q of Y and points x_1 and x_2 in $f^{-1}(Q)$ the set $f^{-1}(Q)$ is connected either between $\{x_1\}$ and $f^{-1}(f(x_2))$ or between $\{x_2\}$ and $f^{-1}(f(x_1))$.

3.1. Theorem. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) (SC1) *implies* (SC2)*.*

(b) (SC2) *implies* (SC3)*.*

(c) *If both X and Y are Hausdorff spaces and if each inverse closed subset of the space X has property* (M)*, then* (SC3) *implies* (SC1)*.*

Proof. (a). Assume (SC1) and suppose on the contrary that (SC2) is not satisfied. Thus there are a connected closed nonempty subset Q of Y and two quasi-components D_1 and D_2 of the inverse image $f^{-1}(Q)$ such that $f(D_1)\setminus f(D_2) \neq \emptyset \neq f(D_2)\setminus f(D_1)$.

Take points $q_1 \in f(D_1)\setminus f(D_2)$ and $q_2 \in f(D_2)\setminus f(D_1)$, and let $p_1 \in D_1$ and $p_2 \in D_2$ be such that $f(p_1) = q_1$ and $f(p_2) = q_2$. Denote by C_1 and C_2 the components of $f^{-1}(Q)$ that contain points p_1 and p_2 respectively. Then by Statement 1.3 we have $C_1 \subset D_1$ and $C_2 \subset D_2$. Condition (SC1) implies that either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$. In the former case we have $q_1 = f(p_1) \in f(C_1) \subset f(C_2) \subset f(D_2)$; if the latter one holds, then similarly $q_2 = f(p_2) \in f(C_2) \subset f(C_1) \subset f(D_1)$. In both we get a contradiction with the choice of q_1 and q_2.

(b). Assume (SC2) and let Q, x_1 and x_2 be as in (SC3). Denote by C_1 and C_2 the quasi-components of $f^{-1}(Q)$ that contain the points x_1 and x_2 respectively. If $f(C_1) \subset f(C_2)$, then $f(x_1) \in f(C_2)$, whence $f^{-1}(f(x_1)) \cap C_2 \neq \emptyset$. By the definition of a quasi-component, the set $f^{-1}(Q)$ is connected between $\{x_2\}$ and a point of $f^{-1}(f(x_1)) \cap C_2$, and so it is connected between $\{x_2\}$ and $f^{-1}(f(x_1))$ (see Statement 1.1). If $f(C_2) \subset f(C_1)$, the proof is the same.

(c). Assume (SC3). Let a nonempty subset Q of Y be connected and closed, and let C_1 and C_2 be quasi-components of $f^{-1}(Q)$. For every $i \in \{1,2\}$ take points $y_i \in f(C_i)$ and choose points $x_i \in C_i$ such that $f(x_i) = y_i$. By continuity of f the set $f^{-1}(Q)$ is a closed subset of X, and therefore it has property (M) by assumption . Thus its quasi-components coincide with components; in particular, C_1 and C_2 are components of $f^{-1}(Q)$. Further, continuity of f implies that the sets $f^{-1}(f(x_i))$ are closed subsets of $f^{-1}(Q)$. If the set $f^{-1}(Q)$ is connected between $\{x_1\}$ and $f^{-1}(f(x_2))$, then property (M) applied to $f^{-1}(Q)$ implies that there is a point $x_2^* \in f^{-1}(f(x_2))$ such that $f^{-1}(Q)$ is connected between $\{x_1\}$ and $\{x_2^*\}$. By the definition of a quasi-component we infer that the points x_1 and x_2^* are in the same quasi-component of $f^{-1}(Q)$, so $x_2^* \in C_1$. Since $f(x_2^*) = f(x_2) = y_2$, and since y_2 is an arbitrary point of C_2, we see that $f(C_2) \subset f(C_1)$. If the set $f^{-1}(Q)$ is connected between $\{x_2\}$ and $f^{-1}(f(x_1))$, we get the opposite inclusion in the same way. □

As a consequence of Statement 1.8 we get a corollary.

3.2. Corollary. *For mappings $f : X \to Y$ between compact Hausdorff spaces X and Y conditions* (SC1) - (SC3) *are equivalent.*

3.3. Corollary. *For a mapping $f : X \to Y$ between continua X and Y each of conditions* (SC1) - (SC3) *is equivalent to semi-confluence of f.*

3.4. Questions. Assume a mapping $f : X \to Y$ between topological spaces X and Y has compact point inverses. Does a) (SC2) imply (SC1), b) (SC3) imply (SC2)?

3.5. Proposition. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) (C1) *implies* (SC1)*;* (C2) *implies* (SC2)*;* (C3) *implies* (SC3)*.*

(b) *If both X and Y are Hausdorff spaces and if each inverse closed subset of the space X has property* (M)*, (in particular if X and Y are compact), then each of conditions* (C1) - (C3) *implies each of* (SC1) - (SC3)*.*

Proof. (a). The first two implications are consequences of the definitions. To show the third one fix arbitrary points x_1 and x_2 in $f^{-1}(Q)$. Putting $y = f(x_2)$ we see that $f^{-1}(Q)$ is connected between $\{x_1\}$ and $f^{-1}(y)$ by (C3) and Statement 1.1.

(b) follows from (a) and from Theorems 2.1 and 3.1. □

3.6. Example. *There exist a metric space X and a surjective mapping $f : X \to [0,1]$ which satisfies condition* (C2) *(thus* (SC2)*) but not* (SC1) *(thus not* (C1)*).*

Proof. Denote by $\mathscr{C}$ the middle-third Cantor set in the closed unit interval $[0,1]$. In the Cartesian coordinates (x,y) in the plane $\mathbb{R}^2$ for each $n \in \mathbb{N}$ put $I_n = \{(x, 1/n) : x \in [0,1]\}$ and $I_0 = \{(x,0) : x \in \mathscr{C}\}$, and define

$$X = \bigcup \{I_n : n \in \{0\} \cup \mathbb{N}\}\,.$$

Let $\varphi : \mathscr{C} \to [0,1]$ be the Cantor-Lebesgue mapping of $\mathscr{C}$ onto $[0,1]$ (see e.g. [E], 3.2. B, p. 146; [W], p. 35) and denote by $\varphi^* : [0,1] \to [0,1]$ the monotone extension of φ (i.e. $\varphi^*|\mathscr{C} = \varphi$ and φ^* is constant on each component of $[0,1]\backslash\mathscr{C}$). Define a mapping $f : X \to [0,1]$ by

$$f((x,y)) = \begin{cases} \varphi(x) & \text{if} \quad (x,y) \in I_0\,, \\ \varphi^*(x) & \text{if} \quad (x,y) \in I_n \quad \text{for } n \in \mathbb{N}\,. \end{cases}$$

Let a connected closed nonempty subset Q of $[0,1]$ be given. The set $f^{-1}(Q)$ has countably many quasi-components C_n where $n \in \{0\} \cup \mathbb{N}$, such that

$$C_n \subset I_n \quad \text{for each} \quad n \in \{0\} \cup \mathbb{N}\,.$$

If $n \in \mathbb{N}$ the sets C_n are continua, so they coincide with the components of $f^{-1}(Q)$, while C_0 is a closed, but in general not connected, subset of I_0. Each singleton $\{(x,0)\} \in C_0$ forms a separate component of $f^{-1}(Q)$. Further, by the definition of the mapping f we have $f(C_n) = Q$ for each $n \in \{0\} \cup \mathbb{N}$, so f satisfies (C2) (and, by Proposition 3.5 (a), also (SC2)). If $Q = [0,1/2]$, then the singletons $\{(0,0)\}$ and $\{(1/3,0)\}$ are two components of $f^{-1}(Q)$, and we see that $f(\{(0,0)\}) = \{0\}$ and $f(\{(1/3,0)\}) = \{1/2\}$, which shows that f does not satisfy (SC1) (and consequently, by Proposition 3.5 (a), it does not satisfy (C1)). □

3.7. Remark. The above Example 3.6 shows that (C2) does not imply (C1) and that (SC2) does not imply (SC1). Furthermore, note that since f satisfies (C2), it satisfies (C3) by Theorem 2.1 (b), and that the set $f^{-1}([0,1/2])$ is an inverse closed subset of the space X which does not have property (M), so the corresponding assumption is essential in Theorem 2.1 (c). Finally, since X of Example 3.6 is Hausdorff while not compact, the same example shows that compactness is indispensable in Corollaries 2.2 and 3.2.

4. Joining mappings.

This class of mappings $f : X \to Y$ between arbitrary topological spaces X and Y can be defined by the following conditions.

(J1) For each connected closed nonempty subset Q of Y and for every two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ we have $f(C_1) \cap f(C_2) \neq \emptyset$.

(J2) For each connected closed nonempty subset Q of Y and for every two quasi-components C_1 and C_2 of the inverse image $f^{-1}(Q)$ we have $f(C_1) \cap f(C_2) \neq \emptyset$.

(J3) For each connected closed nonempty subset Q of Y and points x_1 and x_2 in $f^{-1}(Q)$ there exists a point $y \in Q$ such that $f^{-1}(Q)$ is connected between $\{x_1\}$ and $f^{-1}(y)$ and between $\{x_2\}$ and $f^{-1}(y)$.

4.1. Theorem. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) (J1) *implies* (J2).

(b) (J2) *implies* (J3).

(c) *If both X and Y are Hausdorff spaces and if each inverse closed subset of the space X has property* (M), *then* (J3) *implies* (J1).

Proof. (a). Apply Statement 1.3.

(b). Assume (J2) and let Q, x_1 and x_2 be as in (J3). Denote by C_1 and C_2 the quasi-components of $f^{-1}(Q)$ that contain the points x_1 and x_2 respectively. Then $f(C_1) \cap f(C_2) \neq \emptyset$. Choose a point y in the set $f(C_1) \cap f(C_2)$. Then C_1 and C_2 intersect $f^{-1}(y)$, whence we conclude by Statement 1.4 that condition (J3) is satisfied.

(c). Assume (J3). Let a nonempty subset Q of Y be connected and closed, and let C_1 and C_2 be components of $f^{-1}(Q)$. Choose points $x_1 \in C_1$ and $x_2 \in C_2$. Let y be a point of Q as in (J3). Since the sets $f^{-1}(Q)$ and $f^{-1}(y)$ are closed by continuity of f and since $f^{-1}(Q)$ as a closed and inverse set has property (M) by assumption, then by Statement 1.8 the components C_1 and C_2 are quasi-components of $f^{-1}(Q)$, and there are points x_1^* and x_2^* in $f^{-1}(y)$ such that $f^{-1}(Q)$ is connected between $\{x_1\}$ and $\{x_1^*\}$ as well as between $\{x_2\}$ and $\{x_2^*\}$. By the definition of a quasi-component we infer that $x_1^* \in C_1$ and $x_2^* \in C_2$. Since $f(x_1^*) = f(x_2^*) = y$, it follows that $y \in f(C_1) \cap f(C_2) \neq \emptyset$, and thus (J1) is proved. □

4.2. Corollary. *For mappings $f : X \to Y$ between compact Hausdorff spaces X and Y conditions* (J1), (J2) *and* (J3) *are equivalent.*

4.3. Corollary. *For a mapping $f : X \to Y$ between continua X and Y each of conditions* (J1), (J2) *and* (J3) *is equivalent to the condition that f is joining.*

4.4. Questions. Assume a mapping $f : X \to Y$ between topological spaces X and Y has compact point inverses. Does a) (J2) imply (J1), b) (J3) imply (J2)?

4.5. Proposition. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) (SC1) *implies* (J1); (SC2) *implies* (J2); (SC3) *implies* (J3).

(b) *If both X and Y are Hausdorff spaces and if each inverse closed subset of the space X has property* (M), *(in particular if X and Y are compact), then each of conditions* (SC1) - (SC3) *implies each of* (J1) - (J3).

Proof. (a). The first two implications are consequences of the definitions. To show the third one assume (SC3) and consider two cases. If $f^{-1}(Q)$ is connected between $\{x_1\}$ and $f^{-1}(f(x_2))$, put $y = f(x_2)$, and observe that since $x_2 \in f^{-1}(y)$, the set $f^{-1}(Q)$ is connected between $\{x_2\}$ and $f^{-1}(y)$ simply by definition. The other case, if $f^{-1}(Q)$ is connected between $\{x_2\}$ and $f^{-1}(f(x_1))$, is symmetric to the previous one.

(b) follows from (a) and from Theorems 3.1 and 4.1. □

4.6. Remark. The mapping $f : X \to [0,1]$ of Example 3.6 satisfies (SC2) and so it does (J2) by Proposition 4.5 (a). Using the same components $\{(0,0)\}$ and $\{(1/3,0)\}$ of $f^{-1}([0,1/2])$ it can be observed that f does not satisfy condition (J1). Thus (J2) does not imply (J1), and again, similarly as in Remark 3.7, the same example shows that the assumption concerning property (M) in Theorem 4.1 (c) as well as compactness in Corollary 4.2 are essential.

5. Weakly confluent mappings.

Weak confluence of a mapping $f : X \to Y$ between topological spaces can be formulated as follows.

(WC1) For each connected closed nonempty subset Q of Y there is a component of the inverse image $f^{-1}(Q)$ which is mapped onto Q under f.

(WC2) For each connected closed nonempty subset Q of Y there is a quasi-component of the inverse image $f^{-1}(Q)$ which is mapped onto Q under f.

(WC3) For each connected closed nonempty subset Q of Y there is a point $x \in f^{-1}(Q)$ such that for each point $y \in Q$ the set $f^{-1}(Q)$ is connected between $\{x\}$ and $f^{-1}(y)$.

5.0. Remark. Note that condition (WC2) is just (w′) of [LT], p. 1337 and (w.c.) of [G], p. 113, as well as (WC3) coincides with (w) of [LT], p. 1336.

Using the same argument as in the proofs of previous theorems one can show the following theorem and corollaries. Details are left to the reader.

5.1. Theorem. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) (WC1) *implies* (WC2).

(b) (WC2) *implies* (WC3).

(c) *If both X and Y are Hausdorff spaces and if each inverse closed subset of the space X has property* (M), *then* (WC3) *implies* (WC1).

5.2. Corollary. *For mappings $f : X \to Y$ between compact Hausdorff spaces X and Y conditions* (WC1), (WC2) *and* (WC3) *are equivalent.*

5.3. Corollary. *For a mapping $f : X \to Y$ between continua X and Y each of conditions* (WC1), (WC2) *and* (WC3) *is equivalent to weak confluence of the mapping f.*

The following result is known (see [LT], Corollary 1.4, p. 1337).

5.4. Theorem. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y such that $f^{-1}(y)$ is compact for each $y \in Y$. Then conditions* (WC2) *and* (WC3) *are equivalent.*

5.5. Question. Does (WC2) imply (WC1) for surjective mappings $f : X \to Y$ with compact point inverses between topological spaces X and Y?

Recall that compactness of point inverses is an indispensable assumption in Theorem 5.4, i.e., condition (WC2) is essentially stronger than (WC3); for details see Remarks on p. 1337 of [LT]. Another example showing this is constructed as Example 3.5 of [LT], p. 1342. Since the example is needed for further purposes, we recall it here for the reader convenience. Namely, the following is known.

5.6. Example. *There exists a metric space X and a mapping $f : X \to [0,1]$ such that the sets $f^{-1}(0)$ and $f^{-1}(1)$ are degenerate, each quasi-component of X is compact and none of them is mapped by f onto $[0,1]$. Moreover, for each* *connected nonempty subset Q of $[0,1]$ there is a point $x \in f^{-1}(Q)$ such that for each point $y \in Q$ the set $f^{-1}(Q)$ is connected between $\{x\}$ and $f^{-1}(y)$.*

Proof. In the Cartesian coordinates (x, y) in the plane put

$$\begin{aligned} X = &\{(0,0),(1,0)\} \cup \{(2^{-n},0) : n \in \mathbb{N}\} \cup \{(1-2^{-n},0) : n \in \mathbb{N}\} \\ &\cup \bigcup \{\{(x,2^{-n}) : x \in [2^{-n}, 1-2^{-n}]\} : n \in \mathbb{N}\} \end{aligned}$$

and define $f : X \to [0,1]$ by $f((x,y)) = x$ for $(x,y) \in X$.

To see interrelations between conditions (SC1), (SC2) and (SC3) from one side and conditions (WC1), (WC2) and (WC3) from the other, we begin with an example.

5.7. Example. *There exist a locally compact, arcwise connected metric space X and a surjective mapping $f : X \to [0,1)$ which satisfies condition* (SC1) *(thus* (SC2) *and* (SC3)*) but not* (WC3) *(thus neither* (WC1) *nor* (WC2)*).*

Proof. In the Cartesian coordinates (x, y) in the plane $\mathbb{R}^2$, for each $n \in \mathbb{N}$ put

$$J_n = \{(x, 1/n) : x \in [0, 1-1/n]\} \quad \text{and} \quad J_0 = \{(0,y) : y \in (0,1]\},$$

and define

$$X = \bigcup \{J_n : n \in \{0\} \cup \mathbb{N}\}.$$

Thus X is a locally compact and arcwise connected subset of the plane. Further, let a mapping $f : X \to [0,1)$ be defined by $f((x,y)) = x$ for each point $(x,y) \in X$. The reader can verify that f satisfies the condition (SC1), whence (SC2) and (SC3) follow by Theorem 3.1 (a) and (b). To see that (WC3) does not hold let us take $Q = [1/2, 0)$. Then putting

$$K_n = \{(x, 1/n) : x \in [1/2, 1-1/n]\} \subset J_n \quad \text{for each} \quad n \in \mathbb{N}\backslash\{1\}$$

we have $f^{-1}(Q) = \bigcup\{K_n : n \in \mathbb{N}\backslash\{1\}\}$, and we see that the sets K_n are quasi-components (and components) of $f^{-1}(Q)$. Thus for each point $p \in f^{-1}(Q)$ there is an index $m \in \mathbb{N}\backslash\{1\}$ such that $p = (x, 1/m) \in K_m$. Taking a point $q \in (1-1/m, 1) \subset [0,1)$ we get $f^{-1}(q) = \{(x,y) \in X : y = q\}$. Putting

$$A = \{(x,y) \in f^{-1}(Q) : 1/m \le y\} \quad \text{and} \quad B = \{(x,y) \in f^{-1}(Q) : 1/m > y\}$$

we see that

$$p \in A, \quad f^{-1}(q) \subset B, \quad A \cap B = \emptyset, \quad f^{-1}(Q) = A \cup B$$

and that A and B are simultaneously closed and open in $f^{-1}(Q)$. Thus $f^{-1}(Q)$ is not connected between $\{p\}$ and $f^{-1}(q)$, so (WC3) is not satisfied, whence it follows that also (WC1) and (WC2) are not, by Theorem 5.1 (a) and (b). $\square$

Note that neither the domain nor the range space of the mapping f of the above example is compact. Maćkowiak has shown (see [M], Theorem 3.8, p. 13) that for metric continua X and Y each semi-confluent mapping from X onto Y is weakly confluent. The proof presented there is valid for compact Hausdorff spaces as well. Since for these spaces conditions (SC1), (SC2) and (SC3) are equivalent by Corollary 3.2, as well as conditions (WC1), (WC2) and (WC3) are equivalent by Corollary 5.2, we have the following proposition, part (a) of which is a consequence of Example 5.7, and part (b) is just Maćkowiak's result mentioned above, and formulated in a general setting.

5.8. Proposition. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) *Every of* (SC1), (SC2) *and* (SC3) *implies no one of* (WC1), (WC2) *and* (WC3).

(b) *If both X and Y are compact Hausdorff spaces, then each of conditions* (SC1) - (SC3) *implies each of* (WC1) - (WC3).

One can ask if the implication from semi-confluence to weak confluence holds for larger classes of spaces then compact ones. Remarks below are related to this question.

5.9. Remark. (a) Note that both spaces X and Y in Example 5.7 are locally compact, so compactness cannot be relaxed to local compactness.

(b) It can be verified that closed subsets of the range space X in Example 5.7 have property [M] (and therefore their quasi-components coincide with components by Statement 1.8), so these properties are not strong enough to show the implication in matter.

(c) Since compactness is an invariant under continuous functions, it is enough to assume in Proposition 5.8 (b) only that X is compact. So a question can be asked whether the discussed implication holds if only the range space Y is compact. A negative answer is shown by an example below.

5.10. Example. *There exist a locally compact, arcwise connected metric space X, a metric continuum Y and a surjective mapping $f : X \to Y$ which satisfies condition* (SC1) *(thus* (SC2) *and* (SC3)*) but not* (WC3) *(thus neither* (WC1) *nor* (WC2)*).*

Proof. Both spaces X and Y will be constructed in the plane $\mathbb{R}^2$. Given two points p and q in $\mathbb{R}^2$, we let pq to denote the straight line segment joining p with q. We describe the continuum Y first. In the Cartesian coordinates (x, y) in the plane, for each $n \in \mathbb{N}$ put

$$e = (-1, 0), \quad v = (0, 0), \quad e_0 = (1, 0), \quad e_n = (1, 1/(n+1)),$$
$$Y_n = ee_0 \cup \bigcup\{ve_k : k \in \{1, \ldots, n\}\} \quad \text{and} \quad Y = \bigcup\{Y_n : n \in \mathbb{N}\}.$$

Thus each Y_n is a tree with $n + 2$ end points, and Y is a continuum which is homeomorphic to the well-known harmonic fan (see e.g. [HY], Figure 3-5, p. 109).

Now we will construct the space X. To this aim for each $n \in \mathbb{N}$ let t_n denote the translation of the plane $\mathbb{R}^2$ along the y-axis of n units up, i.e., the parallel displacement by the vector $\langle 0, n\rangle$. Put $X' = \{(-1, y) : y \geq 0\}$ and, for each $n \in \mathbb{N}$, let $X_n = t_n(Y_n)$. Define

$$X = X' \cup \bigcup\{X_n : n \in \mathbb{N}\},$$

and let $f : X \to Y$ be described as follows. The partial mapping $f|X'$ shrinks X' to its initial point e. For each $n \in \mathbb{N}$ the partial mapping $f|X_n$ moves back X_n onto Y_n, i.e., $f|X_n = (t_n|X_n)^{-1} : X_n \to Y_n$. Thus $f(X') = \{e\}$, and for each point $(x, y) \in X_n \subset X'$ we have $f((x, y)) = (x', y')$, where $x' = x$ and $y' = y - n$. One can verify that all the required conditions are satisfied. In particular, taking $Q = \{(x, y) \in Y : 0 \leq y \leq 1/2\}$ we see that none component of $f^{-1}(Q)$ is mapped onto Q under f, so f does not satisfy (WC1). Further, for each point $p \in X_m \cap f^{-1}(Q)$ for some $m \in \mathbb{N}$ one can take a point $q \in (Y_{m+1} \backslash Y_m) \cap Q$ such that $f^{-1}(Q)$ is not connected between $\{p\}$ and $f^{-1}(q)$, and therefore condition (WC3) is not satisfied. □

5.11. Remark. Observe that each closed subset of the domain space X of Example 5.10 has property (M). Thus compactness of X in the implication of Proposition 5.8 (b) cannot be weakened to property (M) for closed subsets of the space X.

5.12. Remark. No example does exist of a mapping $f : X \to Y$ with (SC1) and without (WC3) for which $Y = [0, 1]$. Indeed, let Q be a connected closed nonempty subset of $[0, 1]$. Then $Q = [a, b]$ with $a \leq b$. Let C_a and C_b be components of $f^{-1}(Q)$ such that $a \in f(C_a)$ and $b \in f(C_b)$. Then by (SC1) we have either $f(C_a) \subset f(C_b)$ or $f(C_b) \subset f(C_a)$ and thus in any case both end points a and b are in the image of one component of $f^{-1}(Q)$, whence (WC1) follows. The above observation can be generalized to get a result below. Before we formulate it, we recall a needed concept. A connected closed subset S of a space is said to be *irreducible about a finite set* if there exists a finite set $F \subset S$ such that no connected closed proper subset of S contains F. In particular, if F consists of some two points a and b only, then we say that S is *irreducible between* a and b, or shortly that it is *irreducible*.

5.13. Proposition. *Let $f : X \to Y$ be a surjective mapping between topological Hausdorff spaces X and Y. If Y is compact and if each connected closed nonempty subset of Y is irreducible about a finite set, then*
(a) (SC1) *implies* (WC1)*;*
(b) (SC2) *implies* (WC2).

Proof. In both cases (a) and (b) the proof is the same. We shall argue for (a). So, assume (SC1) and let Q be a connected closed nonempty subset (i.e. a subcontinuum) of Y. Then Q is irreducible about a finite set, say $\{y_1, \ldots, y_n\}$. Let C_1 and C_2 be the components of $f^{-1}(Q)$ such that $y_1 \in f(C_1)$ and $y_2 \in f(C_2)$. Then by (SC1) we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$, and thus in any case both points y_1 and y_2 are in the image of one component of $f^{-1}(Q)$. Label this component $C_{1,2}$ and assume that for some $k \in \{1, 2, \ldots, n\}$ a component $C_{1,2,\ldots,k}$ of $f^{-1}(Q)$ has been found such that $\{y_1, \ldots, y_k\} \subset f(C_{1,2,\ldots,k})$. Let

C_{k+1} be the component of $f^{-1}(Q)$ such that $y_{k+1} \in f(C_{k+1})$. Again by (SC1) we have either $f(C_{k+1}) \subset f(C_{1,2,\ldots,k})$ or $f(C_{1,2,\ldots,k}) \subset f(C_{k+1})$. In the either case all $k+1$ points $y_1, \ldots, y_k, y_{k+1}$ are in the image of one component of $f^{-1}(Q)$, which is then labelled $C_{1,2,\ldots,k+1}$. Thus by induction we infer that there exists a component $C_{1,2,\ldots,n}$ of $f^{-1}(Q)$ such that

$$\{y_1, \ldots, y_n\} \subset f(C_{1,2,\ldots,n}),$$

whence it follows from irreducibility of Q about $\{y_1, \ldots, y_n\}$ that

$$f(C_{1,2\ldots,n}) = Q,$$

i.e., that (WC1) holds. □

5.14. Corollary. *Let $f : X \to Y$ be a surjective mapping between topological Hausdorff spaces X and Y. If each inverse closed subset of the space X has property* (M)*, if Y is compact and if each connected closed nonempty subset of Y is irreducible about a finite set, then* (SC3) *implies* (WC1), (WC2) *and* (WC3).

Proof. Condition (SC3) implies (SC1) by Theorem 3.1 (c). Further, (SC1) implies (WC1) by Proposition 5.13 (a). Finally (WC1) implies (WC2) and (WC3) by Theorem 5.1 (a) and (b). □

5.15. Remark. It follows from Example 5.10 that the assumption concerning the structure of connected close subsets of Y, viz. finiteness of sets connected closed subsets of Y are irreducible about, is essential in Proposition 5.13.

6. Pseudo-confluent mappings.

Pseudo-confluence of a mapping $f : X \to Y$ between topological spaces can be formulated by the following three conditions which are analogous to conditions considered in the previous sections.

(PC1) For each irreducible connected closed subset Q of Y there is a component of the inverse image $f^{-1}(Q)$ which is mapped onto Q under the mapping f.

(PC2) For each irreducible connected closed subset Q of Y there is a quasi-component of the inverse image $f^{-1}(Q)$ which is mapped onto Q under the mapping f.

(PC3) For each irreducible connected closed subset Q of Y and points y_1 and y_2 in Q the set $f^{-1}(Q)$ is connected between $f^{-1}(y_1)$ and $f^{-1}(y_2)$.

Besides, let us consider two conditions discussed in [LT], p. 1336-1337.

(p) For each connected closed nonempty subset Q of Y and for each two points y_1 and y_2 in Q the set $f^{-1}(Q)$ is connected between $f^{-1}(y_1)$ and $f^{-1}(y_2)$.

(p′) For each connected closed nonempty subset Q of Y and for each two points y_1 and y_2 in Q there is a quasi-component C of the inverse image $f^{-1}(Q)$ such that $\{y_1, y_2\} \subset f(C)$.

6.0. Remark. For confluent as well as for weakly confluent mappings some of the considered conditions (C1) - (C3) and (WC1) - (WC3) were the same as certain conditions in [LT] and [G] – see Remarks 2.0 and 5.0. It is not the case for pseudo-confluent mappings. The reason for this is that the starting point for formulation of (PC1) - (PC3) was not the original definition of a pseudo-confluent mapping as given in [LT], p. 1336 using the condition (p), but another condition, formulated in Theorem 5.3 of [LT], p. 1346, which was shown to be equivalent to (p) for mappings from a compact Hausdorff space X onto a compact metric space Y, and which was taken in [M], (iii), p. 25, as the definition of pseudo-confluent mappings between metric continua.

6.1. Theorem. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) (PC1) *implies* (PC2).

(b) (PC2) *implies* (PC3).

(c) (p′) *implies* (p).

(d) (p) *implies* (PC3).

(e) *If the mapping f is such that $f^{-1}(y)$ is compact for each $y \in Y$, then* (p) *implies* (p′).

(f) *If the mapping f is closed, if both X and Y are Hausdorff spaces and if each inverse closed subset of the space X has property* (M), *then* (PC3) *implies* (PC1).

Proof. (a). Apply Statement 1.3.

(b). Assume (PC2). Take an irreducible connected closed subset Q of Y, and let C be a quasi-component of $f^{-1}(Q)$ with $f(C) = Q$. Let y_1 and y_2 be arbitrary points of Q. Then $f^{-1}(y_1) \cap C \neq \emptyset \neq f^{-1}(y_2) \cap C$, whence we conclude by Statement 1.4 that $f^{-1}(Q)$ is connected between $f^{-1}(y_1)$ and $f^{-1}(y_2)$. Thus (PC3) holds.

(c). Apply Statement 1.4.

(d). This implication is obvious.

(e). This is known, see [LT], Corollary 1.4, p. 1337.

(f). Assume condition (PC3) is satisfied. Let a connected closed subset Q of Y be irreducible between points y_1 and y_2. Then $f^{-1}(Q)$ is connected between $f^{-1}(y_1)$ and $f^{-1}(y_2)$. Again by continuity of f the sets $f^{-1}(Q)$, $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are closed. Since $f^{-1}(Q)$ is an inverse set, it has property (M) by assumption, whence if follows by Statement 1.9 that there is a component C of $f^{-1}(Q)$ such that

$$f^{-1}(y_1) \cap C \neq \emptyset \neq f^{-1}(y_2) \cap C\,.$$

So, $\{y_1, y_2\} \subset f(C) \subset Q$. Note that $f(C)$ is connected and, since the mapping f is closed, it is also closed. Therefore by irreducibility of Q we infer that $f(C) = Q$, and so the proof is finished. □

6.2. Corollary. *For mappings $f : X \to Y$ between compact Hausdorff spaces X and Y conditions* (PC1), (PC2) *and* (PC3) *are equivalent, as well as conditions* (p) *and* (p′) *are equivalent, and every of (p) and (p′) implies every of* (PC1), (PC2) *and* (PC3).

The following result is known (see [LT], 5.3, p. 1346).

6.3. Theorem. *Let a mapping $f : X \to Y$ map a compact Hausdorff space X onto a compact metric space Y. Then condition* (PC1) *implies* (p).

6.4. Corollary. *Let a mapping $f : X \to Y$ map a compact Hausdorff space X onto a compact metric space Y. Then conditions* (PC1), (PC2), (PC3), (p) *and* (p$'$) *are equivalent.*

6.5. Corollary. *For a mapping $f : X \to Y$ between continua X and Y each of conditions* (PC1), (PC2), (PC3), (p) *and* (p$'$) *is equivalent to pseudo-confluence of f.*

6.6. Remark. Note that the mapping f of Example 5.6 satisfies the condition (p) (thus (PC3)), while not (PC2). Hence (p) (and (PC3)) implies neither (PC1) nor (PC2). Taking $y_1 = 0$ and $y_2 = 1/2$ we see that there is no quasi-component C of X such that $f(C)$ contains both y_1 and y_2, so neither (p) nor (PC3) implies (p$'$). It follows that compactness of point inverses is an indispensable assumption in the implication (e) of Theorem 6.1.

6.7. Example. *There exists a metric space X and a mapping $f : X \to [0,1]$ such that the condition* (p$'$) *is satisfied, each quasi-component of X is compact and none of them is mapped by f onto $[0,1]$.*

Proof. In the Cartesian coordinates (x,y) in the plane $\mathbb{R}^2$ for each $n \in \mathbb{N}$ put

$$L_n = \{(x, 2^{-n}) : x \in [0, 1-2^{-n}]\} \quad \text{and} \quad R_n = \{(x, 3^{-n}) : x \in [1-3^{-n}, 1]\}\,.$$

Let $X = \{(0,0),(1,0)\} \cup \bigcup\{(L_n \cup R_n) : n \in \mathbb{N}\}$ and define $f : X \to [0,1]$ by $f((x,y)) = x$ for $(x,y) \in X$. $\square$

6.8. Remark. Example 6.7 shows that (p$'$) does not imply (PC2), so it does not imply (PC1), too.

In connection with implications (e)and (f) of Theorem 6.1 we have the following questions.

6.9. Questions. Assume a mapping $f : X \to Y$ between topological spaces X and Y has compact point inverses. Does a) (PC2) imply (PC1), b) (PC3) imply (PC2)?

6.10. Questions. Assume that a mapping $f : X \to Y$ between Hausdorff spaces X and Y is closed, and that each inverse closed subset of the space X has property (M). Does (p) imply (p$'$)?

6.11. Proposition. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) (WC2) *implies* (p$'$)*;* (WC3) *implies* (p).

(b) (WC1) *implies* (PC1)*;* (WC2) *implies* (PC2)*;* (WC3) *implies* (PC3).

(c) *If both X and Y are Hausdorff spaces and if each inverse closed subset of the space X has property* (M), *(in particular if X and Y are compact), then each of conditions* (WC1) - (WC3) *implies* (p), (p$'$) *and each of* (PC1) - (PC3).

Proof. (a). The first implication is obvious. The second one is stated in [LT], p. 1336.

(b). The first two implications are consequences of the definitions. The third one follows from (a) and from (d) of Theorem 6.1. We show a direct proof of this implication. Assume (WC3) and suppose on the contrary that (PC3) does not hold. Then there are an irreducible connected closed subset Q of Y and points y_1 and y_2 of Q such that $f^{-1}(Q)$ is not connected between $f^{-1}(y_1)$ and $f^{-1}(y_2)$. Thus there is a closed and open subset F in $f^{-1}(Q)$ such that $f^{-1}(y_1) \subset F$ and $F \cap f^{-1}(y_2) = \emptyset$. By (WC3) there is a point $x \in f^{-1}(Q)$ such that $f^{-1}(Q)$ is connected between $\{x\}$ and $f^{-1}(y_1)$ for each point $y \in Q$. In particular $f^{-1}(Q)$ is connected between $\{x\}$ and $f^{-1}(y_1)$, so each closed and open subset of $f^{-1}(Q)$ containing $f^{-1}(y_1)$ must contain $\{x\}$. Thus $x \in F$. Then the set $f^{-1}(Q) \setminus F$ is both closed and open in $f^{-1}(Q)$, it contains $f^{-1}(y_2)$ without intersecting the singleton $\{x\}$. Consequently $f^{-1}(Q)$ is not connected between $\{x\}$ and $f^{-1}(y_2)$, contrary to the definition of the point x.

(c). By Theorem 5.1 conditions (WC1) - (WC3) are mutually equivalent. By Proposition 6.10 (a) and (b) each of them implies both (p) and (p$'$) as well as (PC1) - (PC3). □

7. Monotone mappings.

A mapping $f : X \to Y$ between metric continua X and Y is called *monotone* provided that the inverse image of each subcontinuum of Y is a subcontinuum of X. Obviously, each monotone mapping between continua is confluent. Monotoneity of a mapping $f : X \to Y$ between topological spaces X and Y can be defined in several ways. Consider the following ones.

(M1) For each connected closed subset Q of Y the inverse image $f^{-1}(Q)$ is connected.

(M2) For each connected closed subset Q of Y the inverse image $f^{-1}(Q)$ consists of one quasi-component.

(M3) For each connected closed subset Q of Y and points x_1 and x_2 in $f^{-1}(Q)$ the set $f^{-1}(Q)$ is connected between $\{x_1\}$ and $\{x_2\}$.

(M4) For each connected closed subset Q of Y the inverse image $f^{-1}(Q)$ is connected;
(Kuratowski [K], p. 131).

(M5) For each point $y \in Y$ the inverse image $f^{-1}(y)$ is connected;
(Engelking [E], p. 358; Hocking and Young [HY], p. 137).

(M6) For each point $y \in Y$ the inverse image $f^{-1}(y)$ is continuum;
(Whyburn [W], p. 70 and p. 127).

Interrelations between the above conditions are collected in the following theorem.

7.1. Theorem. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) (M1), (M2) *and* (M3) *are equivalent.*

(b) *The following implications hold and none of them can be reversed:*

$$(\mathrm{M6}) \longrightarrow (\mathrm{M5}) \longleftarrow (\mathrm{M4}) \longrightarrow (\mathrm{M1}) .$$

(c) *If Y is a T_1-space, then* (M1) *implies* (M5).

(d) *If the mapping f is either open or closed, then* (M5) *implies* (M4).

(e) *If X and Y are compact Hausdorff spaces, then all six conditions* (M1) - (M6) *are equivalent.*

Proof. (a). Apply Statement 1.2.

(b). All three implications are obvious. We show that no other implication between the considered conditions is possible in general. Let S^1 stand for the unit circle, i.e., the set of complex numbers of module 1. The mapping $f : [0,1) \to S^1$ defined by $f(t) = \exp(2\pi \mathrm{i}t)$ for $t \in [0,1)$ satisfies (M6) (thus (M5)), while neither (M4) nor (M1). Therefore (M6) and (M5) do not imply (M4) or (M1). The mapping f of the real half line $[0,+\infty)$ onto $[0,1]$ defined by $f(t) = t$ for $t \in [0,1]$ and $f(t) = 1$ for $t > 1$ satisfies (M4) (thus (M5) and (M1)) while not (M6). To see that (M1) does not imply (M4) consider the following example.

In the Cartesian coordinates (x,y) in the plane $\mathbb{R}^2$ put $a = (-2,-1)$, $b = (0,-1)$, $c = (0,1)$, and let ab and bc stand for the straight line segments. Putting $S = \{(x,y) \in \mathbb{R}^2 : y = \sin(1/x) \text{ and } x \in (0,1]\}$, define $X = ab \cup S$ and $Y = bc \cup S$. Let a mapping $f : X \to Y$ be defined by the conditions: $f(a) = c$, $f(b) = b$, $f|ab : ab \to bc$ is linear, and $f|S : S \to S$ is the identity. Then both X and Y are connected subsets of the plane, Y is compact, and f is one-to-one. It can be verified that f satisfies (M1). But $\{c\} \cup S$ is a connected subset of Y and $f^{-1}(\{c\} \cup S) = \{a\} \cup S$, so (M4) does not hold.

(c). Singletons are closed in T_1-spaces.

(d). See [E], Theorem 6.1.29, p. 358. Compare [K], §46, I, Theorem 9, p. 131.

(e). This is a consequence of (b), (c) and (d). □

7.2. Proposition. *Let $f : X \to Y$ be a surjective mapping between topological spaces X and Y. Then:*

(a) *Each of conditions* (M1) - (M4) *implies each of* (C1) - (C3).

(b) *If the mapping f is either open or closed, then each of* (M5) *and* (M6) *implies each of* (C1) - (C3).

Proof. (a). Since (M1) obviously implies (C1), this is a consequence of the Theorems 7.1 (a) and (b) and 2.1 (a) and (b).

(b). This follows from Theorem 7.1 (d) and (b) and from (a) above. □

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