ARCHIVUM MATHEMATICUM (BRNO)
Tomus 32 (1996), 13 – 27

ON THE OSCILLATION OF AN mTH ORDER PERTURBED NONLINEAR DIFFERENCE EQUATION

P. J. Y. Wong and R. P. Agarwal

Abstract. We offer sufficient conditions for the oscillation of all solutions of the perturbed difference equation

$$|\Delta^m y(k)|^{\alpha-1}\Delta^m y(k) + Q(k, y(k-\sigma_k), \Delta y(k-\sigma_k), \cdots, \Delta^{m-2}y(k-\sigma_k))$$
$$= P(k, y(k-\sigma_k), \Delta y(k-\sigma_k), \cdots, \Delta^{m-1}y(k-\sigma_k)),\ \ k \ge k_0$$

where $\alpha > 0$. Examples which dwell upon the importance of our results are also included.

1. INTRODUCTION

The theory of difference equations and their applications have been and still are receiving intensive attention. In fact, in the last few years several monographs and hundreds of research papers have appeared, e.g., [1,8] cover more than 450 articles. In this paper we shall consider the mth order perturbed difference equation

$$|\Delta^m y(k)|^{\alpha-1}\Delta^m y(k) + Q(k, y(k-\sigma_k), \Delta y(k-\sigma_k), \cdots, \Delta^{m-2}y(k-\sigma_k))$$
$$= P(k, y(k-\sigma_k), \Delta y(k-\sigma_k), \cdots, \Delta^{m-1}y(k-\sigma_k)),\ \ k \ge k_0 \quad (1.1)$$

where $\alpha > 0$ and Δ is the forward difference operator defined as $\Delta y(k) = y(k+1) - y(k)$. Further, we suppose that $\sigma_k \in Z$ and $\lim_{k\to\infty}(k-\sigma_k) = \infty$. Throughout it is also assumed that there exist real sequences $\{q(k)\}$, $\{p(k)\}$ and a function $f : \Re \to \Re$ such that

(I) $uf(u) > 0$ for all $u \ne 0$;

(II) $$\frac{Q(k, x(k-\sigma_k), \Delta x(k-\sigma_k), \cdots, \Delta^{m-2}x(k-\sigma_k))}{f(x(k-\sigma_k))} \ge q(k),$$

$$\frac{P(k, x(k-\sigma_k), \Delta x(k-\sigma_k), \cdots, \Delta^{m-1}x(k-\sigma_k))}{f(x(k-\sigma_k))} \le p(k) \text{ for all } x \ne 0;$$

and

1991 *Mathematics Subject Classification*: 39A10.
Key words and phrases: oscillatory solutions, difference equations.
Received August 31, 1995.

(III) $\lim_{k\to\infty}[q(k)-p(k)] \ge 0$.

By a solution of (1.1), we mean a nontrivial sequence $\{y(k)\}$ defined for $k \ge \min_{\ell\ge 0}(\ell-\sigma_\ell)$, $\Delta^m y(k)$ is not identically zero, and $y(k)$ fulfills (1.1) for $k \ge k_0$. A solution $\{y(k)\}$ is said to be oscillatory if it is neither eventually positive nor negative, and nonoscillatory otherwise. Throughout, for $i \ge 0$ we shall use the usual factorial notation $k^{(i)} = k(k-1)\cdots(k-i+1)$.

In the literature, numerous oscillation criteria for nonlinear difference and differential equations related to (1.1) have been established, e.g., see [1-4,7,9, 13-20 and the references cited therein]. We refer particularly to [2-4] in which oscillation theorems for higher order nonlinear difference equations are presented. Thandapani and Sundaram [12] have recently considered a special case of (1.1)

$$\Delta^{2m} y(k) + q(k) f(y(k-\sigma_k)) = 0, \ k \ge k_0 \tag{1.2}$$

where $\{q(k)\}$ is an eventually positive sequence. We have extended their work to general higher order equations. In fact, our results include, as special cases, known oscillation theorems not only for (1.2), but also for several other particular difference equations considered in [1]. Further, our results generalize those in [11,19]. Finally, we remark that the paper is partly motivated by the analogy between differential and difference equations, in fact discrete version of the results in [5,6,10] have been developed.

2. PRELIMINARIES

Lemma 2.1. [1, p.29] *Let $1 \le j \le m-1$ and $y(k)$ be defined for $k \ge k_0$. Then,*

(a) $\liminf_{k\to\infty} \Delta^j y(k) > 0$ *implies* $\lim_{k\to\infty} \Delta^i y(k) = \infty$, $0 \le i \le j-1$;
(b) $\limsup_{k\to\infty} \Delta^j y(k) < 0$ *implies* $\lim_{k\to\infty} \Delta^i y(k) = -\infty$, $0 \le i \le j-1$.

Lemma 2.2. [1, p.29] (Discrete Kneser's Theorem) *Let $y(k)$ be defined for $k \ge k_0$, and $y(k) > 0$ with $\Delta^m y(k)$ of constant sign for $k \ge k_0$ and not identically zero. Then, there exists an integer p, $0 \le p \le m$ with $(m+p)$ odd for $\Delta^m y(k) \le 0$ and $(m+p)$ even for $\Delta^m y(k) \ge 0$, such that*

(a) $p \le m-1$ *implies* $(-1)^{p+i}\Delta^i y(k) > 0$ *for all* $k \ge k_0$, $p \le i \le m-1$;
(b) $p \ge 1$ *implies* $\Delta^i y(k) > 0$ *for all large* $k \ge k_0$, $1 \le i \le p-1$.

Lemma 2.3. [1, p.30] Let $y(k)$ be defined for $k \ge k_0$, and $y(k) > 0$ with $\Delta^m y(k) \le 0$ for $k \ge k_0$ and not identically zero. Then, there exists a large integer $k_1 \ge k_0$ such that

$$y(k) \ge \frac{1}{(m-1)!}\,(k-k_1)^{(m-1)}\Delta^{m-1} y(2^{m-p-1}k), \ k \ge k_1$$

where p is defined in Lemma 2.2.

3. MAIN RESULTS

For clarity the conditions used in the main results are listed as follows :

$$f \text{ is continuous and } \liminf_{|u|\to\infty} f(u) > 0, \tag{3.1}$$

$$\sum^{\infty}[q(k)-p(k)]^{1/\alpha} = \infty, \tag{3.2}$$

$$f \text{ is nondecreasing}, \tag{3.3}$$

$$\lim_{k\to\infty} \frac{1}{k^{(m-1)}} \sum_{\ell=k_0}^{k-m} (k-\ell-1)^{(m-1)}[q(\ell)-p(\ell)]^{1/\alpha} = \infty, \tag{3.4}$$

$$\begin{gathered} f \text{ is nondecreasing, } f(uv) \geq Mf(u)f(v) \text{ for } u,v>0 \\ \text{and some positive constant } M, \end{gathered} \tag{3.5}$$

$$\int_0^{\theta} \frac{du}{f(u)^{1/\alpha}} < \infty, \; \int_0^{-\theta} \frac{du}{f(u)^{1/\alpha}} < \infty \text{ for all } \theta > 0, \tag{3.6}$$

$$\sum^{\infty} \left\{ f(k^{(m-1)})[q(k)-p(k)] \right\}^{1/\alpha} = \infty. \tag{3.7}$$

Theorem 3.1. Suppose (3.1) and (3.2) hold.

(a) If m is even or $m = 1$, then all solutions of (1.1) are oscillatory.
(b) If $m(\geq 3)$ is odd, then a solution $y(k)$ of (1.1) is either oscillatory or $\Delta y(k)$ is oscillatory.

Proof. Let $\{y(k)\}$ be a nonoscillatory solution of (1.1), say, $y(k) > 0$ for $k \geq k_1 \geq k_0$. We shall consider only this case because the proof for the case $y(k) < 0$ for $k \geq k_1 \geq k_0$ is similar. Using (I) - (III), it follows from (1.1) that

$$\begin{aligned} |\Delta^m y(k)|^{\alpha-1}\Delta^m y(k) &\leq [p(k)-q(k)]f(y(k-\sigma_k)) \\ &\leq 0, \; k \geq k_1. \end{aligned} \tag{3.8}$$

Hence, we have

$$\Delta^m y(k) \leq 0, \; k \geq k_1. \tag{3.9}$$

Case 1 m is even

In view of (3.9), from Lemma 2.2 (here p is odd and $1 \leq p \leq m-1$, take $i = 1$ in (b)) it follows that

$$\Delta y(k) > 0, \; k \geq k_1. \tag{3.10}$$

Let

$$L = \lim_{k\to\infty} y(k-\sigma_k). \tag{3.11}$$

Then, since $k-\sigma_k \to \infty$ and $y(k)$ is increasing for large k (by (3.10)), we have $L > 0$ and L is finite or infinite.

(i) Suppose that $0 < L < \infty$. Since f is continuous, we get

$$\lim_{k\to\infty} f(y(k-\sigma_k)) = f(L) > 0.$$

Thus, there exists an integer $k_2 \ge k_1$ such that

$$f(y(k-\sigma_k)) \ge \frac{1}{2}\, f(L),\ \ k \ge k_2. \tag{3.12}$$

Now, from (3.8) we get

$$|\Delta^m y(k)|^{\alpha-1}\Delta^m y(k) + [q(k)-p(k)]f(y(k-\sigma_k)) \le 0,\ \ k \ge k_2 \tag{3.13}$$

which in view of (III) and (3.12) leads to

$$|\Delta^m y(k)|^{\alpha-1}\Delta^m y(k) + [q(k)-p(k)]\frac{1}{2}\, f(L) \le 0,\ \ k \ge k_2. \tag{3.14}$$

Using (3.9), inequality (3.14) is equivalent to

$$|\Delta^m y(k)|^{\alpha} \ge [q(k)-p(k)]\frac{1}{2}\, f(L),\ \ k \ge k_2$$

or

$$-\Delta^m y(k) \ge \left\{[q(k)-p(k)]\frac{1}{2}\, f(L)\right\}^{1/\alpha},\ \ k \ge k_2. \tag{3.15}$$

Summing (3.15) from k_2 to $(k-1)$, we obtain

$$\Delta^{m-1} y(k) \le \Delta^{m-1} y(k_2) - \left[\frac{1}{2}\, f(L)\right]^{1/\alpha} \sum_{\ell=k_2}^{k-1} [q(\ell)-p(\ell)]^{1/\alpha}. \tag{3.16}$$

By (3.2), the right side of (3.16) tends to $-\infty$ as $k \to \infty$. Thus, there exists an integer $k_3 \ge k_2$ such that

$$\Delta^{m-1} y(k) < 0,\ \ k \ge k_3.$$

It follows from Lemma 2.1(b) ($j = m-1$) that $y(k) \to -\infty$ as $k \to \infty$. This contradicts the assumption that $\{y(k)\}$ is eventually positive.

(ii) Suppose that $L = \infty$. By (3.1), we have

$$\liminf_{k\to\infty} f(y(k-\sigma_k)) > 0.$$

This implies the existence of an integer $k_2 \ge k_1$ such that

$$f(y(k-\sigma_k)) \ge A,\ \ k \ge k_2 \tag{3.17}$$

for some $A > 0$. In view of (III) and (3.17), it follows from (3.13) that

$$|\Delta^m y(k)|^{\alpha-1}\Delta^m y(k) + [q(k) - p(k)]A \le 0, \ k \ge k_2.$$

The rest of the proof is similar to that of Case 1(i).

Case 2 m is odd

Here, in view of (3.9), in Lemma 2.2 we have p is even and $0 \le p \le m-1$ and hence we cannot conclude that (3.10) is true. Let L be defined as in (3.11). We note that $L > 0$.

(i) Suppose that $\Delta y(k) > 0$ for $k \ge k_1$, i.e., (3.10) holds. Then, L is finite or infinite and the proof follows as in Case 1.

(ii) Suppose that $\Delta y(k) < 0$ for $k \ge k_1$. Then, L is finite and the proof follows as in Case 1(i).

(iii) Suppose that $\Delta y(k)$ is oscillatory. For the special case $m = 1$, (1.1) provides

$$|\Delta y(k)|^{\alpha-1}\Delta y(k) \le [p(k) - q(k)]f(y(k-\sigma_k)) < 0, \ k \ge k_1 \tag{3.18}$$

where in view of (3.2), we have noted and used in the last inequality

$$q(k) - p(k) > 0 \tag{3.19}$$

for sufficiently large k. It follows from (3.18) that

$$\Delta y(k) < 0, \ k \ge k_1$$

which contradicts the assumption that $\Delta y(k)$ is oscillatory.

The proof of the theorem is now complete.

Example 3.1. Consider the difference equation

$$|\Delta^4 y(k)|^2 \Delta^4 y(k) + [y(k+1)]^3\left(b + \frac{3^{12}}{2^9}\right) = b[y(k+1)]^3, \ k \ge 0 \tag{3.20}$$

where $b = b(k, y(k+1), \Delta y(k+1), \Delta^2 y(k+1))$ is any function. Here, $\alpha = 3$ and $m = 4$. Take $k - \sigma_k \equiv (k+1)$ and $f(y) = y^3$. Then, (3.1) clearly holds. Further, we have

$$\frac{Q(k, y(k+1), \Delta y(k+1), \Delta^2 y(k+1))}{f(y(k+1))} = b + \frac{3^{12}}{2^9} \equiv q(k)$$

and

$$\frac{P(k, y(k+1), \Delta y(k+1), \Delta^2 y(k+1), \Delta^3 y(k+1))}{f(y(k+1))} = b \equiv p(k)$$

and so (3.2) is satisfied. It follows from Theorem 3.1(a) that all solutions of (3.20) are oscillatory. One such solution is given by $\{y(k)\} = \{(-1)^k/2^k\}$.

Example 3.2. Consider the difference equation

$$|\Delta y(k)|^{\alpha-1}\Delta y(k) + y(k \pm 2c)\cdot(b + 2^\alpha) = b\ y(k \pm 2c), \ k \ge 0 \tag{3.21}$$

where $\alpha > 0$, c is any fixed integer, and $b = b(k, y(k \pm 2c))$ is any function. Here, $m = 1$. Take $k - \sigma_k \equiv (k \pm 2c)$ and $f(y) = y$. Then, it is obvious that (3.1) holds. Further, we have

$$\frac{Q(k, y(k \pm 2c))}{f(y(k \pm 2c))} = b + 2^\alpha \equiv q(k)$$

and

$$\frac{P(k, y(k \pm 2c))}{f(y(k \pm 2c))} = b \equiv p(k)$$

and so (3.2) is satisfied. Hence, Theorem 3.1(a) ensures that all solutions of (3.21) are oscillatory. One such solution is given by $\{y(k)\} = \{(-1)^k\}$.

Example 3.3. Consider the difference equation

$$|\Delta^3 y(k)|^{\alpha-1}\Delta^3 y(k) + y(k)\left[b + \frac{4^\alpha(2k+3)^\alpha}{k}\right] = b\ y(k),\ \ k \ge 1 \tag{3.22}$$

where $0 < \alpha < 1/2$ and $b = b(k, y(k), \Delta y(k))$ is any function. Here, $m = 3$. Taking $k - \sigma_k \equiv k$ and $f(y) = y$, we note that (3.1) holds. Next,

$$\frac{Q(k, y(k), \Delta y(k))}{f(y(k))} = b + \frac{4^\alpha(2k+3)^\alpha}{k} \equiv q(k)$$

and

$$\frac{P(k, y(k), \Delta y(k), \Delta^2 y(k))}{f(y(k))} = b \equiv p(k)$$

lead to

$$\sum^\infty [q(k) - p(k)]^{1/\alpha} = 4\sum^\infty \left(\frac{2}{k^{1/\alpha-1}} + \frac{3}{k^{1/\alpha}}\right) < \infty$$

and hence (3.2) is not satisfied. The conditions of Theorem 3.1 are violated. In fact, (3.22) has a solution given by $\{y(k)\} = \{(-1)^k k\}$, and we observe that both $\{y(k)\}$ and $\{\Delta y(k)\} = \{(-1)^{k+1}(2k+1)\}$ are oscillatory. This illustrates Theorem 3.1(b).

Theorem 3.2. Suppose (3.3) and (3.4) hold. Then, the conclusion of Theorem 3.1 follows.

Proof. Suppose that $\{y(k)\}$ is a nonoscillatory solution of (1.1), say, $y(k) > 0$ for $k \ge k_1 \ge k_0$. Using (I) - (III), from (1.1) we still get (3.8) and (3.9).

Case 1 m is even

Since (3.9) holds, from Lemma 2.2 (take $i = 1$ in (b)) we obtain (3.10). It follows that

$$y(k) \ge y(k_1) \equiv a,\ \ k \ge k_1. \tag{3.23}$$

In view of (3.23) and the fact that $\lim_{k\to\infty}(k - \sigma_k) = \infty$, there exists an integer $k_2 \ge k_1$ such that

$$y(k - \sigma_k) \ge a,\ \ k \ge k_2. \tag{3.24}$$

Since the function f is nondecreasing (condition (3.3)), (3.24) provides

$$f(y(k-\sigma_k)) \geq f(a) \equiv A, \ k \geq k_2. \tag{3.25}$$

Now, from (3.8) we get (3.13) which on using (3.25) provides

$$|\Delta^m y(k)|^{\alpha-1}\Delta^m y(k) + [q(k)-p(k)]A \leq 0, \ k \geq k_2.$$

In view of (3.9), the above inequality is the same as

$$|\Delta^m y(k)|^{\alpha} \geq [q(k)-p(k)]A, \ k \geq k_2$$

or

$$-\Delta^m y(k) \geq \left\{[q(k)-p(k)]A\right\}^{1/\alpha}, \ k \geq k_2. \tag{3.26}$$

By discrete Taylor's formula [1, p.26], $y(k)$ can be expressed as

$$y(k) = \sum_{i=0}^{m-1} \frac{(k-k_2)^{(i)}}{i!} \ \Delta^i y(k_2) + \frac{1}{(m-1)!} \sum_{\ell=k_2}^{k-m} (k-\ell-1)^{(m-1)} \Delta^m y(\ell)$$

which on rearranging and using (3.26) yields

$$\begin{aligned} &\frac{A^{1/\alpha}}{(m-1)!} \sum_{\ell=k_2}^{k-m} (k-\ell-1)^{(m-1)} [q(\ell)-p(\ell)]^{1/\alpha} \\ \leq &\sum_{i=0}^{m-1} \frac{(k-k_2)^{(i)}}{i!} \ \Delta^i y(k_2) - y(k) \leq \sum_{i=0}^{m-1} \frac{(k-k_2)^{(i)}}{i!} \ \Delta^i y(k_2), \ k \geq k_2. \end{aligned}$$

Dividing both sides by $k^{(m-1)}$, the above inequality becomes

$$\frac{A^{1/\alpha}}{(m-1)!} \ \frac{1}{k^{(m-1)}} \sum_{\ell=k_2}^{k-m} (k-\ell-1)^{(m-1)} [q(\ell)-p(\ell)]^{1/\alpha}$$

$$\leq \sum_{i=0}^{m-1} \frac{(k-k_2)^{(i)}}{i!} \ \frac{1}{k^{(m-1)}} \ \Delta^i y(k_2), \ k \geq k_2. \tag{3.27}$$

By (3.4), the left side of (3.27) tends to ∞ as $k \to \infty$. However, the right side of (3.27) is finite as $k \to \infty$.

Case 2 m is odd

In this case, taking note of (3.9), in Lemma 2.2 we have p is even and $0 \leq p \leq m-1$. Therefore, we cannot ensure that (3.10) holds.

(i) Suppose that $\Delta y(k) > 0$ for $k \geq k_1$, i.e., (3.10) holds. The proof for this case follows from that of Case 1.

(ii) Suppose that $\Delta y(k) < 0$ for $k \geq k_1$. Then, $y(k) \downarrow a \ (> 0)$ and so there exists a $k_2 \geq k_1$ such that (3.24) holds. The proof then proceeds as in Case 1.

(iii) Suppose that $\Delta y(k)$ is oscillatory. Condition (3.4) implies that

$$\frac{1}{k^{(m-1)}}(k-(k-m)-1)^{(m-1)}[q(k-m)-p(k-m)]^{1/\alpha}>0$$

for sufficiently large k, which ensures that (3.19) holds for sufficiently large k. Hence, for the special case $m=1$, we get (3.18) and it is seen from the proof of Theorem 3.1 (Case 2(iii)) that this leads to some contradiction.

The proof of the theorem is now complete.

Example 3.4. Consider the difference equation

$$|\Delta^4 y(k)|^2\Delta^4 y(k)+[y(k+1)]^{15}\left(b+3^{12}\ 2^{12k+3}\right)=b[y(k+1)]^{15},\ k\geq 0 \tag{3.28}$$

where $b=b(k,y(k+1),\Delta y(k+1),\Delta^2 y(k+1))$ is any function. Here, $\alpha=3$, $m=4$, and $f(y)=y^{15}$, which is nondecreasing. Taking $k-\sigma_k\equiv(k+1)$, we have

$$\frac{Q(k,y(k+1),\Delta y(k+1),\Delta^2 y(k+1))}{f(y(k+1))}=b+3^{12}\ 2^{12k+3}\equiv q(k)$$

and

$$\frac{P(k,y(k+1),\Delta y(k+1),\Delta^2 y(k+1),\Delta^3 y(k+1))}{f(y(k+1))}=b\equiv p(k).$$

We find that

$$\begin{aligned}
&\frac{1}{k^{(m-1)}}\sum_{\ell=k_0}^{k-m}(k-\ell-1)^{(m-1)}[q(\ell)-p(\ell)]^{1/\alpha}\\
&\quad=\frac{1}{k^{(3)}}\sum_{\ell=0}^{k-4}(k-\ell-1)^{(3)}\cdot 81\cdot 2^{4\ell+1}\geq\frac{1}{k^{(3)}}\ 3^{(3)}\cdot 81\cdot 2^{4(k-4)+1}\\
&\quad=3^{(3)}\cdot 81\cdot 2^{k-3}\frac{2^{3(k-4)}}{k^{(3)}}\geq 3^{(3)}\cdot 81\cdot 2^{k-3},\ k\geq 7.
\end{aligned}$$

Hence, (3.4) is satisfied. By Theorem 3.2(a) all solutions of (3.28) are oscillatory. One such solution is given by $\{y(k)\}=\{(-1)^k/2^k\}$.

Remark 3.1. It is clear that conditions (3.1) and (3.2) are fulfilled for equation (3.28). Hence, Example 3.4 also illustrates Theorem 3.1(a).

Remark 3.2. Equation (3.21) also satisfies conditions (3.3) and (3.4). Hence, Theorem 3.2(a) ensures that all solutions of (3.21) are oscillatory. We have seen that one such solution is given by $\{y(k)\}=\{(-1)^k\}$.

Remark 3.3. In Example 3.3, the condition (3.3) is satisfied. To check whether condition (3.4) is fulfilled, we note that

$$\frac{1}{k^{(m-1)}}\sum_{\ell=k_0}^{k-m}(k-\ell-1)^{(m-1)}[q(\ell)-p(\ell)]^{1/\alpha}$$

$$=\frac{1}{k^{(2)}}\sum_{\ell=1}^{k-3}(k-\ell-1)^{(2)}\ \frac{4(2\ell+3)}{\ell^{1/\alpha}}\le 4\ \frac{(k-2)^{(2)}}{k^{(2)}}\ \sum_{\ell=1}^{k-3}\left(\frac{2}{\ell^{1/\alpha-1}}+\frac{3}{\ell^{1/\alpha}}\right).$$

Letting $k\to\infty$, we get, in view of $0<\alpha<1/2$,

$$\lim_{k\to\infty}\frac{1}{k^{(2)}}\sum_{\ell=1}^{k-3}(k-\ell-1)^{(2)}\ \frac{4(2\ell+3)}{\ell^{1/\alpha}}\le 4\sum_{\ell=1}^{\infty}\left(\frac{2}{\ell^{1/\alpha-1}}+\frac{3}{\ell^{1/\alpha}}\right)<\infty$$

and hence (3.4) is not satisfied. The conditions of Theorem 3.2 are violated. In fact, it is noted that equation (3.22) has an oscillatory solution given by $\{y(k)\}=\{(-1)^k k\}$ where $\{\Delta y(k)\}$ is also oscillatory. Hence, Example 3.3 also illustrates Theorem 3.2(b).

Theorem 3.3. Suppose $\sigma_k=-\sigma$, $\sigma\ge 1$ and (3.5) - (3.7) hold. Then, the conclusion of Theorem 3.1 follows.

Proof. Again suppose that $\{y(k)\}$ is a nonoscillatory solution of (1.1), say, $y(k)>0$ for $k\ge k_1\ge k_0$. Using (I) - (III), from (1.1) we have

$$|\Delta^m y(k)|^{\alpha-1}\Delta^m y(k)\le[p(k)-q(k)]f(y(k+\sigma))\le 0,\ k\ge k_1 \tag{3.29}$$

and therefore (3.9) holds.

Case 1 m is even

Since (3.9) holds, from Lemma 2.2 (take $i=1$ in (b), $i=m-1$ in (a)) we get for $k\ge k_1$,

$$\Delta y(k)>0,\ \Delta^{m-1}y(k)>0. \tag{3.30}$$

Using $\Delta y(k)>0$ for $k\ge k_1$ and Lemma 2.3, we find that there exists $k_2\ge k_1$ such that

$$\begin{aligned} y(k+\sigma)\ &\ge\ y(k)\\ &\ge\ y\left(2^{p-m+1}\ k\right)\\ &\ge\ \frac{1}{(m-1)!}\left(2^{p-m+1}\ k-k_2\right)^{(m-1)}\Delta^{m-1}y(k)\\ &\ge\ \frac{1}{(m-1)!}\ 2^{(p-m+1)(m-1)}\left(k-2^m\ k_2\right)^{(m-1)}\Delta^{m-1}y(k),\ k\ge k_2. \end{aligned}$$

It follows that

$$\begin{aligned} y(k+\sigma)\ &\ge\frac{1}{(m-1)!}\ 2^{(p-m+1)(m-1)}\ \frac{1}{2^{m-1}}\ k^{(m-1)}\Delta^{m-1}y(k)\\ &=A\ k^{(m-1)}\Delta^{m-1}y(k),\ k\ge 2^{m+1}\ k_2+m-2\equiv k_3 \end{aligned} \tag{3.31}$$

where $A = 2^{(p-m)(m-1)}/(m-1)!$.

In view of (3.5), it follows from (3.31) that

$$f(y(k+\sigma)) \geq f\left(A\ k^{(m-1)}\Delta^{m-1}y(k)\right) \geq Mf(A)f\left(k^{(m-1)}\Delta^{m-1}y(k)\right)$$

$$\geq M^2 f(A) f\left(k^{(m-1)}\right) f\left(\Delta^{m-1}y(k)\right),\ \ k \geq k_3. \tag{3.32}$$

Now, using (3.32) in (3.29) gives

$$|\Delta^m y(k)|^{\alpha-1}\Delta^m y(k) + [q(k)-p(k)]M^2 f(A)f\left(k^{(m-1)}\right) f\left(\Delta^{m-1}y(k)\right) \leq 0,$$
$$k \geq k_3$$

which, on noting that $\Delta^{m-1}y(k) > 0$ for $k \geq k_3$ and (3.9), is equivalent to

$$M^2 f(A) f\left(k^{(m-1)}\right)[q(k)-p(k)] \leq \frac{|\Delta^m y(k)|^\alpha}{f\left(\Delta^{m-1}y(k)\right)},\ \ k \geq k_3$$

or

$$\left\{M^2 f(A) f\left(k^{(m-1)}\right)[q(k)-p(k)]\right\}^{1/\alpha} \leq \frac{-\Delta^m y(k)}{f\left(\Delta^{m-1}y(k)\right)^{1/\alpha}},\ \ k \geq k_3. \tag{3.33}$$

Summing (3.33) from k_3 to k, we get

(3.34)
$$\begin{aligned}\left[M^2 f(A)\right]^{1/\alpha}\sum_{\ell=k_3}^{k}\left\{f(\ell^{(m-1)})[q(\ell)-p(\ell)]\right\}^{1/\alpha} &\leq \sum_{\ell=k_3}^{k}\frac{-\Delta^m y(\ell)}{f\left(\Delta^{m-1}y(\ell)\right)^{1/\alpha}} \\ &\leq \int_0^{\Delta^{m-1}y(k_3)}\frac{du}{f(u)^{1/\alpha}}.\end{aligned}$$

By (3.7), the left side of (3.34) tends to ∞ as $k \to \infty$, whereas the right side is finite by (3.6).

Case 2 m is odd

Here, in view of (3.9), in Lemma 2.2 we have p is even and $0 \leq p \leq m-1$. Hence, instead of (3.30) we can only conclude that

$$\Delta^{m-1}y(k) > 0,\ \ k \geq k_1. \tag{3.35}$$

(i) Suppose that $\Delta y(k) > 0$ for $k \geq k_1$. The proof follows as in Case 1.

(ii) Suppose that $\Delta y(k) < 0$ for $k \geq k_1$. Then, on using Lemma 2.3 we find that there exists $k_2 \geq k_1$ such that

$$\begin{aligned}
& y(k+\sigma) \\
& \geq y(2^{p-m+1}\ k + k + \sigma) \\
& \geq \frac{1}{(m-1)!}\left(2^{p-m+1}\ k + k + \sigma - k_2\right)^{(m-1)} \Delta^{m-1} y\left(2^{m-p-1}(2^{p-m+1}\ k + k + \sigma)\right) \\
& \geq \frac{1}{(m-1)!}\left(2^{p-m+1}\ k + k + \sigma - k_2\right)^{(m-1)} \frac{\Delta^{m-1}y(k)}{\Delta^{m-1}y(k)}\ \Delta^{m-1} y\left(k + 2^{m-1}(k+\sigma)\right) \\
& \geq \frac{\beta}{(m-1)!}\left(2^{p-m+1}\ k - k_2\right)^{(m-1)} \Delta^{m-1} y(k) \\
& \geq \frac{\beta}{(m-1)!}\ 2^{(p-m+1)(m-1)}\left(k - 2^m\ k_2\right)^{(m-1)} \Delta^{m-1} y(k),\ \ k \geq k_2
\end{aligned}$$

where we have also used the fact that $\Delta^{m-1}y(k)$ is nonincreasing (by (3.9)) and

$$\beta = \min_{k \geq k_2} \frac{\Delta^{m-1}y(k + 2^{m-1}(k+\sigma))}{\Delta^{m-1}y(k)} > 0.$$

It follows that

$$\begin{aligned}
y(k+\sigma) & \geq \frac{\beta}{(m-1)!}\ 2^{(p-m+1)(m-1)}\ \frac{1}{2^{m-1}}\ k^{(m-1)} \Delta^{m-1} y(k) \\
& = A\beta\ k^{(m-1)} \Delta^{m-1} y(k),\ \ k \geq k_3
\end{aligned}$$

where A and k_3 are defined in (3.31). The rest of the proof uses a similar argument as in Case 1.

(iii) Suppose that $\Delta y(k)$ is oscillatory. Condition (3.7) implies that

$$\left\{ f(k^{(m-1)})[q(k) - p(k)] \right\}^{1/\alpha} > 0$$

for sufficiently large k. This ensures that (3.19) holds for sufficiently large k. Hence, for the special case $m = 1$, we get (3.18) and we have seen from the proof of Theorem 3.1 (Case 2(iii)) that this leads to some contradiction.

The proof of the theorem is now complete.

Example 3.5. Consider the difference equation

$$|\Delta^2 y(k)|\Delta^2 y(k) + y(k+1)\,[b + 16(k+1)] = b\ y(k+1),\ \ k \geq 0 \tag{3.36}$$

where $b = b(k, y(k+1))$ is any function. Here, $\alpha = 2$ and $m = 2$. Take $k - \sigma_k \equiv (k+1)$ and $f(y) = y$. Then, (3.5) and (3.6) clearly hold. Further, we have

$$\frac{Q(k, y(k+1))}{f(y(k+1))} = b + 16(k+1) \equiv q(k)$$

and

$$\frac{P(k, y(k+1), \Delta y(k+1))}{f(y(k+1))} = b \equiv p(k).$$

Hence,

$$\sum^{\infty} \left\{ f(k^{(m-1)})[q(k) - p(k)] \right\}^{1/\alpha} = \sum^{\infty} [k \cdot 16(k+1)]^{1/2} = \infty$$

and (3.7) is satisfied. It follows from Theorem 3.3(a) that all solutions of (3.36) are oscillatory. One such solution is given by $\{y(k)\} = \{(-1)^k k\}$.

Remark 3.4. In the above example, it is obvious that the conditions of Theorem 3.1 are satisfied. Hence, Example 3.5 also illustrates Theorem 3.1(a).

Remark 3.5. Equation (3.36) clearly satisfies condition (3.3). To see that (3.4) is fulfilled, we note that

$$\begin{aligned}
&\lim_{k\to\infty} \frac{1}{k^{(m-1)}} \sum_{\ell=k_0}^{k-m} (k-\ell-1)^{(m-1)} [q(\ell) - p(\ell)]^{1/\alpha} \\
&\qquad = \lim_{k\to\infty} \frac{1}{k} \sum_{\ell=0}^{k-2} (k-\ell-1)\ 4\sqrt{\ell+1} \\
&\qquad = 4 \lim_{k\to\infty} \left[\sum_{\ell=0}^{k-2} \sqrt{\ell+1} - \frac{1}{k} \sum_{\ell=0}^{k-2} (\ell+1)^{3/2} \right] \\
&\qquad \ge 4 \lim_{k\to\infty} \left[\sum_{\ell=0}^{k-2} \sqrt{\ell+1} - \frac{k-1}{k} \sum_{\ell=0}^{k-2} \sqrt{\ell+1} \right] \\
&\qquad = 4 \lim_{k\to\infty} \frac{1}{k} \sum_{\ell=1}^{k-1} \sqrt{\ell} \\
&\qquad \ge 4 \lim_{k\to\infty} \frac{1}{k} \int_1^k \sqrt{\ell-1}\, d\ell \ = \ \frac{8}{3} \lim_{k\to\infty} \frac{(k-1)^{3/2}}{k} \ = \ \infty.
\end{aligned}$$

Hence, the conditions of Theorem 3.2 are satisfied and Example 3.5 also illustrates Theorem 3.2(a).

Example 3.6. Consider the difference equation

$$|\Delta y(k)|^{\alpha-1} \Delta y(k) + y(k+2c) \cdot (b + 2^\alpha) = b\ y(k+2c),\ \ k \ge 0 \tag{3.37}$$

where $\alpha > 1$, c is any fixed positive integer, and $b = b(k, y(k+2c))$ is any function. Here, $m = 1$. Take $k - \sigma_k \equiv (k+2c)$ and $f(y) = y$. Then, it is obvious that (3.5) and (3.6) hold. Further, we have

$$\frac{Q(k, y(k+2c))}{f(y(k+2c))} = b + 2^{\alpha} \equiv q(k)$$

and

$$\frac{P(k, y(k+2c))}{f(y(k+2c))} = b \equiv p(k).$$

Thus,

$$\sum^{\infty} \left\{ f(k^{(m-1)})[q(k) - p(k)] \right\}^{1/\alpha} = \sum^{\infty} 2 = \infty$$

and (3.7) is fulfilled. It follows from Theorem 3.3(a) that all solutions of (3.37) are oscillatory. One such solution is given by $\{y(k)\} = \{(-1)^k\}$.

Example 3.7. Consider the difference equation

$$(3.38)\quad |\Delta^3 y(k)|^{\alpha-1}\Delta^3 y(k) + [y(k+2)]^{\beta}\left[b + \frac{4^{\alpha}(2k+3)^{\alpha}}{(k+2)^{\beta}}\right] = b[y(k+2)]^{\beta},\ k \geq 0$$

where $\alpha > 0$, β is any odd integer satisfying $|\beta| > \alpha$, and $b = b(k, y(k+2), \Delta y(k+2))$ is any function. We have $k - \sigma_k \equiv (k+2)$, $f(y) = y^{\beta}$, and

$$\frac{Q(k, y(k+2), \Delta y(k+2))}{f(y(k+2))} = b + \frac{4^{\alpha}(2k+3)^{\alpha}}{(k+2)^{\beta}} \equiv q(k),$$

$$\frac{P(k, y(k+2), \Delta y(k+2), \Delta^2 y(k+2))}{f(y(k+2))} = b \equiv p(k).$$

<u>Case 1</u> $\beta > \alpha$

It is clear that (3.5) and (3.7) hold whereas (3.6) does not hold.

<u>Case 2</u> $\beta < -\alpha$

In this case (3.6) holds but (3.5) and (3.7) are not satisfied.

Hence, the conditions of Theorem 3.3 are violated if $|\beta| > \alpha$. In fact, (3.38) has a solution given by $\{y(k)\} = \{(-1)^k k\}$ and both $\{y(k)\}$ and $\{\Delta y(k)\}$ are oscillatory. This example illustrates Theorem 3.3(b).

Remark 3.6. Let $\beta > 2\alpha$ in Example 3.7. Condition (3.1) clearly holds. However,

$$\sum^{\infty} [q(k) - p(k)]^{1/\alpha} = \sum^{\infty} \frac{4(2k+3)}{(k+2)^{\beta/\alpha}} < \infty$$

and so (3.2) is not fulfilled. Hence, the conditions of Theorem 3.1 are violated.

Moreover, we see that (3.3) holds. Since

$$\frac{1}{k^{(m-1)}}\sum_{\ell=k_0}^{k-m}(k-\ell-1)^{(m-1)}[q(\ell)-p(\ell)]^{1/\alpha}$$

$$= \frac{1}{k^{(2)}}\sum_{\ell=0}^{k-3}(k-\ell-1)^{(2)}\ \frac{4(2\ell+3)}{(\ell+2)^{\beta/\alpha}}$$

$$\le 4\ \frac{(k-1)^{(2)}}{k^{(2)}}\ \sum_{\ell=0}^{k-3}\frac{4(2\ell+3)}{(\ell+2)^{\beta/\alpha}} \le 4\sum_{\ell=0}^{\infty}\frac{4(2\ell+3)}{(\ell+2)^{\beta/\alpha}} < \infty,$$

the condition (3.4) is not satisfied. Therefore, the conditions of Theorem 3.2 are violated.

Hence, when $\beta > 2\alpha$ Example 3.7 also illustrates both Theorems 3.1(b) and 3.2(b).

References

[1] Agarwal, R.P., *Difference Equations and Inequalities,* Marcel Dekker, New York, 1992.

[2] Agarwal, R.P., *Properties of solutions of higher order nonlinear difference equations I,* An. St. Univ. Iasi **29**(1983), 85-96.

[3] Agarwal, R.P., *Difference calculus with applications to difference equations,* in General Inequalities, ed. W. Walter, ISNM 71, Birkhaver Verlag, Basel (1984), 95-160.

[4] Agarwal, R.P., *Properties of solutions of higher order nonlinear difference equations II,* An. St. Univ. Iasi **29**(1985), 165-172.

[5] Grace, S.R., Lalli, B.S., *Oscillation theorems for nth order delay equations,* J. Math. Anal. Appl. **91**(1983), 342-366.

[6] Grace, S.R., Lalli, B. S., *Oscillation theorems for damped differential equations of even order with deviating arguments,* SIAM J. Math. Anal. **15**(1984), 308-316.

[7] Hooker. J.W., Patula, W.T., *A second order nonlinear difference equation: oscillation and asymptotic behavior,* J. Math. Anal. Appl. **91**(1983), 9-29.

[8] Lakshmikantham, V., Trigiante, D., *Difference Equations with Applications to Numerical Analysis,* Academic Press, New York, 1988.

[9] Popenda, J. *Oscillation and nonoscillation theorems for second order difference equations,* J. Math. Anal. Appl. **123**(1987), 34-38.

[10] Staikos, V.A., *Basic results on oscillation for differential equations,* Hiroshima Math. J. **10**(1980), 495-516.

[11] Thandapani, E., *Oscillatory behavior of solutions of second order nonlinear difference equations,* J. Math. Phys. Sci. **25**(1991), 451-464.

[12] Thandapani, E., Sundaram, P., *Oscillation theorems for some even order nonlinear difference equations,* preprint.

[13] Wong, P. J. Y., Agarwal, R. P., *Oscillation theorems and existence of positive monotone solutions for second order nonlinear difference equations,* Math. Comp. Modelling **21**(1995), 63-84.

[14] Wong, P. J. Y., Agarwal, R. P., *Oscillation theorems and existence criteria of asymptotically monotone solutions for second order differential equations,* Dynamic Systems and Applications, to appear.

[15] Wong, P. J. Y., Agarwal, R. P., *Oscillatory behavior of solutions of certain second order nonlinear differential equations,* J. Math. Anal. Applic., **198** (1996), 337-354.

[16] Wong, P. J. Y., Agarwal, R. P., *Oscillation theorems for certain second order nonlinear difference equations,* to appear.

[17] Wong, P. J. Y., Agarwal, R. P., *Oscillation and monotone solutions of a second order quasilinear difference equation,* to appear.

[18] Wong, P. J. Y., Agarwal, R. P., *On the oscillation and asymptotically monotone solutions of second order quasilinear differential equations,* Appl. Math. Comp., to appear.

[19] Wong, P. J. Y., Agarwal, R. P., *Oscillations and nonoscillations of half-linear difference equations generated by deviating arguments, Advances in Difference Equations II,* Computers Math. Applic., to appear.

[20] Wong, Z., Yu, J., *Oscillation criteria for second order nonlinear difference equations,* Funk. Ekv. **34**(1991), 313-319.

Patricia J. Y. Wong
Division of Mathematics
Nanyang Technological University
Singapore

Ravi P. Agarwal
Department of Mathematics
National University of Singapore
Singapore