ARCHIVUM MATHEMATICUM (BRNO)
Tomus 32 (1996), 117 – 121

ON THE REGULARITY OF GROUP ALGEBRAS

A. A. Bovdi and T. P. Lángi

Abstract. We describe n-regular and n-weakly regular group algebras. KG is n-regular if and only if one of the following conditions holds:
(1) $char K = 0$ and G is locally finite; or
(2) $char K = p$, G is locally finite, $\Delta^p(G)$ is finite and contains all the elements of G of p-power order and $rad(K\Delta^p(G))^n = 0$.

Introduction

As it is well-known, a ring is said to be Neumann regular if the equation $axa = a$ has a solution $x \in R$ for any $a \in R$, or is characterized so that every finitely generated left ideal of R is generated by an idempotent. There are several generalizations of regularity, for instance, n-weakly regular [4] and n-regular rings [1].

Definition. *A ring R is called n-weakly regular if $a \in aRa^nR$ holds for any $a \in R$.*

Obviously, a ring R is n-weakly regular if and only if the equation $axa^ny = a$ can be solved in R for any $a \in R$.

Definition. *If for any $a_1, \dots a_n \in R$ there exist $x_1, \dots x_n \in R$ with*

$$R(a_1 - a_1x_1a_1)R \dots R(a_n - a_nx_na_n)R = 0$$

then the ring R is called n-regular.

The aim of this paper is to describe n-weakly regular and n-regular group algebras. Recall that a 1-regular ring is precisely a Neumann regular ring, and group rings satisfying this property were described by Auslander [2], Connel [3] and Villamayor [7]: *KG is Neumann regular if and only if G is a locally finite group, K is a Neumann regular ring and the order of any torsion element of G is invertible in K.*

1991 *Mathematics Subject Classification*: 16D40, 16S34, 20C05.
Key words and phrases: group algebra, n-weakly regular ring, n-regular ring.
Research supported by the Hungarian National Foundation for Scientific Research grants No. T014279 and No. T16432.
Received November 28, 1995.

On n-weakly regular group rings we know only some elementary properties [6].

1. n-REGULAR GROUP ALGEBRAS

Let $\Delta(G)$ denote the union of the finite conjugacy classes of G. Clearly, the subgroup $\Delta^p(G)$ generated by the p-elements of $\Delta(G)$ is normal in G.

Let $N(KG)$ be the union of the nilpotent ideals of KG and let $rad(KG)$ denote the prime radical of KG. In the proof of the theorem we use the following result of Passman [5, Theorem 8.1.9 and Theorem 8.1.12]: if K is a field of characteristic $p > 0$, then the ideal $N(KG)$ is nilpotent if and only if the subgroup $\Delta^p(G)$ is finite. Then $N(KG) = rad(K\Delta^p(G))KG$.

Theorem 1. *Let K be a field. The group algebra KG is n-regular if and only if at least one of the following conditions holds:*
(1) $charK = 0$ and G is locally finite; or
(2) $charK = p$ and
(a) G is locally finite,
(b) $\Delta^p(G)$ is finite and contains all the p-elements of G,
(c) $rad(K\Delta^p(G))^n = 0$.

Proof. Let KG be an n-regular group algebra. Then for an arbitrary $a \in KG$ there exist elements $x_1, \dots, x_n \in KG$ with

$$KG(a - ax_1a)KG \dots KG(a - ax_na)KG = 0.$$

It follows that for every prime ideal P of KG there exists i with $a - ax_ia \in P$ and let I_i denote the intersection of all the prime ideals P with $a - ax_ia \in P$. Clearly, we have $rad(KG) = \cap_{i=1}^n I_i$. By induction on t we will prove that there exists an element $b_t \in KG$ with $a - ab_ta \in \cap_{i=1}^t I_i$. This is true for $t = 1$ and we assume that $a - ab_ta \in \cap_{i=1}^t I_i$. Then

$$a-a(b_t+x_{t+1}-b_tax_{t+1})a = (a-ab_ta)(1-x_{t+1}a) = (1-ab_t)(a-ax_{t+1}a) \in \cap_{i=1}^{t+1} I_i$$

and $b_{t+1} = b_t + x_{t+1} - b_tax_{t+1}$. Thus $KG/rad(KG)$ is a regular ring.

Now let $a_1, \dots, a_n \in rad(KG)$ and $b_1, \dots, b_n \in KG$ with

$$KG(a_1 - a_1b_1a_1)KG \dots KG(a_n - a_nb_na_n)KG = 0. \tag{1}$$

Since $b_ia_i \in rad(KG)$, the element $b_ia_i - 1$ has an inverse and by (1)

$$a_1a_2 \dots a_n = (a_1b_1a_1 - a_1)(b_1a_1 - 1)^{-1} \dots (a_nb_na_n - a_n)(b_na_n - 1)^{-1} = 0.$$

We obtain that $rad(KG)^n = 0$.

Let $charK = 0$. It is well-known [5, Theorem 2.3.4] that KG does not contain nilpotent ideals and $rad(KG) = 0$. Thus KG is a regular ring and by Auslander-Connel-Villamayor's theorem G is locally finite.

Let $charK = p$. Then $N(KG)$ is a nilpotent ideal and by Passman's theorem the subgroup $\Delta^p(G)$ is finite. Let $\mathcal{I}(\Delta^p(G))$ denote the ideal generated by all $u-1$, $u \in \Delta^p(G)$. Then

$$K(G/\Delta^p(G)) \cong KG/\mathcal{I}(\Delta^p(G)).$$

Since the factorgroup $G/\Delta^p(G)$ has no finite normal subgroups of order divisible by p, by Passman's theorem [5, Theorem 4.2.13] $KG/\Delta^p(G)$ does not contain nilpotent ideals. Thus the prime radical of KG is contained in $\mathcal{I}(\Delta^p(G))$ and $KG/\mathcal{I}(\Delta^p(G))$ is the homomorphic image of $KG/rad(KG)$. We conclude that $K(G/\Delta^p(G))$ is a regular ring, and by Auslander-Connel-Villamayor's theorem the group $G/\Delta^p(G)$ is locally finite and does not contain elements of order p. Since $\Delta^p(G)$ is a finite group, it implies that G is also locally finite. Clearly, $rad(KG) = N(KG) = rad(K\Delta^p(G))KG$. We obtain that $rad(K\Delta^p(G))^n = 0$ and the necessity of the conditions of the theorem is proved.

If $charK = 0$ and G is locally finite then by Auslander-Connel-Villamayor's theorem KG is a regular ring, and hence it is an n-regular ring.

Now suppose that K is of characteristic p and KG satisfies the conditions (a), (b) and (c). If $a \in KG$ and $H = \langle Supp(a), \Delta^p(G)\rangle$, then the subgroup H is finite, $\Delta^p(G) = \Delta^p(H)$ and by Passman's theorem $N(KH) = rad(K\Delta^p(G))KH$. Since KH has a finite dimension, the radical $rad(KH)$ is a nilpotent ideal and $KH/rad(KH)$ is a semisimple artinian ring. It is well-known that a semisimple artinian ring is a regular ring and for the element a there exists $x \in KH$ with $axa - a \in rad(KH) \subseteq N(KG)$. It is proved then that $KG/N(KG)$ is a regular ring.

If $a_1, \ldots, a_n \in KG$ then for every a_i there exists $x_i \in KG$ with $a_i x_i a_i - a_i \in N(KG)$. Since $N(KG)^n = 0$, we conclude that

$$KG(a_1 - a_1x_1a_1)KG \ldots KG(a_n - a_nx_na_n)KG = 0$$

and KG is an n-regular ring.

□

2. n-WEAKLY REGULAR GROUP ALGEBRAS

A hamiltonian group is a non-abelian group in which every subgroup is normal. Such groups G are characterized as follows: G is a direct product of an elementary abelian 2-group E, an abelian torsion group A in which any element is of odd order, and a quaternion group Q of order 8.

Theorem 2. *Let K be a field and $n \geq 2$ a fixed natural number. The group algebra KG is n-weakly regular if and only if at least one of the following conditions holds:*

(a) *$charK = p$ and G is an abelian torsion group containing no elements of order p;*

(b) $charK = 0$ and G is an abelian torsion group, or a hamiltonian group $G = Q \times E \times A$ that in KA the equation $x^2 + y^2 + z^2 = 0$ has only the trivial solution.

Proof. Let KG be n-weakly regular. Then KG does not contain nilpotent elments and G is torsion. Indeed, in the contrary case there exists $0 \neq b \in KG$ with $b^2 = 0$ and we obtain a contradiction $b \in bRb^nR = 0$. From n-weakly regularity we obtain that if $g \in G$ then $(1-g) = (1-g)x$ for some $x \in KG(1-g)^nKG$, and hence $(1-g)(x-1) = 0$. It is well-known that for an element g of infinite order $1-g$ is not zero divisior in KG, which implies that G is a torsion group.

Clearly, if $charK = p$ and $h \in G$ is of order p then $x = 1 + h + \cdots + h^{p-1}$ is a nilpotent element in KG because $x^2 = px = 0$. We obtain that the characteristic of the field K does not divide the order of any element of G.

Let $H = \langle g \mid g^t = 1\rangle$ be a cyclic subgroup of G. Then the element $y = (1 + \ldots + g^{t-2} + g^{t-1})c(1-g)$ has the property $y^2 = 0$ for any $c \in G$. Since KG has no nilpotent elements, we have $y = 0$ and $c \in N_G(H)$. We proved that each cyclic subgroup is normal in G, and hence G is either hamiltonian or abelian.

Assume that G is a hamiltonian group, and let KG be of characteristic p. Then the characteristic of the field K does not divide the order of any element of G, and KG contains no nilpotent elements, which, by Sehgal's result [8, Proposition 6.1.12], is impossible.

Now suppose that $charK = 0$. Then the quaternion group

$$Q = \langle a, b \mid a^4 = 1, b^2 = a^2, bab^{-1} = a^{-1}\rangle$$

is a subgroup of G, $G = Q \times E \times A$. Let (y_1, y_2, y_3) be a nontrivial solution of the equation

$$x^2 + y^2 + z^2 = 0 \tag{2}$$

in KA. Put $H = \langle Q, Supp(y_1), Supp(y_2), Supp(y_3)\rangle$. Then $H = Q \times A_1$ and A_1 is a finite subgroup of A. By Artin-Wedderburn's theorem we have

$$KA_1 = F_1 \oplus \cdots \oplus F_s \tag{3}$$

and

$$KH = \oplus_{i=1}^{s} F_i Q.$$

By (3) the equation (2) has a nontrivial solution (α, β, γ) at least in one of the fields F_i and

$$x = \alpha(a - a^3) + \beta(a^2b - b) + \gamma(ab - a^3b)$$

is a nilpotent element in KG, which is a contradiction.

In order to prove the converse, suppose that (a) holds. Then $H = \langle Supp(a)\rangle$ is a finite group for any $a \in KG$, and hence KH is a semisimple artinian ring. By Artin-Wedderburn's theorem KH is a direct sum of fields. Obviously, KH is an n-weakly regular group algebra, and KG is also an n-weakly regular ring.

Now suppose that the condition (b) holds. Clearly, it is enough to prove the statement for a finite group G. Because E is an elementary abelian 2-group, by Artin-Wedderburn's theorem

$$KE = K_1 \oplus \cdots \oplus K_s,$$

where $K_i = K$ and

$$K_iA = \oplus_{j=1}^{d} F_{ji}.$$

It is easy to see that

$$KG = \oplus_{i=1}^{s} \oplus_{j=1}^{d} F_{ji}Q$$

and

$$F_{ji}Q \cong F_{ji} \oplus F_{ji} \oplus F_{ji} \oplus F_{ji} \oplus S,$$

where S is the quaternion division algebra over F_{ji}. Thus KG is n-weakly regular. □

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Adalbert Bovdi
Institute of Mathematics,
Kossuth Lajos University,
H-4010 Debrecen, pf. 12,
Hungary

Tamás Lángi
Institute of Mathematics,
Kossuth Lajos University,
H-4010 Debrecen, pf. 12,
Hungary