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NATURAL TRANSFORMATIONS OF SEMI–HOLONOMIC 3–JETS

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Abstract. Let $\bar{J}^3$ be the functor of semi-holonomic 3-jets and $\bar{J}^{3,2}$ be the functor of those semi-holonomic 3-jets, which are holonomic in the second order. We deduce that the only natural transformations $\bar{J}^3 \to \bar{J}^3$ are the identity and the contraction. Then we determine explicitely all natural transformations $\bar{J}^{3,2} \to \bar{J}^{3,2}$, which form two 5-parameter families.

Applying the point of view of the category theory, we can interpret some differential geometric operations as natural transformations of the geometric functors in question, [3]. We are going to discuss the semi-holonomic 3-jets, [1], from such a point of view. Let $\mathcal{M}f_m$ be the category of m-dimensional manifolds and local diffeomorphisms and $\mathcal{M}f$ be the category of all manifolds and all smooth maps, [3]. The construction of the space $\bar{J}^3(M,N)$ of semi-holonomic 3-jets from an m-dimensional manifold M into a manifold N is a functor on the product category $\mathcal{M}f_m \times \mathcal{M}f$. For every local diffeomorphism $f : M \to \bar{M}$ and every smooth map $g : N \to \bar{N}$ we define $\bar{J}^3(f,g) : \bar{J}^3(M,N) \to \bar{J}^3(\bar{M},\bar{N})$ by

$$\bar{J}^3(f,g)(X) = (j^3_y g) \circ X \circ (j^3_x f)^{-1} \tag{1}$$

where $x = \alpha X$ or $y = \beta X$ is the source or the target of $X \in \bar{J}^3(M,N)$, respectively.

In [4] it is deduces for the functor $\bar{J}^2$ of the semi-holonomic 2-jets that all natural transformations $\bar{J}^2 \to \bar{J}^2$ form two one-parameter families, which can be constructed by means of the canonical involution $\bar{J}^2 \to \bar{J}^2$ by J. Pradines, [6], or by means of the difference tensor by I. Kolář, [2]. But in the third order we have a different situation. We recall that the contraction of $\bar{J}^3(M,N)$ means the map

$$X \mapsto j^3_{\alpha X}\widehat{\beta X}$$

where $\widehat{\beta X}$ denotes the constant map of M into $\beta X \in N$.

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Proposition 1. *The only transformations $\bar{J}^3 \to \bar{J}^3$ are the identity and the contraction.*

Proof. Consider first the subcategory $\mathcal{M}f_m \times \mathcal{M}f_n \subset \mathcal{M}f_m \times \mathcal{M}f$. The standard fiber $S = \bar{J}^3_\circ(\mathbb{R}^m, \mathbb{R}^n)_\circ$ is a $G^3_m \times G^3_n$-space, [4]. By (1), the action of $(A, B) \in G^3_m \times G^3_n$ on $X \in S$ is given by the jet composition

$$\overline{X} = B \circ X \circ A^{-1} \tag{2}$$

By [3], the natural transformations $\bar{J}^3 \to \bar{J}^3$ are in bijection with $G^3_m \times G^3_n$-equivariant maps $F : S \to S$.

Write

$$\begin{aligned} A^{-1} &= (a^i_j,\, a^i_{jk},\, a^i_{jkl}) \qquad i, j, k, l, \ldots = 1, \ldots, m \\ B &= (b^p_q,\, b^p_{qr},\, b^p_{qrs}) \qquad p, q, r, s, \cdots = 1, \ldots, n \end{aligned}$$

where the second and third order terms are symmetric in all subscripts, and

$$\begin{aligned} X &= (x^p_i,\, x^p_{ij},\, x^p_{ijk})\,, \\ \overline{X} &= (\bar{x}^p_i,\, \bar{x}^p_{ij},\, \bar{x}^p_{ijk})\,. \end{aligned}$$

Evaluating (2), we find

$$\bar{x}^p_i = b^p_q\, x^q_j\, a^j_i \tag{3}$$

$$\bar{x}^p_{ij} = b^p_{qr}\, x^q_k\, x^r_l\, a^k_i\, a^l_j + b^p_q\,(x^q_{kl}\, a^k_i\, a^l_j + x^q_k\, a^k_{ij}) \tag{4}$$

$$\begin{aligned} \bar{x}^p_{ijk} = {}& b^p_{qrs}\, x^q_l\, x^r_m\, x^s_n\, a^l_i\, a^m_j\, a^n_k + b^p_{qr}[(x^q_l\, x^r_{mn} + x^q_{ln}\, x^r_m \\ &+ x^q_{lm}\, x^r_n)\, a^l_i\, a^m_j\, a^n_k + x^q_l\, x^r_m (a^l_i\, a^m_{jk} + a^l_{ik}\, a^m_j + a^l_{ij}\, a^m_k)] \\ &+ b^p_q[x^q_{lmn}\, a^l_i\, a^m_j\, a^n_k + x^q_{lm}\,(a^l_i\, a^m_{jk} + a^l_{ik}\, a^m_j + a^l_{ij}\, a^m_k) \\ &+ x^q_l\, a^l_{ijk}]\,. \end{aligned} \tag{5}$$

The map F is of the following form

$$\begin{aligned} \bar{x}^p_i &= F^p_i(x^p_i,\, x^p_{ij},\, x^p_{ijk}) = F^p_i\ (x_1,\, x_2,\, x_3)\,, \\ \bar{x}^p_{ij} &= F^p_{ij}(x^p_i,\, x^p_{ij},\, x^p_{ijk}) = F^p_{ij}\,(x_1,\, x_2,\, x_3)\,, \\ \bar{x}^p_{ijk} &= F^p_{ijk}(x^p_i,\, x^p_{ij},\, x^p_{ijk}) = F^p_{ijk}(x_1,\, x_2,\, x_3)\,. \end{aligned}$$

The equivariance condition for F^p_i reads

$$\begin{aligned} b^p_q\, F^q_j(x_1,\, x_2,\, x_3)\, a^i_j = {}& F^p_i\big(b^p_q\, x^q_j\, a^j_i\,,\, b^p_{qr}\, x^q_k\, x^r_l\, a^k_i\, a^l_j \\ &+ b^p_q(x^q_{kl}\, a^k_i\, a^l_j + x^q_k\, a^k_{ij}),\, b^p_{qrs}\, x^q_l\, x^r_m\, x^s_n\, a^l_i\, a^m_j\, a^n_k \\ &+ b^p_{qr}[(x^q_l\, x^r_{mk} + x^q_{ln}\, x^r_m + x^q_{lm}\, x^r_n)\, a^l_i\, a^m_j\, a^n_k + x^q_l\, x^r_m \\ &(a^l_i\, a^m_{jk} + a^l_{ik}\, a^m_j + a^l_{ij}\, a^m_k)] + b^p_q[x^q_{lmn}\, a^l_i\, a^m_j\, a^n_k \\ &+ x^q_{lm}(a^l_i\, a^m_{jk} + a^l_{ik}\, a^m_j + a^l_{ij}\, a^m_k) + x^q_l\, a^l_{ijk}]\big)\,. \end{aligned} \tag{6}$$

Similar conditions hold for F^p_{ij} and F^p_{ijk} as well.

We shall heavily use the homogeneous function theorem, [3], p.213. Taking into account the canonical injection $G^1_n \subset G^3_n$, the equivariance of F^p_i with respect to the homotheties in G^1_n yields

$$kF^p_i(x_1,\, x_2,\, x_3) = F^p_i(kx_1,\, k^2x_2,\, k^3x_3)\,.$$

By the homogeneous function theorem, F^p_i is linear in x_1 and independent of x_2, x_3. Using the homotheties in G^1_m and G^1_n, we deduce for F^p_{ij}

$$\begin{aligned} k^2\, F^p_{ij}(x_1,\, x_2,\, x_3) &= F^p_{ij}(kx_1,\, k^2x_2,\, k^3x_3)\\ k\quad F^p_{ij}(x_1,\, x_2,\, x_3) &= F^p_{ij}(kx_1,\, kx_2,\, kx_3)\,. \end{aligned}$$

Hence F^p_{ij} is linear in x_2 and independent of x_1, x_3.
If we apply both homotheties to F^p_{ijk}, we obtain

$$\begin{aligned} k^3F^p_{ijk}(x_1,\, x_2,\, x_3) &= F^p_{ijk}(kx_1,\, k^2x_2,\, k^3x_3)\,,\\ k\quad F^p_{ijk}(x_1,\, x_2,\, x_3) &= F^p_{ijk}(kx_1,\, kx_2, kx_3)\,. \end{aligned}$$

Hence F^p_{ijk} is linear in x_3 and independent of x_1, x_2.

Taking into account the generalized invariant tensor theorem, [3], p. 230, the equivariancy with respect to canonical injection of $G^1_m \times G^1_n$ into $G^3_m \times G^3_n$ yields

(7) $$\bar{x}^p_i = kx^p_i \qquad k \in R$$

(8) $$\bar{x}^p_{ij} = a\, x^p_{ij} + b\, x^p_{ji} \qquad a, b \in R$$

(9) $$\bar{x}^p_{ijk} = c\, x^p_{ijk} + d\, x^p_{jik} + e\, x^p_{jki} + f\, x^p_{kji} + g\, x^p_{ikj} + h\, x^p_{kij} \qquad c, d, e, f, g, h \in R\,.$$

Next we shall discuss the kernel of the jet projection $\pi^3_1 : G^3_m \times G^3_n \to G^1_m \times G^1_n$. In the second order we obtain the following two possibilities from [4]

I. $\quad k = 0\,, \qquad a + b = 0$

II. $\quad k = 1\,, \qquad a + b = 1$

This leads to the two cases, [4],

I. $\quad \bar{x}^p_i = 0\,, \qquad \bar{x}^p_{ij} = k(x^p_{ij} - x^p_{ji}) \qquad k \in R$

II. $\quad \bar{x}^p_i = x^p_i\,, \qquad \bar{x}^p_{ij} = tx^p_{ij} + (1-t)x^p_{ji} \qquad t \in R$

In the third order, we have

(10)
$$\begin{aligned} &F^p_{ijk} + b^p_{qrs}\, F^q_i\, F^r_j\, F^s_k + b^p_{qr}\, (F^q_i\, F^r_{jk} + F^q_{ik}\, F^r_j + F^q_{ij}\, F^r_k)\\ &+ b^p_{qr}\, F^q_l\, F^r_m\, (\delta^l_i\, a^m_{jk} + a^l_{ik}\, \delta^m_j + a^l_{ij}\, \delta^m_k) + F^p_{lm}\, (\delta^l_i\, a^m_{jk} + a^l_{ik}\, \delta^m_j\\ &+ a^l_{ij}\, \delta^m_k) + F^p_l\, a^l_{ijk} = c\, \bar{x}^p_{ijk} + d\, \bar{x}^p_{jik} + e\, \bar{x}^p_{jki} + f\, \bar{x}^p_{kji} + g\, \bar{x}^p_{ikj} + h\, \bar{x}^p_{kij} \end{aligned}$$

and

$$\bar{x}^p_{ijk} = b^p_{qrs}\, x^q_i\, x^r_j\, x^s_k + b^p_{qr}(x^q_i\, x^r_{jk} + x^q_{ik}\, x^r_j + x^q_{ij}\, x^r_k) \\ + x^p_{ijk} + x^p_{lm}(\delta^l_i\, a^m_{jk} + a^l_{ik}\, \delta^m_j + a^l_{ij}\, \delta^m_k) + x^p_l\, a^l_{ijk}\,. \tag{11}$$

If we put $b^p_{qr} = 0$, $a^l_{ijk} = 0$, $b^p_{qrs} = 0$, we deduce from (8), (10), (11)

$$b = 0, \quad a = c + g, \quad d + c = 0, \quad f + h = 0\,. \tag{12}$$

In the second order, we have deduced, [4],

$$k = a + b\,. \tag{13}$$

Consider first the case $k = 0$. By (13) we find $a = 0$, so that we have $F^p_i = F^p_{ij} = 0$. From (8), (10), (11) we further deduce

$$\begin{aligned} c + d + e &= 0 \qquad & f + g + h &= 0 \\ c + d + g &= 0 & e + f + h &= 0 \\ c + g + h &= 0 & d + e + f &= 0 \end{aligned} \tag{14}$$

The only solution of (12) - (14) is $a = b = c = d = e = f = g = h = 0$. Hence $F^p_{ijk} = 0$ as well. This is the contraction.

In the case $k = 1$ we have $a = 1$, so that $F^p_i = x^p_i$, $F^p_{ij} = x^p_{ij}$. Analogously as above we deduce

$$\begin{aligned} c + d + e &= 1 \qquad & f + g + h &= 0 \\ c + d + g &= 1 & e + f + h &= 0 \\ c + g + h &= 1 & d + e + f &= 0 \end{aligned} \tag{15}$$

The only solution is $c = 1$, $d = e = f = g = h = 0$. Hence $F^p_{ijk} = x^p_{ijk}$, which is the identity.

Finally, the case of the whole category $\mathcal{M}f_m \times \mathcal{M}f$ is reduced to $\mathcal{M}f_m \times \mathcal{M}f_n$ in the same way as in the proof of Proposition 1 in [4]. □

There is a more rich structure of natural transformations in the case of the subspace $\bar{J}^{3,2}(M,N) \subset \bar{J}^3(M,N)$ characterized by the property that the underlying 2-jet is holonomic. Even $\bar{J}^{3,2}$ is a bundle functor on the category $\mathcal{M}f_m \times \mathcal{M}f$. By [5], in the second order we have

$$F^p_i = k\, x^p_i\,, \quad F^p_{ij} = k\, x^p_{ij} \tag{16}$$

with two possibilities $k = 0$ and $k = 1$. In the first case, we deduce similarly as above

$$\begin{aligned}F^p_i &= 0, \quad F^p_{ij} = 0\,,\\ F^p_{ijk} &= d(x^p_{jik} - x^p_{ijk}) + e(x^p_{jki} - x^p_{ijk}) + f(x^p_{kji} - x^p_{ijk})\\ &\quad + g(x^p_{ikj} - x^p_{ijk}) + h(x^p_{kij} - x^p_{ijk})\end{aligned} \tag{17}$$

with arbitrary real parameters d, e, f, g, h. In the second case, we obtain in the same way

$$\begin{aligned}F^p_i &= x^p_i, \quad F^p_{ij} = x^p_{ij}\\ F^p_{ijk} &= x^p_{ijk} + d(x^p_{jik} - x^p_{ijk}) + e(x^p_{jki} - x^p_{ijk}) + f(x^p_{kji} - x^p_{ijk})\\ &\quad + g(x^p_{ikj} - x^p_{ijk}) + h(x^p_{kij} - x^p_{ijk})\end{aligned} \tag{18}$$

Thus, we can summarize by

Proposition 2. *All natural transformations $\bar J^{3,2} \to \bar J^{3,2}$ form the two 5-parameter families (17) and (18).*

We are going to characterize this result geometrically. We recall that for every $X \in \bar J^{3,2}(M,N)$ there exists a unique $sX \in J^3(M,N)$ such that X and sX are equivalent with respect to curves, [2]. The coordinate form of sX is $(x^p_i, x^p_{ij}, x^p_{(ijk)})$, where the round bracket denotes symmetrization. Since $\bar J^{3,2}(M,N)$ is an affine bundle over $J^2(M,N)$ with the associated vector bundle $TN \otimes \otimes^3 T^*M$, we have

$$\Delta X = X - sX \in TN \otimes \otimes^3 T^* M\,. \tag{19}$$

On the other hand we know that all natural transformations ν of $TN \otimes \otimes^3 T^*M$ into itself form a 6-parameter family, which is linearly generated by all permutations of the subscripts, [3]. Then are verifies directly that in the case $k = 1$ all natural transformations (18) are of the form

$$X \mapsto s(X) + \nu(\Delta X)\,. \tag{20}$$

Only 5 parameters are essential in (20).

In the case $k = 0$, we shall use the canonical injection $i : TN \otimes \otimes^3 T^* M \to \bar J^{3,2}(M,N)$, the coordinate form of which is $i(x^p_{ijk}) = (0, 0, x^p_{ijk})$. Clearly, all natural transformations (17) can be interpreted as

$$X \mapsto i(\nu(\Delta X))\,. \tag{21}$$

Even in (21) only 5 parameters are essential.

Remark. The results that the only natural transformations of $\bar J^3$ into itself as well as of J^r into itself, $r \geqslant 2$, [5], are the identity and the contraction suggest the conjecture that the same is true for every $\bar{J}^r$, $r \geqslant 3$. However, this is not correct. According to an oral communication by I. Kolář, all natural transformation of $\bar{J}^4$ into itself form two 3-parameter families. This result was deduced analytically by the methods of the present paper and both families were characterized geometrically in terms of the geometry of the fourth iterated tangent bundle $TTTTM$.

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