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CONDITIONS FOR THE ABSENCE OF POSITIVE SOLUTIONS OF A FIRST ORDER DIFFERENTIAL INEQUALITY WITH A SINGLE DELAY

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Abstract. A sufficient integral condition for the absence of eventually positive solutions of a first order stable type differential inequality with one nondecreasing retarded argument is given. In the special case of equality the result becomes an oscillation criterion.

1. Introduction

Let N denote the set of natural numbers $\{1, 2, \dots, \}$, $N_0 = \{0\} \cup N$, R the set of all real numbers, R_+ the set of all positive real numbers and $C[X, Y]$ the set of all continuous functions with domain X and range contained in Y.

It is well-known [1, p.16] that for $M, \tau \in C[R_+, R_+]$, $\tau(t) < t$ and $\lim\limits_{t\to\infty} \tau(t) = +\infty$ the inequality

$$x'(t) \le -M(t)x(\tau(t)) \tag{1$\le$}$$

has no eventually positive solution, if $\underline{\lim}_{t\to\infty} \int\limits_{\tau(t)}^{t} M(s)\, ds > \frac{1}{e}$. The development to this theorem is described in the notes [1, p.68]. This paper is concerned with the more general case $\underline{\lim}_{t\to\infty} \int\limits_{\tau(t)}^{t} M(s)\, ds \ge \frac{1}{e}$.

The first result in this direction has me reached in December 1989 in a letter of Á. Elbert [2, p.T813]. All solutions of

$$x'(t) = -M(t)x(t-1) \tag{2=}$$

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oscillate, if $M(t) = \frac{1}{e} + \chi(t)$, $\chi(t) > 0$ a nonincreasing function with $\int^{\infty} \chi(t) \mathrm{d}\, t = +\infty$.

In [2] this result is extended to the inequality of neutral type

$$(3\leq) \qquad x'(t) - cx'(t-\tau) \leq -M(t)x(t-1).$$

We mention only a specialization of the result in [2] to the equation (2=).

All solutions of (2=) oscillate, if $M(t) = \frac{1}{e} + \chi(t)$, $\chi(t) \geq 0$, $\int\limits_t^{t+1} \chi(\sigma)\ \mathrm{d}\sigma > 0$ nonincreasing and $\int\limits^{\infty} \chi(\sigma)\mathrm{d}\sigma = +\infty$.

In [3] it is shown that this statement remains true without the assumption $\int\limits_t^{t+1} \chi(\sigma)\ \mathrm{d}\sigma$ is a nonincreasing function.

Elbert and Stavroulakis present in [4] an interesting investigation for the equation (1=) with variable delay. An essential role plays a class A_λ of coefficient functions. But their results does not contain Theorem 1 in [3], and therefore in this paper we will extend Theorem 1 of [3] to the case of variable delay. Theorem 1 and Theorem 2 of [3] and Theorem 1 of [4] are special cases of the result in this paper.

2. Some lemmas

A solution of the inequality (1≤) on an interval I is an absolutely continuous function on I satisfying the inequality almost everywhere on I. Clearly on an initial set must be given an initial function. But the nature of the initial function is without meaning for our asymptotic investigation.

We assume that the function H is always locally summable without further mentioning.

Lemma 1. *Let x be a positive solution of (1≤) in the interval $I := [T, \infty)$, $\tau : I \to R$ a nondecreasing continuous function; $\tau(t) < t$, $t \in I$; $\lim\limits_{t\to\infty} \tau(t) = \infty$, $\tau_1(t) := \tau(t)$; $\tau_{n+1}(t) := \tau(\tau_n(t))$, $n \in N$; $T_0 := T$; $T_{k+1} := \min\{t; \tau(t) = T_k\}$, $k \in N_0$; $P \geq T_3$; $M(t) \geq H(t) \geq 0$, $t \in I$; $\int\limits_{\tau(t)}^{t} H(\sigma)\ d\sigma \geq \frac{1}{e}$ if $\tau(t) \geq T$.*

Then holds $G(P) := \ln \dfrac{x(\tau(P))}{x(P)} \leq 2(1 + \ln 2)$.

Proof. From (1≤) it follows that $x(t)$ is nonincreasing in $T_1 \leq t < \infty$. Denote Q the greatest point such that $\int\limits_Q^P H(\sigma)\ \mathrm{d}\sigma = \frac{1}{2e}$. Then $\tau(P) < Q < P$ and $x'(t) \leq -H(t)x(\tau(P))$, $Q \leq t \leq P$. Using $x(P) \geq 0$ and integrating the latter estimation for $x'(t)$ from Q to P we obtain $-x(Q) \le x(P) - x(Q) \le -x(\tau(P)) \int\limits_Q^P H(\sigma)\mathrm{d}\sigma =$ $-\frac{1}{2e} x(\tau(P))$ or $\frac{x(\tau(P))}{x(Q)} \le 2e$. Choose S such that $\int\limits_P^S H(\sigma)\,\mathrm{d}\sigma = \frac{1}{2e}$. We see $\int\limits_Q^S H(\sigma)\,\mathrm{d}\sigma = \frac{1}{e}$ and from the assumption $\int\limits_{\tau(S)}^S H(\sigma)\mathrm{d}\sigma \ge \frac{1}{e}$ and the definition of Q it follows $\tau(S) \le Q$ and $x'(t) \le -H(t)x(Q)$, $P \le t \le S$. How before we get

$-x(P) \le x(S) - x(P) \le -\frac{1}{2e} x(Q)$ or $\frac{x(Q)}{x(P)} \le 2e$.

Multiplying the two inequalities containing $2e$ on the right-hand side we obtain $\frac{x(\tau(P))}{x(P)} \le 4e^2$ or $G(P) \le 2(1 + \ln 2)$. Lemma 1 is proved.

Define a function $g : [T_2, \infty) \to R$ by $g(t) := \min\{G(s); \tau(t) \le s \le t\}$.

Lemma 2. *Under the assumptions of Lemma 1 g is nondecreasing on $[T_2, \infty)$.*

Proof. From (1$\le$) we conclude

$$\frac{x'(t)}{x(t)} \le -H(t)\frac{x(\tau(t))}{x(t)}, \quad T_1 \le t < \infty.$$

Integration from $\tau(t)$ to t yields

$$-G(t) \le -\int\limits_{\tau(t)}^t H(\sigma)e^{G(\sigma)}\mathrm{d}\sigma, \quad T_2 \le t < \infty.$$

Using the definition of g we see

$$G(t) \ge e^{g(t)} \int\limits_{\tau(t)}^t H(\sigma)\mathrm{d}\sigma, \quad T_2 \le t < \infty. \tag{4}$$

Assume that there exist two points t and u with $T_2 \le t < u < \infty$ and $g(t) > G(u)$. Choose $c \ne 1$, $g(t) > c > G(u)$. G is a continuous function. Put $S = \min\{s, G(s) = c, t \le s < \infty\}$. We have $t < S$ and $G(S) = g(S) = c$. However, due to (4) we would obtain $c = G(S) \ge e^{g(S)}\frac{1}{e} = e^c\frac{1}{e} > ec\frac{1}{e} = c$. This is impossible. Consequently $g(t) \le G(u)$, $T_2 \le t \le u < \infty$ and therefore $g(t) \le g(v)$, $T_2 \le t \le v < \infty$. Lemma 2 is proved.

Define $F : [T_2, \infty) \to R$ by $F(t) := \min\{s; G(s) = g(t), \tau(t) \le s \le t\}$.

Lemma 3. *Under the assumptions of Lemma 1 it holds that $F(t) < t$, $T_2 < t < \infty$.*

Proof. Assume $F(t) = t$ for a point t with $T_2 < t < \infty$. This implies $G(s) > g(t)$, $\tau(t) \le s < t$. Since G is continuous, there exists a $\delta > 0$ such that $t - \delta > T_2$ and $G(s) > g(t)$, $\tau(t) - \delta \le s < t$. Hence $g(t-\delta) > g(t)$ in contradiction to Lemma 2. Lemma 3 is proved.

Put $F_0(t) := t$, $F_1(t) := F(t)$ and $F_{n+1}(t) := F(F_n(t))$, if $F_n(t) \ge T_2$, $n \in N$.

Lemma 4. *Under the assumptions of Lemma 1 and $F_{2n-1}(t) > T_2$ it holds that $F_{2n}(t) < \tau_n(t)$, $n \in N$.*

Proof. $n = 1$. We have $F(t) > T_2$. If $F(t) = \tau(t)$, we conclude using Lemma 3 $F_2(t) = F(F(t)) < F(t) = \tau(t)$. If $F(t) > \tau(t)$, it follows $G(s) > g(t)$, $\tau(t) \le s < F(t)$. From Lemma 2 we have $g(F(t)) \le g(t)$ and from Lemm 3 $F_2(t) = F(F(t)) < F(t)$. Hence $G(F_2(t)) = G(F(F(t))) = g(F(t)) \le g(t) < G(s)$, $\tau(t) \le s < F(t)$. Therefore $F_2(t) < \tau(t)$. Lemma 4 is proved in case $n = 1$.
Let us now assume that the statement of Lemma 4 is true for the natural number $n = k$. $F_{2k+1}(t) > T_2$ is equivalent to $F_1(F_{2k}(t)) > T_2$. Using the case $n = 1$ we conclude $F_2(F_{2k}(t)) < \tau(F_{2k}(t))$. Clearly with $F_{2k+1}(t) > T_2$ it is also $F_{2k-1}(t) > T_2$. Our assumption for $n = k$ shows $F_{2k}(t) < \tau_k(t)$. τ is nondecreasing. Therefore it follows $\tau(F_{2k}(t)) \le \tau(\tau_k(t)) = \tau_{k+1}(t)$. Hence $F_{2(k+1)}(t) < \tau_{k+1}(t)$. Lemma 4 is proved by induction.

Remark. Define $\delta := \min\{s - \tau(s); T \le s \le t\}$. Then follows under the assumptions of Lemma 4 $\tau_n(t) \le t - n\delta$, $n \in N$.

Proof. $\delta \le s - \tau(s)$, $T \le s \le t$ or equivalently $\tau(s) \le s - \delta$, $T \le s \le t$. This shows $\tau(t) \le t - \delta$, $\tau_2(t) \le \tau(t-\delta) \le t - \delta - \delta = t - 2\delta$ and so on. Since $\tau_n(t) > F_{2n}(t) \ge \tau(F_{2n-1}(t)) \ge \tau(T_2) = T_1$ we may continue up to $\tau_n(t) \le t - n\delta$. The remark is proved.

A consequence of the remark is that for each point $t > T_2$ there exists a natural number n depending on t such that $F_n(t) \le T_2$.

Lemma 5. *Under the assumptions of Lemma 1 it holds that $g(t) \ge \frac{1}{e}$, $T_4 \le t < \infty$.*

Proof. (4) shows $G(t) \ge 0$, $T_2 \le t < \infty$. Hence $g(t) \ge 0$, $T_3 \le t < \infty$. Again from (4) we obtain $G(t) \ge \int\limits_{\tau(t)}^{t} H(\sigma)\mathrm{d}\sigma \ge \frac{1}{e}$, $T_3 \le t < \infty$. Therefore, $g(t) \ge \frac{1}{e}$, $T_4 \le t < \infty$. Lemma 5 is proved.

Definition. $\chi \in \mathrm{Piag}[t_0,\infty)$ iff χ is a generalized function on $[t_0,\infty)$ and $\int\limits_{t_1}^{t_2} \chi(\sigma)\,\mathrm{d}\sigma$ is defined in such a way that for all t_1, t_2, t_3 with $t_0 \le t_1 < t_2 < t_3 < \infty$ $\int\limits_{t_1}^{t_2} \chi(\sigma)\mathrm{d}\sigma \ge 0$ and $\int\limits_{t_1}^{t_2} \chi(\sigma)\mathrm{d}\sigma + \int\limits_{t_2}^{t_3} \chi(\sigma)\mathrm{d}\sigma = \int\limits_{t_1}^{t_3} \chi(\sigma)\mathrm{d}\sigma$.

The abbreviation Piag comes from positive integral additive generalized function, although we postulate only nonnegative.

3. The main result

Theorem 1. *Let x be a solution of ($1\le$) in the interval $J := [t_0,\infty)$, $\tau : J \to R$ a nondecreasing continuous function; $\tau(t) < t$, $t \in J$, $\lim\limits_{t\to\infty} \tau(t) = \infty$; $M(t) \ge H(t) \ge 0$, $t \in J$; $\chi \in \mathrm{Piag}\,[t_0,\infty)$; $\int\limits_{\tau(t)}^{t} H(\sigma)\,d\sigma \ge \dfrac{1}{e} + \int\limits_{\tau(t)}^{t} \chi(\sigma)\,d\sigma$, $t_0 \le \tau(t) < \infty$; $\int\limits_{t_0}^{\infty} \chi(\sigma)\,d\sigma = \infty$. Then x is not eventually positive.*

Proof. Suppose the contrary. Then there exists a $T \ge t_0$ with $x(t) > 0$ for $t \ge T$. Choose t such that $\int\limits_{T_5}^{t} \chi(\sigma)\mathrm{d}\sigma > 2(1+\ln 2)$ and a natural number n such that $T_4 \le F_n(t) \le T_5$. By Lemma 4 this is possible. Using Lemma 1, (4), $e^x \ge ex$ and the assumption on H, the definition of $F(t)$, $\int\limits_{\tau(t)}^{F(t)} \chi(\sigma)\mathrm{d}\sigma \ge 0$, the iteration of the inequality $G(t) \ge G(F(t))\Big(1 + e\int\limits_{F(t)}^{t} \chi(\sigma)d\sigma\Big)$, $\prod\limits_{\nu=1}^{n}(1+a_\nu) \ge \sum\limits_{\nu=1}^{n} a_\nu$ for $a_\nu \ge 0$, $\sum\limits_{\nu=1}^{n} \int\limits_{b_\nu}^{b_{\nu-1}} \chi(\sigma)\mathrm{d}\sigma = \int\limits_{b_n}^{b_0} \chi(\sigma)\mathrm{d}\sigma$, Lemma 5, the choice of $F_n(t)$, the choice of t we obtain $2(1+\ln 2) \ge G(t) \ge e^{g(t)} \int\limits_{\tau(t)}^{t} H(\sigma)\mathrm{d}\sigma \ge e g(t)\Big(\frac{1}{e} + \int\limits_{\tau(t)}^{t} \chi(\sigma)\mathrm{d}\sigma\Big)$

$$= G(F(t))\Big(1 + e\int\limits_{\tau(t)}^{t} \chi(\sigma)d\sigma\Big) \ge G(F(t))\Big(1 + e\int\limits_{F(t)}^{t} \chi(\sigma)\mathrm{d}\sigma\Big)$$

$$\geq G(F_n(t))\prod_{\nu=1}^{n}\Big(1+e\int\limits_{F_\nu(t)}^{F_{\nu-1}(t)}\chi(\sigma)\mathrm{d}\sigma\Big)\geq G(F_n(t))\sum_{\nu=1}^{n}e\int\limits_{F_\nu(t)}^{F_{\nu-1}(t)}\chi(\sigma)\mathrm{d}\sigma$$

$$=G(F_n(t))e\int\limits_{F_n(t)}^{t}\chi(\sigma)\mathrm{d}\sigma\geq\int\limits_{F_n(t)}^{t}\chi(\sigma)\mathrm{d}\sigma\geq\int\limits_{T_5}^{t}\chi(\sigma)\mathrm{d}\sigma>2(1+\ln 2).$$

This contradiction completes the proof of Theorem 1.

A sufficient condition for $\chi\in\operatorname{Piag}[t_0,\infty)$ is $\chi=\chi_1+\chi_2$, where $\chi_1:[t_0,\infty)\to[0,\infty)$ denotes a locally summable function and $\chi_2(t)=\sum\limits_{\nu=1}^{\infty}c_\nu\delta(t-s_\nu)$ with (s_n) a increasing sequence, $\lim\limits_{n\to\infty}s_n=\infty$, $c_n\geq 0$, $n\in N$, δ the δ-distribution and the integral $\int\limits_a^b\chi(\sigma)\mathrm{d}\sigma$ defined in the following manner.

$$\int\limits_a^b\chi(\sigma)\mathrm{d}\sigma=\int\limits_a^b\chi_1(\sigma)\mathrm{d}\sigma+\int\limits_a^b\chi_2(\sigma)\mathrm{d}\sigma \text{ and}$$

$$\int\limits_a^b\chi_2(\sigma)\mathrm{d}\sigma=\lim_{\varepsilon\to 0+}\int\limits_{a-\varepsilon}^{b-\varepsilon}\chi_2(\sigma)\mathrm{d}\sigma=\sum_{\substack{\nu\\ a\leq s_\nu<b}}c_\nu.$$

With $\chi_1:=0$ and $\chi_2(t):=\sum\limits_{\nu=0}^{\infty}c_\nu\delta(t-\tilde{T}_\nu)$, where $\tilde{T}_0:=T_0\geq t_0$, $\tilde{T}_{k+1}:=\max\{t;\tau(t)=\tilde{T}_k\}$, $k\in N_0$, we obtain from Theorem 1

Theorem 2. *Let x be a solution of (1$\leq$) in the interval $J:=[t_0,\infty)$, $\tau:J\to R$ a nondecreasing continuous function; $\tau(t)<t$, $t\in J$; $\lim\limits_{t\to\infty}\tau(t)=\infty$, $M(t)\geq H(t)\geq 0$, $t\in J$; $\int\limits_{\tau(t)}^{t}H(\sigma)d\sigma\geq\frac{1}{e}+c_k$, $\tilde{T}_k<t\leq\tilde{T}_{k+1}$, $c_k\geq 0$, $k\in N_0$; $\sum\limits_{\nu=0}^{\infty}c_\nu=+\infty$. Then x is not eventually positive.*

Theorem 2 extends [4, Theorem 1].

An immediate consequence of Theorem 1 is

Theorem 3. *Under the conditions of theorem 1 let x be a solution of (1=) in the interval $J:=[t_0,\infty)$. Then x is oscillatory.*

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