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OSCILLATION OF LINEAR FUNCTIONAL EQUATIONS OF HIGHER ORDER

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Abstract. The paper contains sufficient conditions under which all solutions of linear functional equations of the higher order are oscillatory.

1. Introduction

Let $\Re$ be the set of real numbers and let I denote an unbounded subset of $\Re_+ = [0, \infty)$. By g^m we mean the m-th iterate of the function $g : I \to I$, i.e.

$$g^0(t) = t, \qquad g^{m+1}(t) = g(g^m(t)), \qquad t \in I, \quad m = 0, 1, ...$$

In the whole of this paper upper indices at the sign of a function will denote iterations. In each instance we have the relation $g^1(t) = g(t)$. Exponents of a power of a function will be written after a bracket containing the whole expression for the function.

We consider the oscillatory behavior of solutions of functional equations of the form

$$\text{(E)} \qquad Q_0(t)x(t) + Q_1(t)x(g(t)) + Q_2(t)x(g^2(t)) + ... + Q_{m+1}(t)x(g^{m+1}(t)) = 0,$$

where $Q_k : I \to \Re$ for $k = 0, 1, ..., m+1, m \geq 1$, and $g : I \to I$ are given functions and x is an unknown real valued function. We also assume that

$$(1) \qquad g(t) \neq t \qquad \text{and} \qquad lim_{t\to\infty} g(t) = \infty, \qquad t \in I.$$

By a solution of equation (E) we mean a function $x : I \to \Re$ such that $\sup\{|x(s)| : s \in I_{t_0} = [t_0, \infty) \cap I\} > 0$ for any $t_0 \in \Re_+$ and x satisfies (E) on I.

A solution x of equation (E) is called oscillatory if there exists a sequence of points $\{t_n\}_{n=1}^{\infty}, t_n \in I$, such that $lim_{n\to\infty} t_n = \infty$ and $x(t_n)x(t_{n+1}) \leq 0$ for $n = 1, 2, ...$ Otherwise it is called nonoscillatory.

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In contrast with the extensive development of the oscillation theory of differential and difference equations (for example see [2], [4] and the references contained therein), the authors are of the opinion that at this time in the literature there are no known oscillation criteria for functional equations. The purpose of this paper is to obtain sufficient conditions under which all solutions of (E) are oscillatory.

First let us observe that existence of oscillatory solutions of equation (E) is connected with the sign of the functions Q_i $(i = 0, 1, ..., m+1)$ on I. For example, it is easy to prove that either $Q_i(t) > 0$ or $Q_i(t) < 0$ for $i = 0, 1, ..., m+1$, $t \in I$, implies that equation (E) possesses only oscillatory solutions. If one of the coefficients Q_i has an opposite sign then others, i.e. if there exists $j \in \{0, 1, ..., m+1\}$ such that $Q_j(t) < 0$ and $Q_i(t) > 0$, $i \in \{0, 1, ...m+1\} - \{j\}$ then equation (E) can possess both oscillatory and nonoscillatory solutions. For example, the functional equation

$$3x(t) - 5x(t+\pi) + x(t+2\pi) + x(t+3\pi) = 0, \qquad t \in [0, \infty)$$

has an oscillatory solution $x = cos2t$ and a nonoscillatory solution $x = t+1$. So, a question arises: if the last case holds, under what additional conditions on the coefficients Q_i every solution of (E) will be oscillatory. We present some answers to this question in case

$$Q_1(t) < 0 \quad \text{and} \quad Q_i(t) > 0 \quad (i = 0, 2, 3, ..., m+1) \quad \text{for} \quad t \in I.$$

Without loss of generality we may assume that $Q_1(t) = -1, \quad t \in I$.
Then equation (E) takes the form

$$\text{(L)} \quad x(g(t)) = Q_0(t)x(t) + Q_2(t)x(g^2(t)) + ... + Q_{m+1}(t)x(g^{m+1}(t)), \qquad m \geq 1.$$

In the proofs of our theorems the following lemmas will be useful.

Lemma 1. *Consider the functional inequality*

$$(2) \qquad x(g(t)) \geq P(t)x(t) + Q(t)x(g^{k+1}(t)), \qquad k \geq 1,$$

where $P, Q : I \to \Re_+$ and g satisfies condition (1). If

$$(3) \qquad \liminf_{I \ni t \to \infty} \sum_{i=0}^{k-1} Q(g^i(t)) \prod_{j=1}^{k} P(g^{i+j}(t)) > \left(\frac{k}{k+1}\right)^{k+1}$$

then the functional inequality (2) has not positive solutions for large $t \in I$.

Proof. Suppose that x is a nonoscillatory positive solution of (2) and let $x(t) > 0$ for $t \in I_{t_1}$, $t_1 > 0$. Then also, in view of (1), there exists a point $t_2 \in I_{t_1}$ such that $x(g^i(t)) > 0$ for $t \in I_{t_2}$ and $i \in \{1, 2, ..., k+1\}$. Therefore from (2) we have for $t \in I_{t_2}$

$$x(g(t)) \geq P(t)x(t)$$

which gives for $i \in \{1, 2, ..., k+1\}$

$$x(g^i(t)) \geq x(t) \prod_{j=0}^{i-1} P(g^j(t)). \tag{4}$$

Using now (4) in (2) one gets

$$x(g(t)) \geq P(t)x(t) + x(g(t))Q(t) \prod_{j=1}^{k} P(g^j(t)) \tag{5}$$

and

$$Q(t) \prod_{j=1}^{k} P(g^j(t)) \leq 1 - P(t)\frac{x(t)}{x(g(t))}.$$

By iteration for $i = 0, 1, ..., k-1$ we have

$$Q(g^i(t)) \prod_{j=1}^{k} P(g^{i+j}(t)) \leq 1 - P(g^i(t))\frac{x\left(g^i(t)\right)}{x\left(g^{i+1}(t)\right)}.$$

Summing now both sides of the above inequality from $i = 0$ to $i = k-1$ we obtain

$$\sum_{i=0}^{k-1} Q(g^i(t)) \prod_{j=1}^{k} P(g^{i+j}(t)) \leq k - \sum_{i=0}^{k-1} P(g^i(t))\frac{x\left(g^i(t)\right)}{x\left(g^{i+1}(t)\right)}.$$

Since

$$\begin{aligned}\frac{1}{k}\sum_{i=0}^{k-1} P(g^i(t))\frac{x\left(g^i(t)\right)}{x\left(g^{i+1}(t)\right)} &\geq \left\{\prod_{i=0}^{k-1} P(g^i(t))\frac{x\left(g^i(t)\right)}{x\left(g^{i+1}(t)\right)}\right\}^{\frac{1}{k}} \\ &= \left\{\frac{x(t)}{x\left(g^k(t)\right)}\prod_{i=0}^{k-1} P(g^i(t))\right\}^{\frac{1}{k}}\end{aligned}$$

therefore

$$\sum_{i=0}^{k-1} Q(g^i(t)) \prod_{j=1}^{k} P(g^{i+j}(t)) \leq k\left\{1 - \left[\frac{x(t)}{x(g^k(t))}\prod_{i=0}^{k-1} P(g^i(t))\right]^{\frac{1}{k}}\right\}. \tag{6}$$

Let

$$\left(\frac{k}{k+1}\right)^{k+1} = A < \liminf_{I \ni t \to \infty} \sum_{i=0}^{k-1} Q(g^i(t)) \prod_{j=1}^{k} P(g^{i+j}(t)).$$

Then there exist a constant $B \in (A, 1)$ and a point $t_3 \in I_{t_2}$ such that

$$A < B \leq \sum_{i=0}^{k-1} Q(g^i(t)) \prod_{j=1}^{k} P(g^{i+j}(t)) \qquad \text{for} \quad t \in I_{t_3}. \tag{7}$$

Choose now the least natural number M such that

$$\left(\frac{B}{A}\right)^M > \frac{k}{B} \tag{8}$$

which is possible because of $B > A$. From (6) we have

$$\frac{B}{k} \leq 1 - \left[\frac{x(t)}{x(g^k(t))} \prod_{i=0}^{k-1} P(g^i(t))\right]^{\frac{1}{k}}.$$

Thus

$$\begin{aligned} \frac{x(t)}{x(g^k(t))} \prod_{i=0}^{k-1} P(g^i(t)) \leq \left(1 - \frac{B}{k}\right)^k &\leq \frac{1}{B} \max_{A<B<1} B \left(1 - \frac{B}{k}\right)^k \\ &= \frac{1}{B} \left(\frac{k}{k+1}\right)^{k+1} = \frac{A}{B}. \end{aligned}$$

Hence we have

$$x(g^k(t)) \geq \frac{B}{A} x(t) \prod_{i=0}^{k-1} P(g^i(t))$$

and

$$x(g^{k+1}(t)) \geq \frac{B}{A} x(g(t)) \prod_{j=1}^{k} P(g^j(t)).$$

Repeating we have

$$x(g^{k+1}(t)) \geq \left(\frac{B}{A}\right)^2 x(g(t)) \prod_{j=1}^{k} P(g^j(t)).$$

Similarly we get

$$x(g^{k+1}(t)) \geq \left(\frac{B}{A}\right)^M x(g(t)) \prod_{j=1}^{k} P(g^j(t)), \tag{9}$$

where M is the same as in (8). From (2) and (9) we have

$$\begin{aligned} x(g(t)) &\geq P(t)x(t) + Q(t) \left(\frac{B}{A}\right)^M x(g(t)) \prod_{j=1}^{k} P(g^j(t)) \\ &\geq Q(t) \left(\frac{B}{A}\right)^M x(g(t)) \prod_{j=1}^{k} P(g^j(t)). \end{aligned}$$

Thus

$$1 \geq \left(\frac{B}{A}\right)^{M} Q(t) \prod_{j=1}^{k} P(g^{j}(t)).$$

Therefore we have for $i \in \{0, 1, ..., k-1\}$

$$1 \geq \left(\frac{B}{A}\right)^{M} Q(g^{i}(t)) \prod_{j=1}^{k} P(g^{i+j}(t)).$$

Summing now from $i = 0$ to $i = k-1$ we get

$$k \geq \left(\frac{B}{A}\right)^{M} \sum_{i=0}^{k-1} Q(g^{i}(t)) \prod_{j=1}^{k} P(g^{i+j}(t))$$

and by (7)

$$k \geq \left(\frac{B}{A}\right)^{M} B.$$

But this contradicts (8). Thus the proof of the lemma is complete. □

A slight modification in the proof of Lemma 1 leads to the following result

Lemma 2. *If*

$$\liminf_{I \ni t \to \infty} \sum_{i=0}^{k-1} Q(g^{i}(t)) \prod_{j=1}^{k} P(g^{i+j}(t)) > \left(\frac{k}{k+1}\right)^{k+1},$$

then the functional inequality

$$x(g(t)) \leq P(t)x(t) + Q(t)x(g^{k+1}(t)), \qquad k \geq 1, \tag{10}$$

where $P, Q : I \to \Re_{+}$ and g satisfies (1) has not negative solutions for large $t \in I$.

Corollary 1. *If (3) holds, then the functional equation*

$$x(g(t)) = P(t)x(t) + Q(t)x(g^{k+1}(t)), \qquad k \geq 1, \tag{S}$$

where P, Q, g are the same as in Lemmas 1 and 2, has only oscillatory solutions.

Remark 1. From Corollary 1 in the case $k = 1$ we get Theorem 1 of [1].

2. Main results

In this section we study sufficient conditions for the oscillation of all solutions of equation (L). Further we consider equation (L) for $m > 1$ because for $m = 1$ equation (L) resolve itself into equation (S). As usually we take $\prod_{j=k}^{k-1} a_j = 1$. Moreover, for convenience, we will assume that inequalities about values of functions are satisfied for all large $t \in I$.

We give now two independent conditions for oscillation of all solutions of equation (L).

Theorem 1. *Let*

$$\liminf_{I\ni t\to\infty} \sum_{k=2}^{m+1} Q_k(t) \prod_{j=1}^{k-1} Q_0(g^j(t)) > \frac{1}{4}. \tag{11}$$

Then every solution of equation (L) is oscillatory.

Proof. Suppose that (L) has a nonoscillatory solution x and let $x(t) > 0$. Then also, in view of assumption (1) about function g, $x(g^i(t)) > 0$, $i \in \{1, 2, ..., m+1\}$. Thus from equation (L) we get

$$x(g(t)) \geq Q_0(t)x(t)$$

which gives for $k = 3, 4, ..., m+1$

$$x(g^k(t)) \geq x(g^2(t)) \prod_{j=2}^{k-1} Q_0(g^j(t)). \tag{12}$$

Using now (12) in equation (L) we obtain

$$x(g(t)) \geq Q_0(t)x(t) + x(g^2(t)) \sum_{k=2}^{m+1} Q_k(t) \prod_{j=2}^{k-1} Q_0(g^j(t)).$$

In view of Lemma 1 and (11) the last inequality has not positive solutions, which contradicts the fact that $x(t) > 0$ for sufficiently large $t \in I$.
When $x(t) < 0$ the proof is similar but we apply Lemma 2 in it. Thus the proof is complete. $\square$

We give now another oscillation criterion for equation (L).

Theorem 2. *If*

$$\liminf_{I\ni t\to\infty} \sum_{i=0}^{m-1} G(g^i(t)) \prod_{j=1}^{m} Q_0(g^{i+j}(t)) > \left(\frac{m}{m+1}\right)^{m+1}, \tag{13}$$

where

$$G(t) = \sum_{k=2}^{m} Q_k(t) Q_{m-k+2}(g^{k-1}(t)) + Q_{m+1}(t),$$

then all solutions of equation (L) oscillate.

Proof. Assume that x is an eventually positive solution of equation (L). Then, as in proof of Theorem 1, the following inequality is true

$$x(g^k(t)) > 0 \qquad \text{for} \qquad k \in \{1, 2, ..., m+1\}.$$

Thus from equation (L) we obtain

$$x(g(t)) \geq Q_{m-k+2}(t)x(g^{m-k+2}(t)) \qquad \text{for} \qquad k = 2, ..., m,$$

which gives

$$x(g^k(t)) \geq x(g^{m+1}(t))Q_{m-k+2}(g^{k-1}(t)) \qquad \text{for} \qquad k = 2, ..., m. \tag{14}$$

¿From equation (L) and inequality (14) we have

$$x(g(t)) \geq Q_0(t)x(t) + x(g^{m+1}(t)) \left\{ \sum_{k=2}^{m} Q_k(t)Q_{m-k+2}(g^{k-1}(t)) + Q_{m+1}(t) \right\} =$$
$$= Q_0(t)x(t) + G(t)x(g^{m+1}(t)).$$

Using now Lemma 1 for the above inequality we get a contradiction with the fact that x is a positive solution of (L). Similarly for $x(t) < 0$ in view of Lemma 2 we get a contradiction. Thus the proof is complete. □

Remark 2. One can observe that conditions (11) and (13) for oscillation are independent. For example, the following third order functional equation

$$5x(t) - 10tx(t+1) + 5t(t+1)x(t+2) + [t]^3x(t+3) = 0, \qquad t \in \Re_+,$$

has only oscillatory solutions, since condition (11) of Theorem 1 is fulfilled. However, condition (13) of Theorem 2 is not satisfied. Consider now the functional equation

$$5x(t) - 10tx(t+1) + t(t+1)x(t+2) + 6[t]^3x(t+3) = 0, \qquad t \in \Re_+.$$

Then condition (13) is fulfilled. In this case condition (11) is not satisfied.

3. Applications

In this section we show an application of the main results of this paper to recurrence equations. Consider a recurrence equation of the form

$$x(n+1) = a_0(n)x(n) + a_2(n)x(n+2) + ... + a_{m+1}(n)x(n+m+1), \tag{RE}$$

where $n \in N = \{1, 2, ...\}$, $m \geq 1$ is a natural number, $a_k : N \to \Re_+$ for $k = 0, 2, ..., m+1$. Apply now Theorems 1 and 2 to equation (RE) to obtain the following results

Corollary 2. *If*

$$\liminf_{n\to\infty} \sum_{k=2}^{m+1} a_k(n) \prod_{j=1}^{k-1} a_0(n+j) > \frac{1}{4},$$

then every solution of equation (RE) is oscillatory.

Corollary 3. *Let*

$$\liminf_{n\to\infty} \sum_{i=0}^{m-1} \prod_{j=1}^{m} a_0(n+i+j) \left\{ \sum_{k=2}^{m} a_k(n+i) a_{m-k+2}(n+k-1+i) + a_{m+1}(n+i) \right\} >$$

$$> \left(\frac{m}{m+1}\right)^{m+1}. \tag{15}$$

Then every solution of equation (RE) oscillates.

Remark 3. If in equation (RE) we take $a_0(n) = 1$, $a_k(n) = 0$ for $k = 2, 3, ..., m$, then from condition (15) we get a similar result as in Theorem 1 of [3].

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