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HIGHER ORDER CONTACT OF REAL CURVES IN A REAL HYPERQUADRIC

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Abstract. Let Φ be an hermitian quadratic form, of maximal rank and index $(n,1)$, defined over a complex $(n+1)$ vectorial space V. Consider the real hyperquadric defined in the complex projective space P^nV by

$$Q = \{[\varsigma] \in P^nV,\ \Phi(\varsigma) = 0\},$$

let G be the subgroup of the special linear group which leaves Q invariant and D the $(2n-2)$ distribution defined by the Cauchy Riemann structure induced over Q. We study the real regular curves of constant type in Q, transversal to D, finding a complete system of analytic invariants for two curves to be locally equivalent under transformations of G.

The real hypersurfaces of real codimension one are the boundaries of domains in a complex manifold. Among the non-degenerate real hypersurfaces in C^{n+1} the simplest and most important are the real hyperquadrics. S. Chern and J. Moser show how the geometry of a general non-degenerate real hypersurface can be considered as a generalization of a real hyperquadric [4].

Here we consider an hermitian quadratic form Φ, of maximal rank, and index $(n,1)$, defined over a complex $(n+1)$-vectorial space V, and Q the real hyperquadric defined in the complex projective space P^nV by the equation

$$Q = \{[\zeta] \in P^nV, \quad \Phi(\zeta) = 0\}.$$

The concept of Frenet frames for holomorphic curves in complex projective spaces played an important role in the classical theory of the equidistribution of these curves. In [5] it is investigated under what conditions on a Hermitian manifold every holomorphic curve in the manifold has a Frenet frame.

The purpose of the present article is to study, the real curves of constant type in the hyperquadric Q, finding a complete system of analytic invariants for two

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curves to be locally equivalent under transformations of the subgroup G, of the special linear group which leave the hyperquadric Q invariant. Moreover we find the Frenet frames for these curves.

This problem is an special case of the equivalence of submanifolds in homogeneous spaces, which has been extensively studied by E. Cartan [2,3], G. Jensen [11], M. Green [7], P. Griffiths [8] and A. Rodrigues [14], among others. Using contact theory and the action of the group on each contact manifold of order k, a transversal section to the orbits of maximal dimension, can be naturally obtained.

1. The real projective hyperquadric

Let Φ be an hermitian quadratic form of maximal rank, and index $(n, 1)$, defined over a complex $(n+1)$-vector space V, and $\{h_o, \cdots, h_n\}$ a base of V such that the expression of Φ is given as

$$\Phi(\zeta) = \Phi(\zeta^0, \cdots, \zeta^n) = (\zeta^0)^2 + \cdots + (\zeta^{n-1})^2 - (\zeta^n)^2, \tag{1.1}$$

we can obtain this basis, reducing Φ to its normal form. Consider in V the basis f_α defined by

$$f_0 = i(h_n - h_o), \quad f_\alpha = h_\alpha, \quad f_n = h_n + h_0, \quad 1 \leq \alpha \leq n-1.$$

The form Φ, in this basis, goes into

$$\Phi(\zeta) = \zeta^\alpha \overline{\zeta^\alpha} + i(\zeta^n \overline{\zeta^0} - \overline{\zeta^n} \zeta^0), \quad \zeta \in V, \tag{1.1'}$$

and the matrix of its representation is

$$\begin{pmatrix} 0 & 0 & i \\ 0 & I_{n-1} & 0 \\ -i & 0 & 0 \end{pmatrix} \quad . \tag{1.2}$$

If $<,>$ denotes the bilinear form asociated to the quadratic form Φ, then we have

$$< \zeta, \nu > = {}^t(\overline{\nu}) A(\zeta), \qquad \zeta, \nu \in V, \tag{1.3}$$

where (ν) is the matrix $n \times 1$ of the components of ν in the basis f_α.

Let $G \subset SL(n+1, C)$ be the subgroup which leaves Φ invariant. We represent $g \in G$ in the basis f_α by the matrix (g^α_γ), $0 \leq \alpha, \gamma \leq n$, and its column vector by g_γ, then

$$g \in G \Leftrightarrow (\Phi \circ g)(\zeta) = \Phi(\zeta) \quad \Leftrightarrow \quad {}^t\overline{\zeta}{}^t\overline{g} A g \zeta = {}^t\overline{\zeta} A \zeta \Leftrightarrow (< g_\gamma, g_\alpha >) = A,$$

then $g \in G$ satisfies the relations, for $\quad 0 \leq \gamma \leq n, \quad 1 \leq \alpha, \beta \leq n-1,$

$$< g_0, g_\gamma > = -\delta^\gamma_n, \quad < g_\beta, g_\alpha > = -\delta^\alpha_\beta, \quad < g_n, g_\gamma > = i\delta^\gamma_0, \quad \det g = 1. \tag{1.4}$$

The Lie algebra $\mathcal{G}$ of G is given by

(1.6)

$$\mathcal{G} = \left\{ \ell \in T_eG : \quad \ell = \begin{pmatrix} \ell_0^0 & \ell_1^0 & \dots & \ell_{n-1}^0 & \ell_{\underline{n}}^0 \\ \ell_0^1 & & & & -i\overline{\ell_1^0} \\ \vdots & & (\ell_\beta^\alpha) & & \vdots \\ \ell_0^{n-1} & & & & -i\overline{\ell_{n-1}^0} \\ \ell_0^n & i\overline{\ell_O^1} & \dots & i\overline{\ell_{n-1}^0} & -\overline{\ell_0^0} \end{pmatrix} ; \quad \begin{array}{c} \ell_\beta^\alpha + \overline{\ell_\alpha^\beta} = 0, \\ \mathrm{tr}\ell = 0, \\ \ell_n^0, \ell_0^n \in \Re \end{array} \right\},$$

and $\dim \mathcal{G} = n^2 + 2n$.

The canonical form ω over G with components ω_γ^α respect to the usual basis $I_\gamma^\alpha \in \mathcal{G}(n+1, C)$, satisfies the relation,

$$\omega_x(v) = \sum_0^n \omega_{\gamma x}^\alpha(v) I_\gamma^\alpha \quad v \in T_xG, \qquad \mathrm{d}\omega_\beta^\alpha + \omega_\gamma^\alpha \wedge \omega_\beta^\gamma = 0. \tag{1.7}$$

Let Q be the $(2n-1)$-dimensional real hyperquadric [5], defined in the complex projective space P^nV by the equation

$$Q = \{[\zeta] \in P^nV, \quad \Phi(\zeta) = 0\}.$$

The group G acts on P^nV by $g.[\zeta] = [g.\zeta]$, and the quadric Q is invariant by the action of G on P^nV. Moreover, G acts transitively on Q, indeed: given $p_0 = [(1, \cdots, 0)]$ and $[\zeta] = [(\zeta^0, \cdots, \zeta^n)] \in Q$, we have

$$\begin{array}{c} \text{if } \zeta^0 \neq \text{ then} \\ g_0.p_0 = [\zeta] \end{array}, \quad \text{with } g_0 = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ \frac{\zeta^1}{\zeta^0} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \frac{\zeta^{n-1}}{\zeta^0} & 0 & \dots & 1 & 0 \\ s & i\overline{\frac{\zeta^1}{\zeta^0}} & \dots & i\overline{\frac{\zeta^{n-1}}{\zeta^0}} & 1 \end{pmatrix}, \quad \begin{array}{c} \text{where } s \in C, \\ \mathrm{Im}s = \frac{1}{2}\sum |\frac{\zeta^\alpha}{\zeta^0}|^2 \end{array},$$

$$\text{if} \quad \zeta^0 = 0, \qquad \text{then} \quad g_1.p_0 = [\zeta], \quad \text{with} \quad g_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The isotropy group G^0 at p_0, is $G^0 = \{g \in G : \ g_0^\alpha = 0, 1 \leq \alpha \leq n\}$ and its Lie algebra $\mathcal{G}^0$ are given by

$$\mathcal{G}^0 = \left\{ \ell \in \mathcal{G} : \ell = \begin{pmatrix} \ell_0^0 & \dots & \ell_n^0 \\ 0 & (\ell_\beta^\alpha) & \vdots \\ 0 & 0 & -\overline{\ell_0^0} \end{pmatrix} \right\}.$$

The map $\psi^0 : g \in G \longmapsto g.p_0 \in Q$, defines an isomorphism: $G/G^o \simeq Q$.

In the following we will agree that small Greek indices α run from 1 to $n-1$, the indices α_k run from k to $n-1$, unless otherwise specified, and we will use the summation convention.

2. Action of G on higher order contact elements of Q

Let $C^s_q Q$ be the manifold of contact element of order s, and dimension 1 at $q \in Q$, and $C^s Q$ the manifold of all contact elements $C^s_q Q$, with $q \in Q$. For $k \leq s$ we consider the canonical projection $\pi^s_k : C^s Q \to C^k Q$ given by $C^s_q \Gamma \mapsto C^k_q \Gamma$, with $\Gamma \subset Q$ an 1-dimensional submanifold [6, 12, 15, 16].

Denote by i^s, the canonical immersion $i^s : \Gamma \longrightarrow C^s Q$, defined by $q \in \Gamma \mapsto C^s_q \Gamma$, and $i^{1,s} : C^{s+1} Q \to C^1(C^s Q)$ defined as $C^{s+1}_q \Gamma \mapsto C^1_{C^s_q \Gamma} C^s \Gamma$.

The action $G \times Q \to Q$ induces an action $G \times C^s Q \to C^s Q$ given by $g.C^s_q \Gamma = C^s_{g.q} g.\Gamma$. The forms ω^α_0 which vanish on $T_e G^0$, allow us to define a basis $\{\tilde{\omega}^\alpha_0\}$ of $T^*_{p_0} Q$ as follows. Given $\tilde{v} \in T_{p_0} Q$, let

$$\tilde{\omega}^\alpha_0(\tilde{v}) = \omega^\alpha_{0\,e}(v_e), \quad \text{where} \quad v \in \mathcal{G}, \quad \text{and} \quad T_e \psi^0_e(v) = \tilde{v}. \tag{2.1}$$

The forms $\tilde{\omega}^\alpha_0$ are well defined: indeed,

$$\begin{array}{c} u, v \in \mathcal{G} \\ T_e\psi^0_e(u_e) = T_e\psi^0_e(v_e) \end{array} \Leftrightarrow \begin{array}{c} T_e\psi^0_e(u_e - v_e) = o \\ u_e - v_e \in T_e G^0 \simeq \mathcal{G}^0, \end{array} \quad \text{then} \quad \begin{array}{c} \omega^\alpha_{0\,e}(u_e - v_e) = o \\ \text{and } \tilde{\omega}^\alpha_0(\tilde{u}) = \tilde{\omega}^\alpha_0(\tilde{v}), \end{array}$$

the dimension of $T^*_{p_0} Q$ is $2n-1$, then $\{\tilde{\omega}^\alpha_0\}$ define a basis of $T^*_{p_0} Q$, and the action of the Lie algebra $\mathcal{G}^0$ on $T^*_{p_0} Q$ is given, in coordinates, as follows.

Proposition 2.1. *The Lie algebra $\mathcal{G}^0$ acts on $T^*_{p_0} Q$ as follows:*

$$(\ell, \tilde{\omega}) \in \mathcal{G}^0 \times T^*_{p_0} Q \mapsto \ell.\tilde{\omega}, \quad \ell.\tilde{\omega}(\tilde{v}) = -d\omega|_e(\ell_e, v_e), \text{ with } v \in \mathcal{G}, \quad T_e\psi^0(v_e) = \tilde{v},$$

and its expression in coordinates is

$$\begin{array}{lllllll} \ell.\omega^\alpha_0 & = & \ell^\gamma_\alpha \omega^\gamma_0 & + & (-\ell^0_0 + \ell^\alpha_\alpha)\omega^\alpha_0 & - & i\overline{\ell^0_\alpha}\omega^n_0 \\ \ell.\omega^n_0 & = & & & & - & 2Re\ell^0_n \omega^n_0. \end{array} \qquad \begin{array}{c} 1 \leq \alpha, \gamma \leq n-1, \\ \gamma \neq \alpha \end{array} \tag{2.2}$$

Proof. Let us express

$$\ell.\omega^\alpha_0 = (a^\beta_0 + ib^\beta_0)\omega^\beta_0 + c\omega^n_0, \qquad a^\beta_0, b^\beta_0 \in \Re, \tag{2.3}$$

and consider a basis of vectors in $T_e G$ dual of the basis ω^α_0. Applying (2.3) to such basis, and using (1.7) we obtain the result. □

We observe in (2.2) that the subspace $D_{p_0} \subset T_{p0} Q$ defined by $\tilde{\omega}^n_0 = 0$ is invariant by G^0, since $\mathcal{G}^0$ transforms ω^n_0 in a multiple of itself, and G^0 is connected.

The transitivity of the action of G on Q, allow us to define a $(2n-2)$-dimensional distribution over Q, as follows

$$D : p \in Q \longmapsto (l_g)_*(D_{p_0}), \qquad \text{where } g \in G \text{ and } l_g(p_0) = g.p_0 = p.$$

To study the real curves in Q, it is natural to consider two cases: the curves tangent to the distribution D at all its points, and the curves transversal to D

at all points. In this paper we consider the first case. We will use the following theorem about Lie groups.

Theorem 2.1. *Let G be a Lie group that acts on a smooth manifold M and $\mathcal{G}$ its Lie algebra. Let $\chi(M)$ denote the smooth (C^∞) vector fields on M, and let F be the map:*

$$F:\mathcal{G}\longrightarrow \chi(M),\qquad \ell\longmapsto F_\ell,\quad with F_\ell(x)=\frac{d}{dt}|_{t=o}(\exp(t\ell).x)\quad ,$$

then we have:

a) The integral curve $y(x)$ of the field F_ℓ, at the point $x\in M$, is contained in the orbit $G(x)$ of x.

b) The action of G on M is transitive if and only if for any $x\in M$, and for any $v\in T_xM$, there exists $\ell\in\mathcal{G}$ such that $F_\ell(x)=v$. □

2.1. Action of G on 1-order contact elements of Q

In this paragraph we study the action of the group G on 1-order contact elements of Q, which project on directions transversal to the distribution D.

Let $\mathcal{H}^1$ be the fiber of the contact elements of order 1 transversal to D, which project onto p_0, i. e.,

$$\mathcal{H}^1=\{X^1\in C^1_{p_o}Q:\qquad \tilde\omega_0^n|X^1\neq 0\},$$

where $\tilde w_0^n|X^1$ denotes the restriction of $\tilde\omega_0^n$ to the 1-dimensional subspace defined by the contact element X^1. Consider on $\mathcal{H}^1$ the coordinates defined as in [13],

$$\tilde\omega_0^\alpha=\lambda_0^\alpha\tilde\omega_0^n\quad \lambda_0^\alpha\in C,\quad 1\le\alpha\le n-1, \tag{2.4}$$

and express X^1 in coordinates as $X^1=(\lambda_0^1,\cdots,\lambda_0^{n-1})$. Denote by $\tilde C^1Q$ all the contact elements of order 1, transversal to D.

Proposition 2.2. *Let $F^0:\mathcal{G}^0\rightarrow\chi(\mathcal{H}^1)$, $F_\ell(X)=\frac{d}{dt}|_{t=o}(\exp(t\ell).X)$, then given $X=(\lambda_0^1,\cdots,\lambda_0^{n-1})$ we have*

$$\begin{aligned}F_\ell^0(X)=&\left(\sum_{\gamma=2}^{n-1}\ell_1^\gamma\lambda_0^\gamma+(-\ell_0^0+\ell_1^1+2Re\ell_0^0)\lambda_0^1-i\overline{\ell_1^0}\right)\frac{\partial}{\lambda_0^1}|_X+\cdots\\&+\left(\sum_{\gamma\neq\alpha}\ell_\alpha^\gamma\lambda_0^\gamma+(-\ell_0^0+\ell_\alpha^\alpha+2Re\ell_0^0)\lambda_0^\alpha-i\overline{\ell_\alpha^0}\right)\lambda_0^{n-1}\frac{\partial}{\partial\lambda_0^\alpha}|_X+\cdots\\&+\left(\sum_{\gamma=1}^{n-2}\ell_{n-1}^\gamma\lambda_0^\gamma+(-\ell_0^0+\ell_{n-1}^{n-1}+2Re\ell_0^0)-i\overline{\ell_{n-1}^0}\right)\frac{\partial}{\lambda_{n-1}^1}|_X.\end{aligned}\tag{2.5}$$

Proof. Given $\ell\in\mathcal{G}^0$, let $r(t)=\exp t\ell.X$, which is expressed in coordinates as

$$r(t)=(\lambda_0^1(t),\cdots,\lambda_0^{n-1}(t)),\quad\text{where}\quad \omega_0^\alpha|r(t)=\lambda_0^\alpha(t)\omega_0^n|r(t),$$

deriving with respect to t and evaluating at $t=0$, we have

$$(\ell.\omega_0^\alpha)(X)=\frac{d}{dt}|_{t=0}\lambda_0^\alpha(t)\omega_0^n(X)+\lambda_0^\alpha(t)(\ell.\omega_0^n)(X),$$

now applying (2.2) we obtain the result. □

Proposition 2.3. *The group G acts transitively on $\tilde{C}^1Q$.*

Proof. Since the action of G on Q is transitive, it is sufficient to prove that the action of G^0 on $\tilde{C}^1_{p_0}Q$, the contact elements of order 1 transversal to D which project onto p_0, is transitive.

Now given $X\in\tilde{C}^1_{p_0}Q$, by Proposition 2.2 we can choose $\ell_1,\cdots,\ell_{n-1}\in\mathcal{G}^0$, such that $F^0_{\ell_1}(X)$, $\cdots$, $F^0_{\ell_{n-1}}(X)$ generate $T_X\mathcal{H}^1$, then by theorem 2.1. we have that G^0 acts transitively on $\mathcal{H}^1$. □

Proposition 2.4. *Let $X_0^1\in C^{1,1}_{p_0}Q$ be given by $X_0^1=(0,\cdots,0)$, then*
i) the Lie algebra $\mathcal{G}^1$ of the isotropy group $G^1\subset G^0$ of X_0^1 is given as

$$\mathcal{G}^1=\{\ell\in\mathcal{G}^0:\ell_1^0=\cdots=\ell_{n-1}^0=0\},\quad i.e.,$$

$$\mathcal{G}^1=\left\{\ell=\begin{pmatrix}\ell_0^0 & 0 & \ell_n^0\\ 0 & (\ell_\beta^\alpha) & 0\\ 0 & 0 & -\overline{\ell_0^0}\end{pmatrix};\quad \ell_\beta^\alpha+\overline{\ell_\alpha^\beta}=0,\quad tr\ell=0\right\},$$

ii) $\dim\mathcal{G}^1=\dim\mathcal{G}^0-2n+2$

Proof. By Proposition 2.2., we have

$$F_\ell^0(X_0^1)=0\quad\Leftrightarrow\quad \ell_1^0=\cdots=\ell_\alpha^0=\cdots=\ell_{n-1}^0=0.$$ □

Let $\mathcal{O}^1=G.X_0^1$ be the orbit of the action of G on $C^{1,1}Q$ which contains X_0^1. Then, the map

$$\psi^1:G\to\mathcal{O}^1,\qquad\text{given by}\qquad \psi^1(g)=g.X_0^1$$

induces a diffeomorphism $\mathcal{O}^1\simeq G/G^1$. The forms $\omega_0^\alpha,\omega_0^n,\omega_\alpha^0$, vanishing on G^1, define $4n-3$ linearly independent real forms which can be projected onto $T_{X_0^1}\mathcal{O}^1$, using a similar argument as in (2.1), and the projected forms, denoted by $\tilde{\omega_0^\alpha},\tilde{\omega_\alpha^0},\tilde{\omega_0^n}$, define a basis of $T^*_{X_0^1}\mathcal{O}^1$.

Given $\ell\in\mathcal{G}^1$ proceeding as in (2.2), we can prove that

$$\ell.\tilde{\omega}_\alpha^0=-\sum_{\gamma\neq\alpha}\ell_\alpha^\gamma\tilde{\omega}_\alpha^0+(\ell_\alpha^0-\ell_\alpha^\alpha)\tilde{\omega}_\alpha^0, \tag{2.6}$$

2.2 Action of G on the 2-order contact elements

Let $\mathcal{H}^2$ be the fiber of the contact elements of order 2 which project onto X_0^1. Denote by $i : C^2Q \to C^1(C^1Q)$ the canonical immersion and $\pi_0^1 : C^1Q \to Q$, and $\pi_0^{1,1} : C^1(C^1Q) \to C^1Q$, the canonical projections. Then

$$i(\mathcal{H}^2) = \{X^2 \in C^2_{p_0}Q : \qquad \tilde{\omega_0^\alpha}|X^2 = 0, \qquad 2 \le \alpha \le n-1\},$$

indeed, given $X_p^2 \in C_p^2Q$ the image $i(X_p^2)$ is identified with an 1-dimensional subspace in $T_{\pi_0^{1,1}(X_p^2)}C^1Q$. Then $(\pi_0^1)_* i(X_p^2)$ is identified with a subspace in T_pQ. The following can easily be verified, using coordinates

if $X^{1,1} \in C^1(C^1Q)$, exists $X^2 \in C^2Q, i(X^2) = X^{1,1} \Leftrightarrow T(\pi_0^1)(X^{1,1}) = \pi_0^{1,1}(X^{1,1})$.

□

Similarly as in (2.4), consider coordinates in $\mathcal{H}^2$ defined as

$$X^2 = (\lambda_1^0, \cdots, \lambda_{n-1}^0), \text{ where } \tilde{\omega_\alpha^0}|X^2 = \lambda_\alpha^0 \tilde{\omega_0^n}|X^2.$$

Let $\tilde{C}^2Q$ be the contact elements of order 2, transversal to D, which project onto $\tilde{C}^1Q$.

Proposition 2.5. *There are two types of orbits, $\tilde{\mathcal{O}}^2$, $\hat{\mathcal{O}}^2$ by the action of G on $\tilde{C}^2Q$, defined as*

$$\tilde{\mathcal{O}}^2 = G.\tilde{X}_0^2, \qquad \textit{with } \pi_0^2(\tilde{X}_0^2) = X_0^1, \quad \lambda_1^0(\tilde{X}_0^2) = 1, \quad \lambda_\alpha^0(X_0^2) = 0,$$
$$\hat{\mathcal{O}}^2 = G.\hat{X}_0^2, \qquad \textit{with } \pi_0^2(\hat{X}_0^2) = X_0^1, \quad \lambda_\alpha^0(\hat{X}_0^2) = 0, \quad 1 \le \alpha \le n-1.$$

Proof. Since G^0 acts transitively on $\tilde{C}^1Q$ it is sufficient to prove that there are two types of orbits by the action of G^1 on $\mathcal{H}^2$.

Let $F^1 : \mathcal{G}^1 \to \chi(\mathcal{H}^2)$ be defined as

$$F_\ell^1(X^2) = \frac{d}{dt}|_{t=o}(\exp{(t\ell)}.X^2) \quad ,$$

using coordinates we have $F_\ell^1(X^2) = \sum B_\alpha^0 \frac{\partial}{\partial \lambda_\alpha^0}|_{X^2}$.

Now, given $X^2 = (\lambda_1^0, \cdots, \lambda_{n-1}^0)$, and $\ell \in \mathcal{G}^1$, let $r(t) = \exp t\ell.X^2$ be expressed in coordinates as $r(t) = (\lambda_1^0(t), \cdots, \lambda_{n-1}^0(t))$, where $\omega_\alpha^0|r(t) = \lambda_\alpha^0(t)\omega_0^n|r(t)$,

deriving the last expression with respect to t at 0, and using (2.6) we have,

$$F_\ell^1(X^2) = (-\sum_{\alpha \ne 1} \ell_1^\alpha \lambda_\alpha^0 + (-\ell_0^0 + \ell_1^1 - 2Re\ell_0^0)\lambda_1^0, \cdots, \\ -\sum_{\alpha \ne n-1} \ell_{n-1}^\alpha \lambda_\alpha^0 + (-\ell_0^0 + \ell_{n-1}^{n-1} - 2Re\ell_0^0)\lambda_{n-1}^0). \tag{2.7}$$

Now

i) if $\lambda_\alpha^0 \ne 0$, for same α, then we can find $2n-2$ fields $\ell_1, \cdots, \ell_{2n-2} \in \mathcal{G}^1$ such that $\{F_{\ell_j}^1(X^2)\}$ generate $T_{X^2}\mathcal{H}^2$, and by Theorem 2.1 we conclude that G^2 acts transitively on the contact elements of order 2 with $\lambda_\alpha^0 \ne 0$, for some α. These elements can be represented as $\tilde{\mathcal{O}}^2 = G.\tilde{X}_0^2$, with $\pi_0^2(\tilde{X}_0^2) = X_0^1$, $\lambda_1^0(\tilde{X}_0^2) = 1$, $\lambda_\alpha^0(\tilde{X}_0^2) = 0$.

ii) If $\lambda_\alpha^0 = 0$, for all α we have the element $\hat{X}_0^2$, defined by $\pi_0^2(\hat{X}_0^2) = X_0^1$ and $\lambda_\alpha^0(\hat{X}_0^2) = 0$. Using (2.7) we have $F_\ell^1(\hat{X}^2) = 0$, for $\ell \in \mathcal{G}^1$, then $\mathcal{G}^2 = \mathcal{G}^1$ and $\hat{\mathcal{O}}^2 = G^1$ is an orbit of the action of G^1 on $\mathcal{H}^2$. $\square$

Proposition 2.6. *The Lie algebra of the element $\tilde{X}_0^2$ is given by*

$$\tilde{\mathcal{G}}^2 = \left\{ \ell = \begin{pmatrix} iA_0^0 & 0 & 0 & \ell_0^0 \\ 0 & iA_0^0 & 0 & 0 \\ 0 & 0 & (\ell_{\beta_2}^{\alpha_2}) & 0 \\ 0 & 0 & 0 & iA_0^0 \end{pmatrix}; \quad \begin{matrix} 2 \le \alpha_2, \beta_2 \le n-1 \\ \\ \ell_{\beta_2}^{\alpha_2} + \overline{\ell_{\alpha_2}^{\beta_2}} = 0 \\ tr\ell = 0 \quad , \quad A_0^0 \in \Re \end{matrix} \right\},$$

in the case $n > 2$;

$$\tilde{\mathcal{G}}^2 = \left\{ \ell = \begin{pmatrix} 0 & 0 & \ell_n^0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad \textit{in the case n=2.}$$

Proof. Using (2.7) we have

$$F_\ell^1(\tilde{X}_0^2) = 0 \Leftrightarrow -\ell_0^0 + \ell_1^1 - Re\ell_0^0 = \ell_2^1 = \cdots = \ell_{n-1}^1 = 0,$$

now $\ell_1^1 + \overline{\ell_1^1} = 0$ and $-\ell_0^0 + \ell_1^1 - Re\ell_0^0 = 0$ implies $Re\ell_0^0 = 0$ and $\ell_1^1 = \ell_0^0$.

If $n = 2$, the condition $tr\ell_0^0 = 0$ implies $\ell_0^0 = 0$. $\square$

For $n > 2$, using a similar argument as in (2.1), the forms
$\omega_0^\alpha, \omega_\alpha^0, Re\omega_0^0, \omega_1^1 - iIm\omega_0^0, \omega_{\alpha_2}^1, \omega_0^n,$
can be projected onto $T_{\tilde{X}_0^2}\tilde{\mathcal{O}}^2$ and define a basis of $T^*_{\tilde{X}_0^2}\tilde{\mathcal{O}}^2$.

2.3 Action of G on the 3-order contact elements

Let $\mathcal{H}^3 = \{X^3 \in C^3Q : \quad \pi_2^3(X^3) = \tilde{X}_0^2\}$. We identify $\mathcal{H}^3$ with its image by $i : C^3Q \to C^1(C^2Q)$, and we have that $X^3 \in \mathcal{H}^3$ if and only if

$$\omega_1^0|X^3 = \omega_0^n|X^3 \neq 0, \quad \omega_0^\alpha|X^3 = \omega_{\alpha_2}^0|X^3 = 0.$$

We consider coordinates in $\mathcal{H}^3$ defined as

$$(\omega_0^0 - \omega_1^1)|X^3 = (a_1^1 + ib_1^1)\omega_0^n|X^3, \quad \omega_{\alpha_2}^1|X^3 = \lambda_{\alpha_2}^1\omega_0^n|X^3.$$

Proposition 2.7. *There are no fixed points by the action of G on the contact elements of order 3, which are projected on the contact elements transversal to D.*

Proof. Given $\ell \in \mathcal{G}^2$ similarly to (2.3), we can show that

$$\ell.(\omega_0^0 - \omega_1^1) = \ell_n^0\omega_0^n, \quad \ell.\omega_{\alpha_2}^1 = \sum \ell_{\alpha_2}^\gamma\omega_\gamma^1 + (\ell_1^1 - \ell_\alpha^\alpha)\omega_\alpha^1. \tag{2.8}$$

Let $\mathcal{F}^3 : \mathcal{G}^2 \to \mathcal{X}(\mathcal{H}^3)$ be defined as in Theorem 2.1, then using (2.8), we have for $X^3 = (a_1^1 + ib_1^1, \lambda_2^1, \cdots, \lambda_{n-1}^1)$

$$F_\ell^3(X^3) = \ell_n^0 \frac{\partial}{\partial a_1^1}|X^3 + \left((\ell_1^1 - \ell_2^2)\lambda_2^1 + \sum_{\gamma \neq 2} \ell_2^\gamma \lambda_\gamma^1 \right) \frac{\partial}{\partial \lambda_2^1}|X^3 + \cdots$$

$$+ \left(\sum_{\gamma \neq n-1} \ell_{n-1}^\gamma \lambda_\gamma^1 + (\ell_1^1 - \ell_{n-1}^{n-1})\lambda_{n-1}^1 \right) \frac{\partial}{\partial \lambda_{n-1}^1}|X^3 \quad . \tag{2.9}$$

Now, given $\ell \in \mathcal{G}^2$ with $\ell_n^0 \neq 0$, we have that $F_\ell^3(X^3) \neq 0$, for all $X^3 \in \mathcal{H}^3$. Then there are no contact elements of order 3 such that $\mathcal{G}^2 = \mathcal{G}^3$. Moreover if $\ell \in \mathcal{G}^3$ then $\ell_3^0 = 0$ and $\dim \mathcal{G}^3 < \mathcal{G}^2$, for all $X^3 \in \mathcal{H}^3$. □

The following Proposition gives, a real differential invariant of order 3 [9, 7, 10], defined by a transversal section to the orbits of maximal dimension on $\tilde{C}^3 Q$.

Proposition 2.8. *If $n = 2$ we have a real invariant τ_3 of order three defined by*

$$(\omega_0^0 - \omega_1^1)|X^3 = 2i\tau_3 \omega_0^2|X^3.$$

Proof. If $n = 2$ then $F_\ell^3(\tilde{X}^3) = \ell_2^0 \frac{\partial}{\partial a_1^1}|X^3$. Then the orbits of the action of G^2 on $\mathcal{H}^3$ are on the coordinate axis a_1^1. A transversal section to these orbits is given by

$$I^3 = \{X^3 \in \mathcal{H}^3 : a_1^1 = 0\}, \quad i.e. \quad (\omega_0^0 - \omega_1^1)|X^3 = 2i\tau_3\omega_0^2|X^3, \quad \tau_3 \in \Re.$$ □

We will denote $E_0^0, E_\alpha^\alpha, E_{\alpha_2}^{\alpha_2}, J_n^0, J_\beta^\alpha$ the fields defined by the matrix

$$E_0^0 = \begin{array}{c} \begin{array}{ccccc} & & 3 & & \end{array} \\ \begin{pmatrix} i & & & & \\ & i & & & \\ & & -3i & & \\ & & & 0 & \\ & & & & i \end{pmatrix} \end{array}, \quad E_{\alpha_2}^{\alpha_2} = \begin{array}{c} \begin{array}{cccc} & \alpha_2 & \alpha_2+1 & \end{array} \\ \begin{pmatrix} 0 & & & \\ & -i & & \\ & & i & \\ & & & 0 \end{pmatrix} \end{array},$$

$$E_\beta^\alpha = \begin{array}{c} \begin{array}{cccc} & \alpha & \beta & \end{array} \\ \begin{pmatrix} & & & 0 \\ & & i & \\ & i & & \\ 0 & & & \end{pmatrix} \end{array}, \quad J_\beta^\alpha = \begin{array}{c} \begin{array}{cccc} & \alpha & \beta & \end{array} \\ \begin{pmatrix} & & & 0 \\ & & 1 & \\ & -1 & & \\ 0 & & & \end{pmatrix} \end{array}, \quad J_n^0 = \begin{array}{c} \begin{array}{ccc} & & n \end{array} \\ \begin{pmatrix} & & 1 \\ & 0 & \\ 0 & & \end{pmatrix} \end{array} \quad .$$

Proposition 2.9. *Let* $\tilde{\mathcal{H}}^3 = \{X^3 \in \mathcal{H}^3 : \lambda_{\alpha_2}^1 \neq 0, \quad \textit{for some } \alpha_2\}$, *and* $n > 2$. *If $X_o^3 \in \tilde{\mathcal{H}}^3$, then the orbit $\tilde{\mathcal{O}}^3 = G.X_0^3$ is given by*

$$\tilde{\mathcal{O}}^3 = \{X^3 \in \tilde{\mathcal{H}}^3 : b_1^1(X^3) = b_1^1(X_0^3); \; \sum |\lambda_{\alpha_2}^1(X^3)|^2 = \rho^2, \; \rho^2 = \sum |\lambda_{\alpha_2}^1(X_0^3)|^2\}.$$

Proof. By (2.7), $F^3_{J^0_n}(X^3) = \frac{\partial}{\partial a^1_1}|_{X^3}$, and given $\ell \in \mathcal{G}^3$, F^3_ℓ vanishing in the coordinate $\frac{\partial}{\partial b^1_1}$. Then, to study the distribution D^3 on $\mathcal{H}^3$, generated by the fields $\{F^3_\ell(X^3), \quad \ell \in \mathcal{G}^3\}$, it is enough to consider the components in the coordinates $\frac{\partial}{\partial \lambda^1_\alpha}|_{X^3}$, $2 \leq \alpha \leq n-1$. Now for $X^3 = (a^1_1 + ib^1_1, \lambda^1_2, \cdots, \lambda^1_n)$,

$$
\begin{array}{lllllllll}
F^3_{E^0_0}(X^3) = (0, & i4\lambda^1_2 \ , & \lambda^1_3, & \cdots, i\lambda^1_\alpha, & i\lambda^1_{\alpha+1}, & \cdots, & i\lambda^1_\beta, & \cdots, & i\lambda^1_{n-1}),\\
F^3_{E^\alpha_\alpha}(X^3) = (0, & 0 \ , & 0, & \cdots, i\lambda^1_\alpha, & -i\lambda^1_{\alpha+1}, & \cdots, & 0, & \cdots, & 0),\\
F^3_{J^\alpha_\beta}(X^3) = (0, & 0 \ , & 0, & \cdots, -\lambda^1_\beta, & 0, & \cdots, & \lambda^1_\alpha, & \cdots, & 0),\\
F^3_{E^\alpha_\beta}(X^3) = (0, & 0 \ , & 0, & \cdots, i\lambda^1_\beta, & 0 & , \cdots, & i\lambda^1_\alpha, & \cdots, & 0).
\end{array}
$$

If we consider the identification of the tangent to the fiber with $\Re^{2n-2}$ and $C^{n-1} \simeq \Re^{2n-2}$ given by $\lambda^\alpha_\beta = a^\alpha_\beta + ib^\alpha_\beta \to (a^\alpha_\beta, b^\alpha_\beta)$, we have that the vectors defined by $F^3_{E^0_0}$, $F^3_{E^{\alpha_2}_{\alpha_2}}$, $F^3_{J^\alpha_\beta}$ and $F^3_{E^\alpha_\beta}$, are perpendicular to the vector with components given by X^3, moreover, for $a^1_\alpha \neq 0$ (respect. $b^1_\alpha \neq 0$), it is possible to find a $(2n-5) \times (2n-5)$-square matrix A, inside the matrix B, defined by $F^3_{E^0_0}$, $F^3_{E^{\alpha_2}_{\alpha_2}}$, $F^3_{J^\alpha_\beta}$ and $F^3_{E^\alpha_\beta}$, i.e.,

$$
B = \begin{pmatrix}
0 & b^1_2 & -b^1_3 & 0 & \ldots & 0 & 0 & 0 & -a^1_2 & a^1_3 & 0 & \ldots & 0 & 0\\
0 & -a^1_3 & a^1_2 & 0 & \ldots & 0 & 0 & 0 & -b^1_3 & b^1_2 & 0 & \ldots & 0 & 0\\
0 & -b^1_3 & -b^1_2 & 0 & \ldots & 0 & 0 & 0 & a^1_3 & a^1_2 & 0 & \ldots & 0 & 0\\
. & . & . & . & \ldots & . & . & . & . & . & . & \ldots & & .\\
. & . & . & . & \ldots & . & . & . & . & . & . & \ldots & & .\\
0 & -a^{n-1} & 0 & 0 & \ldots & a^1_2 & 0 & 0 & -b^1_{n-1} & 0 & 0 & \ldots & b^1_2 & 0\\
0 & -b^{n-1} & 0 & 0 & \ldots & -b^1_2 & 0 & 0 & a^1_{n-1} & 0 & 0 & \ldots & a^1_2 & 0
\end{pmatrix}
$$

with the determinant of $A \neq 0$. Indeed, for $a^1_\alpha \neq 0$ (respectively $b^1_\alpha \neq 0$) we can find a matrix A with det $(A) = a^1_\alpha((a^1_\alpha)^2 + c), \quad c \geq 0$. Then, the integral submanifold of the distribution, generated by the fields which define B through X^3_0, is given by

$$
\mathcal{O}^3 = \{X^3 \in \mathcal{H}^3 : \quad \sum_2^{n-1} |\lambda^1_\alpha(X^3)|^2 = \rho^2, \quad b^1_1 = b^1_1(X^3_0), \quad \rho^2 = \sum_2^{n-1} |\lambda^1_\alpha(X^3_0)|^2. \quad \Box
$$

A transversal section to the orbits of the action of G^3 on $\mathcal{H}^3_0$ is given by

$$
I^3_0 = \left\{ \quad X^3 \in \mathcal{H}^3 : \quad \begin{array}{ll} \mathrm{Re}\lambda^1_2(X^3_0) > 0, & \mathrm{Im}\lambda^1_2(X^3) = 0 \\ \lambda^1_{\alpha_3}(X^3) = 0, & a^1_1(X^3) = 0 \end{array} \right\}.
$$

We will consider coordinates ρ_3 and τ_3 in I^3_0 given by

$$
(2.10) \qquad \begin{array}{rcll}
(\omega^0_0 - \omega^1_1)|X^3 & = & i\tau_3\omega^n|X^3 & \tau_3 \in \Re, \\
\omega^1_2|X^3 & = & \rho_3\omega^n_0|X^3, & \rho_3 \in \Re, \quad \rho_3 > 0^{\cdot}
\end{array}
$$

The section I^3 defines two real invariants (see [7]) of C^3Q, and they correspond with the point where the orbit of X^3 meets I^3.

Using (2.7), we have that the Lie algebra $\tilde{\mathcal{G}}^3$, corresponding to the element $X^3_{\tau_3,\rho_3} \in I^3$, is defined as

$$\tilde{\mathcal{G}}^3 = \left\{ \ell = \begin{pmatrix} iA_0^0 \mathrm{Id}_{3\times 3} & 0 & 0 \\ 0 & (\ell^{\alpha_3}_{\beta_3}) & 0 \\ 0 & 0 & iA_0^0 \end{pmatrix}; \quad \begin{array}{c} \ell^{\alpha_3}_{\beta_3} + \overline{\ell^{\beta_3}_{\alpha_3}} = 0 \\ \mathrm{tr}\ell = 0, \quad A_0^0 \in \Re \end{array} \right\},$$

where $\mathrm{Id}_{3\times 3}$ denotes the 3×3 identity matrix. We have that dim $\mathcal{G}^3$=dim $\mathcal{G}^2 - 1$.

An important observation is that the isotropy algebra of $X^3_{\tau_3,\rho_3}$ does not depend on τ_3 and ρ_3. Then a curve $\Gamma \subset Q$, transversal to D, with $C^3\Gamma \subset \tilde{C}^3 Q$, is a curve of constant type [7].
In the case $n = 3$, from the relation $\mathrm{tr}\ell = 0$ we have that $\mathcal{G}^3 = 0$.

2.4 Action of G on 4-order contact elements

Similarly to paragraph 2.3, we consider an element $X_0^3 = X^3_{\tau_3,\rho_3}$, and $\mathcal{O}^3 = G.X_0^3$. The map $\psi^3 : G \longrightarrow \mathcal{O}^3, \quad \longmapsto g.X_0^3$, allows us to project the forms

$$\omega_0^{\alpha_1}, \omega_0^n, \omega^0_{\alpha_1}, \omega_0^0 - \omega_1^1, \omega^1_{\alpha_2}, \omega^0_n, \omega_0^0 - \omega_2^2, \omega^2_{\alpha_3}, \tag{2.11}$$

on $T_{X_0^3}\mathcal{O}^3$. Moreover, the real forms

$$d\tau_3, \quad d\rho_3, \tag{2.12}$$

given by the functions τ_3 and ρ_3, define a basis of $T^*_{X_0^3} I^3$. If $\tilde{C}^3 Q$ denotes the contact elements of order 3 transversal to D, which project onto $\tilde{\mathcal{O}}^2$, then we have

$$T_{X_0^3}\tilde{C}^3 Q = T_{X_0^3}\tilde{\mathcal{O}}^3 \oplus T_{X_0^3} I^3,$$

and we can define a basis of $T^*_{X_0^3}\tilde{C}^3 Q$ extending the forms (2.11), (2.12), as follows

$$\omega^\alpha_\beta | T_{X_0^3} I^3 = 0, \qquad d\tau_3 | T_{X_0^3}\tilde{\mathcal{O}}^3 = 0, \qquad d\rho_3 | T_{X_0^3}\tilde{\mathcal{O}}^3 = 0. \tag{2.13}$$

Let $\mathcal{H}^4 = \{X^4 \in C^4 Q : \quad \pi_2^4(X^4) = X_0^3\}$, defined by

$$\begin{array}{rclcrcl} \omega_0^{\alpha_1}|X^4 &=& 0 &,& (\omega_0^0 - \omega_1^1)|X^4 &=& i\tau_3 \quad \omega_0^n|X^4 \quad , \\ \omega_1^0|X^4 &=& \omega_0^n|X^4 &,& \omega_2^1|X^4 &=& \rho_3 \quad \omega_0^n|X^4 \quad , \\ \omega^0_{\alpha_1}|X^4 &=& 0 &,& \omega^1_{\alpha_2}|X^4 &=& 0 \quad . \end{array}$$

We introduce coordinates in $\mathcal{H}^4$ as follows

$$\begin{array}{rclcrcl} \omega_n^0|X^4 &=& a_n^0\omega_0^n|X^4 &,& (\omega_0^0 - \omega_2^2)|X^4 &=& ib_2^2\omega_0^n|X^4 \quad , \\ \omega^2_{\alpha_3}|X^4 &=& \lambda^2_{\alpha_3}\omega_0^n|X^4 &,& && \\ d\tau_3|X^4 &=& \tau_3^1\omega_0^n|X^4 &,& d\rho_3|X^4 &=& \rho_3^1\omega_0^n|X^4 \quad . \end{array}$$

If $\mathcal{G}^3$ denotes the Lie algebra of the isotropy group of X_0^3, then similarly to (2.8), we can prove

$$\begin{array}{lclclcl} \ell.(\omega_0^0-\omega_2^2) & = & 0 & , & \ell.\omega_n^0 & = & 0 \quad , \\ \ell.\omega_{\alpha_3}^2 & = & -\sum_{\gamma\neq\alpha_3}^{n-1}\ell_{\alpha_3}^{\gamma}\omega_\gamma^2-(\ell_2^2-\ell_{\alpha_3}^{\alpha_3})\omega_{\alpha_3}^2 & , & & & \\ \ell.d\tau_3 & = & 0 & , & \ell.d\rho_3 & = & 0 \quad . \end{array}$$

To prove $\ell.d\tau_3 = \ell.d\rho_3 = 0$, we use the fact that the isotropy algebra $\mathcal{G}^3$ is independent of ρ_3, and τ_3.

Proceeding as in Proposition 2.8, we can prove that a transversal section to the orbits of the action of G on $\mathcal{H}^4$, is given by

$$I^4=\left\{X^4\in\mathcal{H}^4: \quad Re\lambda_3^2(X^4)>0, \quad Im\lambda_3^2(X^4)=0, \quad \lambda_{\alpha_4}^2=0,\right\}$$

so we consider coordinates in I^4 defined as

$$\begin{array}{rclcrcll} (\omega_0^0-\omega_2^2)|X^4 & = & i\tau_4\omega_0^n|X^4 & , & \omega_n^0|X^4 & = & \mu_4\omega_0^n|X^4 & , \\ \omega_3^2|X^4 & = & \rho_4\omega_0^n|X^4 & , & & & & \\ d\tau_3^1|X^4 & = & \tau_3^2\omega_0^n|X^4 & , & d\rho_3^1|X^4 & = & \rho_3^2\omega_0^n|X^4. & \end{array}$$

The invariants μ_4, τ_4, ρ_4 correspond to the coordinates $a_1^1, b_2^2, Re\lambda_3^2$.

The Lie algebra of the isotropy group of order 4 is given by:

$\tilde{\mathcal{G}}^4=\mathcal{G}^3$, if $\lambda_{\alpha_3}^2(X^4)=0$, for all α_3,

$$\tilde{\mathcal{G}}^4=\left\{\ \ell=\begin{pmatrix} iA_0^0 Id_{4\times4} & 0 & 0 \\ 0 & (\ell_{\beta_3}^{\alpha_3}) & 0 \\ 0 & 0 & iA_0^0 \end{pmatrix}; \quad \begin{array}{c} \ell_{\beta_3}^{\alpha_3}+\overline{\ell_{\alpha_3}^{\beta_3}}=0 \\ \mathrm{tr}\ell=0, \quad A_0^0\in\Re \end{array} \right\},$$

if $\lambda_{\alpha_3}^2(X^4)\neq0$ for some α_3, and $X_0^4=X^4_{\tau_3\rho_3\tau_4\rho_4\mu_4\tau_3^1\rho_3^1}$.

Also we observe here that the isotropy algebra of X_0^4 does not depend on the invariants $\tau_3, \rho_3, \tau_4, \rho_4, \mu_4, \tau_3^1, \rho_3^1$.
In the case $n=4$, using the relation $\mathrm{tr}\ell=0$, we have that $\mathcal{G}^4=0$.

2.5 Action of G on the k-order contact elements

Similarly to paragraph 2.4, and using induction we can construct for $k\geq5$, a transversal section I^k to the orbits of maximal dimension, as follows

$$(2.14)\quad \begin{array}{rclcrcll} (\omega_0^0-\omega_{k-2}^{k-2})|X^k & = & i\tau_k\omega_0^n|X^k & , & \omega_{k-1}^{k-2}|X^k & = & \rho_k\omega_0^n|X^k & , \\ d\tau_{k-1}|X^k & = & \tau_{k-1}^1\omega_0^n|X^k & , & d\rho_{k-1}|X^k & = & \rho_{k-1}^1\omega_0^n|X^k & , \\ \cdots & & \cdots & & \cdots & & \cdots & \\ d\tau_4^{k-5}|X^k & = & \tau_4^{k-4}\omega_0^n|X^k & , & d\rho_4^{k-5}|X^k & = & \rho_4^{k-4}\omega_0^n|X^k & , \\ d\mu_4^{k-5}|X^k & = & \mu_4^{k-4}\omega_0^n|X^k & , & & & & \\ d\tau_3^{k-4}|X^k & = & \tau_3^{k-3}\omega_0^n|X^k & , & d\rho_3^{k-4}|X^k & = & \rho_3^{k-3}\omega_0^n|X^k & . \end{array}$$

The Lie algebra, of the isotropy group of a k-order contact element is given by

$$\hat{\mathcal{G}}^k = \tilde{\mathcal{G}}^{k-1}, \qquad \text{if } \lambda^{\alpha_{k-2}}_{\alpha_{k-1}} = 0, \quad \text{for all } \alpha_{k-2},\, \alpha_{k-1},$$

$$\tilde{\mathcal{G}}^k = \left\{ \ell = \begin{pmatrix} iA_0^0 \mathrm{Id}_{k\times k} & 0 & 0 \\ 0 & (\ell^{\alpha_{k-1}}_{\beta_{k-1}}) & 0 \\ 0 & 0 & iA_0^0 \end{pmatrix}; \begin{array}{c} \ell^{\alpha_{k-1}}_{\beta_{k-1}} + \overline{\ell^{\beta_{k-1}}_{\alpha_{k-1}}} = 0, \\ \mathrm{tr}\ell = 0, \quad A_0^0 \in \Re \end{array} \right\}, \quad \text{if } X^k \in I^k.$$

3. Equivalence of regular curves in Q

In this paragraph we give the structure equations for a regular curve in Q, and necessary and sufficient conditions for the equivalence of two regular curves.

Let Γ be a curve in Q transversal to the distribution D at all its points, viewed as a connected 1-dimensional submanifold. Given $p \in \Gamma$, denote by $C^k_p\Gamma$ the contact element of order k of Γ at p, and $G^0_{C^k_p\Gamma}$ the isotropy group by the induced action of G on C^kQ. We say that p is a *regular point* if:

For $n = 2$, $\quad \dim G^0_{C^3_p}\Gamma = 0$,

For $n \geq 3$, $\quad \dim G^0_{C^n_p}\Gamma = 0$,

If p is regular and k is the smallest order of contact with $G^0_{C^k_p\Gamma} = 0$, we say that p is a *regular point* of order k.

Now if p is a regular point of order k then all the points in a neighborhood of p are regulars of order k. We can see this, using the following results about Lie Groups [1].

Proposition 3.1. *Let G be a Lie group acting on a smooth manifold M. For $p \in M$, let G_p be the isotropy group of p and $d(p)$ the dimension of G_p. Then, given $p_0 \in M$, there exists a neighborhood V of p_0 in M, such that $d(p) \leq d(p_0)$ for all $p \in V$.*

We will say that a curve $\Gamma \subset Q$ is *regular* if all its points are regular.

Let I^k be the transversal section defined in (2.14) and $A = G.I^k$. Then we have the following,

Theorem 3.1. *Let $\Gamma \subset Q$ be a curve, and $p \in \Gamma$. Then Γ is regular of order k in p if and only if $C^k_p\Gamma \in A$.*

Proof. By definition of I^k we have: if $X^k_0 \in C^kQ$, and $X^k_0 \notin A$ then $\dim G^0_{X^k} < \dim G^0_{X^k_0}$, for all $X^k \in A$. In consequence, if $C^k_p\Gamma \notin A$ then $\dim G^0_{C^k_p\Gamma} > 0$ and p is no regular. The reciprocal is immediate by the definition of A. $\square$

Theorem 3.2. *Let $h : I^k \times G \to A$ be defined as $h(X, g) = g.X$. Then h has maximal rank.*

Proof. The manifold I^k is transversal to the orbits of the action of G on A. Then, if $X \in I^k$ we have,

$$T_XA = T_XI^k \oplus T_X\mathcal{O}_X, \text{ where } \mathcal{O}_X = G.X,$$

now given $g \in G$

$$Tl_g(T_X A) = Tl_g(T_X I^k) \oplus Tl_g(T_X \mathcal{O}_X) = T_{g.X} g I^k \oplus T_{g.X} \mathcal{O}_X,$$

moreover

$$T_{g.X} I^k \cap T_{g.X} \mathcal{O}_X = Tl_g(T_X I^k \cap T_X \mathcal{O}_X) = 0,$$

then $\dim(Tl_g(T_X A)) = \dim A$ and h has maximal rank. $\square$

Corollary 3.1. *Given $X \in A$ there exists $X_0 \in I^k$, $g_0 \in G$ and neighborhoods $\mathcal{U} \subset A$, $U \subset I^k$, $\mathcal{B} \subset G$ of X, X_0 and g_0 respectively such that*

$$h|_{U \times \mathcal{B}} : U \times \mathcal{B} \longrightarrow \mathcal{U} \text{ is a diffeomorphism,}$$

then there exist smooth sections $\eta : \mathcal{U} \to U \subset I^k$, $\sigma : \mathcal{U} \to \mathcal{B} \subset G$, such that

$$(h|_{U \times \mathcal{B}})^{-1} = (\eta, \sigma). \tag{3.1}$$

Remarks.

1. If $\Gamma : J \in \Re \to Q$ is a regular curve of order k, then $C^k \Gamma \subset A$, and given $X = C^k_p \Gamma$ the section σ allows us to define an immersion $\tilde{\sigma} = \sigma \circ C^k \Gamma$, of the curve Γ in G. Then we can transport the invariants forms ω^α_β defined on G to the curve Γ, as follows

$$\tilde{\omega}^\alpha_\beta = \tilde{\sigma}^* \omega^\alpha_\beta.$$

2. Given $X \in I^k$ the forms ω^α_β, which vanish on G^k, were projected onto $T_{X^k} \mathcal{O}^k$ using the map $\phi^k : g \in G \longmapsto g.X^k \in \mathcal{O}^k$.

For $v \in T_{X^k} \mathcal{O}_{X^k}$ we have defined $\tilde{\omega}^\alpha_\beta(v) = \omega^\alpha_\beta(V)$, where $V \in T_e G$ and $T\psi^j(V) = v$. The forms $\tilde{\omega}^\alpha_\beta$ can be extended similarly to (2.13), to the space $T_{X^k}(G.I^k)$.

3. Let $\Gamma \in Q$ be a regular curve of order k, and $p \in \Gamma$ a regular point of order k then for $q \in \Gamma$ there are $g \in G$, $X^k \in I^k$, such that $C^k_q \Gamma = g.X^k$. The left invariants forms $\tilde{\omega}^\alpha_\beta$ which vanish on G^k, with the exception of $\frac{\omega^0_0 - \overline{\omega^0_0}}{2i}$, were defined on $\Gamma'(0) \in T_p Q$ as

$$\tilde{\omega}^\alpha_\beta(\Gamma'(t)) = \tilde{\omega}^\alpha_\beta(\frac{d}{dt}|_{t=0} C^{k-1} \Gamma(t)).$$

We can transport the form $Im\omega^\alpha_\beta$ to the curve Γ, using the section $\tilde{\sigma}$. Then we define

$$\sigma^* Im\omega^0_0(\Gamma'(t)) = i\tau_n(t) \tilde{\sigma}^* \omega^n_0(\Gamma'(t)), \quad \tau_n(t) \in \Re. \tag{3.2}$$

It is clear, by the definition of I^k that for $n = 2$

$$(3.3)\qquad \begin{array}{rcl c rcl c} \tilde\omega_0^2(\Gamma'(0)) & \neq & 0 & , & \tilde\omega_1^0(\Gamma'(0)) & = & \tilde\omega_0^2(\Gamma'(0)) & , \\ \tilde\omega_0^1(\Gamma'(0)) & = & 0 & , & \tilde\omega_2^0(\Gamma'(0)) & = & \mu_4\tilde\omega_0^2(\Gamma'(0)) & , \\ Re\tilde\omega_0^0(\Gamma'(0)) & = & 0 & , & Im\omega_0^0(\Gamma'(0)) & = & \tau_3(\Gamma'(0)) & , \end{array}$$

for $n > 2$

$$(3.4)\qquad \begin{array}{rcl c rcl c} \tilde\omega_0^n(\Gamma'(0)) & \neq & 0 & , & \tilde\omega_1^0(\Gamma'(0)) & = & \tilde\omega_0^n(\Gamma'(0)) & , \\ \tilde\omega_{\alpha_2}^0(\Gamma'(0)) & = & 0 & , & Re\tilde\omega_0^0(\Gamma'(0)) & = & 0 & , \\ \tilde\omega_{\beta_j}^{\alpha_j}(\Gamma'(0)) & = & 0 & , & \tilde\omega_n^0(\Gamma'(0)) & = & \mu_4\tilde\omega_0^2(\Gamma'(0)) & , \end{array}$$

$$(\tilde\omega_0^0 - \tilde\omega_{\alpha_j-2}^{\alpha_j-2})(\Gamma'(0)) = i\tau_{\alpha_j}\omega_0^n(\Gamma'(0)), \quad \tilde\omega_{\alpha_j-2}^{\alpha_j-1}(\Gamma'(0)) = \rho_{\alpha_{j-2}}\omega_0^n(\Gamma'(0)) \quad .$$

Where $\rho_{\alpha_j}, \mu_{\alpha_j}, \tau_{\alpha_j}$, $\quad 3 \leq j \leq n-1$, are the invariants of the orbits defined in the section 2.5.

3.1 The structure equations

Let

$$\mathrm{e}_\alpha : G \to C^{n+1}, \qquad \text{given by} \qquad \mathrm{e}_\alpha(g_0, \cdots, g_n) = g_\alpha,$$

where $g = (g_\beta^\alpha) = (g_0, \cdots, g_n)$, then we have $\quad d\mathrm{e}_\alpha = \sum \omega_\beta^\alpha \mathrm{e}_\beta$.
If $\Gamma : J \to Q$ is a regular curve transversal to D, we say that Γ is *parametrized by arc length* if $\omega_o^n(\Gamma'(t)) = 1$, for $t \in J$. A parametrization of Γ by arc length is given by

$$s(t) = \int_{t_0}^t \omega_0^n(\Gamma'(t))dt.$$

Given a regular curve parametrized by arc length , from (3.3) we have,

$$\text{for n=2,} \qquad \begin{array}{rcl} d\mathrm{e}_0 & = & i\tau_3 ds\mathrm{e}_0 + ds\mathrm{e}_2 \\ d\mathrm{e}_1 & = & ds\mathrm{e}_0 - 2i\tau_3 ds\mathrm{e}_1 \\ d\mathrm{e}_2 & = & \mu_4 d\mathrm{e}_0 - ids\mathrm{e}_1 + i\tau_3 ds\mathrm{e}_2. \end{array}$$

For $n \geq 3$ the structure equations are given by the matrix,

$$\begin{pmatrix} i\tau_n ds & ds & & & & & & & \mu_4 ds \\ & i\delta_3 ds & \rho_3 ds & & & & & & \\ & -\rho_3 ds & i\delta_4 ds & . & & & & & \\ & & -\rho_4 ds & . & . & & & & \\ & & & . & . & . & & & \\ & & & & . & . & \rho_{n-1}ds & & \\ & & & & & . & i\delta_n ds & & \\ & & & & & & -\rho_n ds & i\epsilon ds & \\ ds & & & & & & & & i\tau_n ds \end{pmatrix}.$$

where $\quad \tau_\alpha, \mu_\alpha, \rho_\alpha, \quad$ are invariants of the orbits and

$$\tau_n = \frac{Im\tilde\omega_0^0}{\tilde\omega_0^n}, \qquad \delta_\alpha = \tau_n - \tau_\alpha, \qquad \epsilon = \sum_{\alpha=3}^n \tau_\alpha - n\tau_n.$$

Remark. *The form ds coincides with the Levi form for Cauchy-Riemann structures, [5].* In D_{p_0} we have a complex structure defined by the operator

$$I : v \in D_{p_0} \mapsto iv \in D_{p_0},$$

and the Levi form defined by $(\alpha(v,v))^2 = \frac{1}{2} d\omega_0^n(v, Iv)$. Using (1.7) we have,

$$d\omega_0^n = -2 \sum \overline{\omega_0^\gamma}(v)\omega_0^\gamma(v), \quad \text{then } (\alpha(v,v))^2 = \sum \overline{\omega_0^\gamma}(v)\omega_0^\gamma(v).$$

The following Theorem about Lie groups, and the structure equations defined above, allow us to prove the following Theorem of equivalence of curves.

Theorem 3.3. *Let G be a n-dimensional Lie group and $\omega^1, \cdots, \omega^n$ a basis of right invariants forms of G. Let $S_1, S_2 \in G$ be two m-dimensional connected submanifolds, with $n < m$. Then, there exists an element $g \in G$ such that $R_g(S_1) = S_2$ if and only if there exists a diffeomorphism $\psi : S_1 \to S_2$, which preserves the forms ω^j, $\psi^*(\omega^j|s_1) = \omega^j|S_2$.*
Proof. [7].

Theorem 3.4. *Let $\Gamma^1, \Gamma^2 \subset Q$ be two regular curves of order k, transverses to D, parametrized by length arc. Let $p_1 \in \Gamma^1$, $p_2 \in \Gamma^2$. Then there exists an element $g \in G$ such that locally, $l_g(\Gamma^1) = \Gamma^2$, $g.p_1 = p_2$ if and only if there exists a local diffeomorphism*

$$\psi : \Gamma_1 \to \Gamma_2, \qquad \textit{such that} \quad \begin{array}{ll} \tau_\alpha^1 \circ \psi = \tau_\alpha^2, & \mu_\alpha^1 \circ \psi = \mu_\alpha^2, \\ \rho_\alpha^1 \circ \psi = \rho_\alpha^2, & \nu_\alpha^1 \circ \psi = \nu_\alpha^2, \end{array}$$

where $\tau_\alpha^j, \mu_\alpha^j, \rho_\alpha^j, \nu_\alpha^j$, $j = 1, 2$, are the invariants asociated to Γ^j.

Proof. Let $\tilde{\sigma}^1, \tilde{\sigma}^2$ be the local sections of Γ^1, Γ^2 respectively in G, defined by the local diffeomorphism given in (3.1). These sections define two connected submanifolds $\tilde{\Gamma}^1, \tilde{\Gamma}^2 \in G$, and Γ^1, Γ^2 are locally equivalent if and only if $\tilde{\Gamma}^1, \tilde{\Gamma}^2$ are locally equivalent, by the following commutative diagram,

$$\begin{array}{ccccccc} J^1 & \xrightarrow{\gamma^1} & \Gamma^1 \subset Q & \xrightarrow{i^k} & C^k Q & \xrightarrow{\sigma^1} & G \\ & & \big\downarrow l_g & & & & \big\downarrow l_g \\ J^2 & \xrightarrow{\gamma^2} & \Gamma^2 \subset Q & \xrightarrow{i^k} & C^k Q & \xrightarrow{\sigma^2} & G \end{array}.$$

Using Theorem 3.3, we have that $\tilde{\Gamma}^1 = \sigma^1 \circ i^k \circ \gamma^1$, and $\tilde{\Gamma}^2 = \sigma^2 \circ i^k \circ \gamma^2$, are equivalents if and only if there exists a local diffeomorphism ψ which preserves the forms ω^α. Since the curves are parametrized by arc length, we have using the structure equations that the forms are preserved if and only if the invariants $\tau_\alpha^j, \mu_\alpha^j, \rho_\alpha^j, \nu_\alpha^j$, $j = 1, 2$, are preserved. $\square$

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