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A CONTACT METRIC MANIFOLD SATISFYING A CERTAIN CURVATURE CONDITION

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Abstract. In the present paper we investigate a contact metric manifold satisfying (C) $(\bar\nabla_{\dot\gamma}R)(\cdot,\dot\gamma)\dot\gamma = 0$ for any $\bar\nabla$-geodesic γ, where $\bar\nabla$ is the Tanaka connection. We classify the 3-dimensional contact metric manifolds satisfying (C) for any $\bar\nabla$-geodesic γ. Also, we prove a structure theorem for a contact metric manifold with ξ belonging to the k-nullity distribution and satisfying (C) for any $\bar\nabla$-geodesic γ.

1. Introduction

A Riemannian manifold $M = (M, g)$ with Riemannian metric tenor g is called (E.Cartan [6]) a locally symmetric space if M satisfies $\nabla R = 0$, where ∇ is the Levi-Civita connection. In [1] a locally symmetric space M is characterized by the remarkable property that the Jacobi operator field $R_{\dot\gamma} = R(\cdot,\dot\gamma)\dot\gamma$ is diagonalizable by a ∇-parallel orthonormal frame field along γ and their eigenvalues are constant along γ for any geodesic γ on M.

On the other hand, T.Takahashi ([11]) introduced the notion of Sasakian locally ϕ-symmetric spaces which may be considered as the analogues of locally Hermitian symmetric spaces. A contact metric locally ϕ-symmetric space is defined as a generalization of the notion of the Sasakian locally ϕ-symmetric spaces and investigated by D.E.Blair ([3]).

In [9], we have introduced a class of contact metric manifolds satisfying

$$\text{(C)} \qquad (\bar\nabla_{\dot\gamma}R)(\cdot,\dot\gamma)\dot\gamma = 0$$

for any unit $\bar\nabla$-geodesic $\gamma(\bar\nabla_{\dot\gamma}\dot\gamma = 0)$, where $\bar\nabla$ is a linear connection such that the structure tensors are parallel. We note that the connection coincides with the Tanaka connection ([13]) on a strongly pseudo-convex integrable CR-manifold whose structure is determined by a given contact metric structure, particularly for 3-dimensional contact metric manifolds and contact metric manifolds with

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the structure vector field ξ belonging to the k-nullity distribution (see section 1), and also note that the geodesics of the Levi-Civita connection and the Tanaka connection do not coincide in general. We easily observe that a contact metric manifold satisfies the condition (C) for any $\bar{\nabla}$-geodesic γ if and only if the Jacobi operator field $R_{\dot{\gamma}}$ is diagonalizable by a $\bar{\nabla}$-parallel orthonormal frame field along γ and their eigenvalues are constant along γ for any $\bar{\nabla}$-geodesic γ in the manifold.

The present paper is a continuation of the preceding papers [8], [9] in which we proved that a 3-dimensional contact metric manifold satisfying the condition (C) for any $\bar{\nabla}$-geodesic γ is locally ϕ-symmetric (in the sense of D.E.Blair). In the present paper, we determine all 3-dimensional contact metric manifolds satisfying the condition (C) for any $\bar{\nabla}$-geodesic γ. Namely, we prove

Theorem A. *Let M be a 3-dimensional contact metric manifold. If M satisfies the condition (C) for any $\bar{\nabla}$-geodesic γ, then M is a Sasakian locally ϕ-symmetric or a contact metric manifold of constant sectional curvature.*

It was proved ([5]) that a 3-dimensional Sasakian ϕ-symmetric space (simply connected and complete Sasakian locally ϕ-symmetric space) is isometric to the unit sphere S^3 in $\mathbb{E}^4$, $SU(2)$, the universal covering space $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$ or the Heisenberg group H, each with a special left invariant metric (see [15]). Also, it was proved ([4]) recently that a 3-dimensional contact metric locally symmetric space is of constant sectional curvature 0 or 1. Thus from Theorem A we have

Corollary B. *Let M be a simply connected and complete 3-dimensional contact metric manifold. If M satisfies the condition (C) for any $\bar{\nabla}$-geodesic γ, then M is isometric to the unit sphere S^3 in $\mathbb{E}^4$, $SU(2)$, the universal covering space $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$ or the Heisenberg group H, each with a special left invariant metric, or the Euclidean space $\mathbb{E}^3$.*

A contact metric on $\mathbb{E}^3$, for example, is explicitly expressed as $\mathbb{R}^3(x^1,x^2,x^3)$ with $\eta = \frac{1}{2}(\cos x^3 dx^1 + \sin x^3 dx^2)$ and $g_{ij} = \frac{1}{4}\delta_{ij}$. Also, in section 3 we prove that

Theorem C. *Let $M^{2n+1} (n \geq 2)$ be a contact metric manifold with ξ belonging to the k-nullity distribution. If M satisfies the condition (C) for any $\bar{\nabla}$-geodesic γ, then M is a Sasakian locally ϕ-symmetric space or M is locally the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant sectional curvature equal to 4.*

We remark that a contact manifold $M^{2n+1} (n \geq 2)$ can not admit a contact metric structure of vanishing curvature (cf. pp. 115 in [2]). All manifolds in the present paper are assumed to be connected and of class C^∞.

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2. Preliminaries

A $(2n+1)$-dimensional manifold M^{2n+1} is said to be a contact manifold if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, we have a unique vector field ξ, which is called the characteristic vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there exists a Riemannian metric g and a $(1,1)$-tensor field ϕ such that

$$\eta(X) = g(X,\xi), \quad d\eta(X,Y) = g(X,\phi Y), \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

where X and Y are vector fields on M. From (2.1) it follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

A Riemannian manifold M equipped with structure tensors (ϕ,ξ,η,g) satisfying (2.1) is said to be a contact metric manifold and is denoted by $M = (M,\phi,\xi,\eta,g)$. Given a contact metric manifold M, following D.E.Blair([2]), we define a $(1,1)$-tensor field h by $h = -\frac{1}{2}L_\xi\phi$, where L denotes Lie differentiation. Then we may observe that h is symmetric and satisfies

$$h\xi = 0 \quad \text{and} \quad h\phi = -\phi h, \tag{2.3}$$

$$\nabla_X \xi = -\phi X - \phi h X, \tag{2.4}$$

where ∇ is Levi-Civita connection. From (2.3) and (2.4) we see that each trajectory of ξ is a geodesic. We denote by R Riemannian curvature tensor defined by

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for all vector fields X, Y, Z. Along a trajectory of ξ, the Jacobi operator $R_\xi = R(\cdot,\xi)\xi$ is a symmetric $(1,1)$-tensor field. We have

$$(trace\ R_\xi) = g(Q\xi,\xi) = 2n - (trace\ h^2), \tag{2.5}$$

$$\nabla_\xi h = \phi - \phi R_\xi - \phi h^2, \tag{2.6}$$

(cf.[2] or [3]) where Q is Ricci $(1,1)$-tensor on M.

A contact metric manifold for which ξ is Killing is called a K-contact metric manifold. It is easy to see that a contact metric manifold is K-contact if and only if $h = 0$. For a contact metric manifold M one may define naturally an almost complex structure on $M \times \mathbb{R}$. If this almost complex structure is integrable, M is said to be Sasakian. A Sasakian manifold is characterized by a condition

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X \tag{2.7}$$

for all vector fields X and Y on the manifold.

Let M be a contact metric manifold. It is well-known that M is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.8}$$

for all vector fields X and Y ([2]).

Let T be a $(1,2)$-tensor field on M defined by

$$(2.9)\ T_XY = -\frac{1}{2}\phi(\nabla_X\phi)Y + \frac{1}{2}\eta(Y)(\phi X + \phi hX) - \eta(X)\phi Y - g(\phi X + \phi hX, Y)\xi.$$

Particularly, for a Sasakian manifold, from (2.7) and (2.9) we see that

$$T_XY = g(X,\phi Y)\xi + \eta(Y)\phi X - \eta(X)\phi Y, \tag{2.10}$$

where X and Y are vector fields on M. Using the tensor field T, we define a linear connection $\bar\nabla$ on M by

$$\bar\nabla_XY = \nabla_XY + T_XY \tag{2.11}$$

(cf. [7] or [8]). Then the linear connection $\bar\nabla$ has the torsion given by $T_XY - T_YX$. Using (2.1), (2.2) and (2.3), we have

$$\bar\nabla\phi = 0, \quad \bar\nabla\xi = 0, \quad \bar\nabla\eta = 0, \quad \bar\nabla g = 0. \tag{2.12}$$

We remark that the above connection $\bar\nabla$ coincides with the Tanaka connection (defined in [12]) on a strongly pseudo-convex integrable CR-manifold whose structure is determined by a contact metric manifold which satisfies $(\nabla_X\phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)$ for any vector fields X and Y (see Proposition 2.1 in [15]). The tangent space T_pM of M at each point $p \in M$ is decomposed as $T_pM = \mathfrak{D}_p \oplus \{\xi\}_p$ (direct sum), where we denote $\mathfrak{D}_p = \{v \in T_pM | \eta(v) = 0\}$. Then $\mathfrak{D} : p \to \mathfrak{D}_p$ defines a distribution orthogonal to ξ. Let γ be a $\bar\nabla$-geodesic parametrized with the arc-length parameter s, where a $\bar\nabla$-geodesic means a geodesic with respect to $\bar\nabla$. From (2.9) and (2.11) we see that a $\bar\nabla$-geodesic does not coincide with a ∇-geodesic in general. We denote $\dot\gamma = \gamma_*(\frac{d}{ds})$ and by γ_* the differential of $\gamma : I \to M$. Define the Jacobi operator $R_{\dot\gamma}$ by $R_{\dot\gamma} = R(\cdot,\dot\gamma)\dot\gamma$ along γ. $R_{\dot\gamma}$ is a symmetric $(1,1)$-tensor field along γ. Moreover, from (2.12) we observe that $\eta(\dot\gamma)$ is constant along γ, and thus a $\bar\nabla$-geodesic whose tangent initially belongs to $\mathfrak{D}$ remains in $\mathfrak{D}$. We call such a $\bar\nabla$-geodesic which is tangent to $\mathfrak{D}$ a *horizontal* $\bar\nabla$*-geodesic*.

Now, recall the definition of a Sasakian locally ϕ-symmetric space ([11]).

Definition 2.1. A Sasakian manifold $M = (M,\phi,\xi,\eta,g)$ is said to be locally ϕ-symmetric if $\phi^2(\nabla_VR)(X,Y)Z = 0$ for all vector fields $V,X,Y,Z \in \mathfrak{D}$.

As a generalization of the above Sasakian one, a contact metric locally ϕ-symmetric space is defined by D.E.Blair([3]) by the same condition. In [7] we characterized a Sasakian locally ϕ-symmetric space by following

Theorem 2.2. *A Sasakian manifold M is locally ϕ-symmetric if and only if M satisfies the condition (C) for any horizontal $\bar\nabla$-geodesic.*

Concerning Theorem 2.2 we prove

Theorem 2.3. *A Sasakian manifold M is locally ϕ-symmetric if and only if M satisfies the condition (C) for any $\bar{\nabla}$-geodesic γ.*

Proof. From (2.8) and (2.12) we see that

$$(\bar{\nabla}_\xi R)(Y,X)\xi = 0$$

for all vector fields X and Y on M. Then, taking account of Theorem 2.2, it suffices to prove $g((\bar{\nabla}_\xi R)(Y,V)V,X) = 0$ for all vector fields $V,X,Y \in \mathfrak{D}$. It follows from (2.10) and (2.11) that

(2.13)
$$\begin{aligned} g((\bar{\nabla}_\xi R)(Y,V)V,X) =& (\nabla_\xi R)(Y,V)V,X) - g(\phi R(Y,V)V,X) + g(R(\phi Y,V)V),X) \\ &+ g(R(X,\phi V)V,Y) + g(R(X,V)\phi V,Y) \end{aligned}$$

for all vector fields $V,X,Y \in \mathfrak{D}$. From (2.8) and the second Bianchi identity, we have

(2.14)
$$\begin{aligned} ((\nabla_\xi R)(Y,V)V,X) =& g(\phi Y,V)g(V,X) - g(\phi Y,X)g(V,V) + g(R(V,X)\phi Y,V) \\ &+ g(\phi V,X)g(V,Y) - g(R(V,X)\phi V,Y). \end{aligned}$$

Thus, from (2.13) and (2.14), we have

(2.15)
$$\begin{aligned} ((\bar{\nabla}_\xi R)(Y,V)V,X) =& g(\phi Y,V)g(V,X) - g(\phi Y,X)g(V,V) + 2g(R(V,X)\phi Y,V) \\ &+ g(\phi V,X)g(V,Y) - 2g(R(V,X)\phi V,Y) \\ &+ g(R(Y,V)\phi V,X) - g(\phi R(Y,V)V,X) \end{aligned}$$

for all vector fields $V,X,Y \in \mathfrak{D}$. From the definition of the curvature tensor, taking account of (2.4) and (2.7), we obtain

(2.16)
$$R(Y,X)\phi Z - \phi R(Y,X)Z = g(\phi Y,Z)X - g(X,Z)\phi Y - g(\phi X,Z)Y + g(Y,Z)\phi X,$$

where X, Y and Z are vector fields on M. By using (2.16), from (2.15) we see that $g((\bar{\nabla}_\xi R)(Y,V)V,X) = 0$ for all vector fields $V,X,Y \in \mathfrak{D}$. □

S. Tanno ([13]) defined the k-nullity distribution of Riemannian manifold (M,g), for a real number k, by

$$\begin{aligned} N(k) : p \longrightarrow N_p(k) = \{z \in T_pM \,|\, R(x,y)z = k(g(y,z)x - g(x,z)y) \\ \text{for any} \quad x,y \in T_pM\}, \end{aligned}$$

and he proved

Proposition 2.4. *Let $M = (M, \phi, \xi, \eta, g)$ be a contact metric manifold. If ξ belong to the k-nullity distribution, then $k \leq 1$. If $k < 1$, then M admits three mutually orthogonal and integral distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$, defined by the eigenspaces of h, where $\lambda = \sqrt{1-k}$.*

In [8], we proved

Theorem 2.5. *Let M be a contact metric manifold with ξ belonging to the k-nullity distribution. Then M is locally ϕ-symmetric (in the sense of D.E.Blair) if and only if M satisfies the condition (C) for any horizontal $\bar{\nabla}$-geodesic.*

Since a contact metric manifold M with ξ belonging to the 1-nullity distribution is a Sasakian manifold, the above Theorem 2.5 is a extension of Theorem 2.2. For a contact metric manifold with ξ belonging to the 0-nullity distrbution, D.E. Blair ([2]) proved

Theorem 2.6. *Let M be a contact metric manifold with ξ belonging to the 0-nullity distribution. Then M is locally the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive constant sectional curvature equal to 4.*

3. 3-dimensional contact metric manifolds

In this section we prove Theorem A. Recently, it was proved in [14] that a 3-dimensional contact metric manifold always satisfies

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX) \tag{3.1}$$

for all vector fields X, Y.

Lemma 3.1. *A 3-dimensional contact metric manifold is Sasakian if and only if $h = 0$.*

Proof. Assume that M^3 is a contact metric manifold. Then from (2.7) and (3.1) we get $g(hX, Y)\xi - \eta(Y)hX = 0$. Taking account of (2.3), we have $g(hX, Y) = 0$ for all vector fields X, Y on M and hence, we have $h = 0$. The converse is obvious. □

Now we prove Theorem A.

Proof of Theorem A. Let $M^3 = (M^3, \phi, \xi, \eta, g)$ be a 3-dimensional contact metric manifold satisfying the condition (C) for any $\bar{\nabla}$-geodesic γ. It is well-known that the curvature tensor R of a 3-dimensional Riemannian manifold is expressed by

$$\begin{aligned} R(Y, X)Z =& \rho(X, Z)Y - \rho(Y, Z)X + g(X, Z)QY - g(Y, Z)QX \\ & - \frac{1}{2}\tau\{g(X, Z)Y - g(Y, Z)X\} \end{aligned} \tag{3.2}$$

for all vector fields X, Y, Z, where $\rho(Y, X) = g(QY, X)$ and τ is the scalar curvature of the manifold.

From (3.2) and the assumption we have

$$\begin{aligned} 0 =&(\bar\nabla_x R)(y,x)x \\ &= (\bar\nabla_x \rho)(x,x)y - (\bar\nabla_x \rho)(y,x)x + g(x,x)(\bar\nabla_x Q)y - g(y,x)(\bar\nabla_x Q)x \\ &- \frac{1}{2}(x\tau)\{g(x,x)y - g(y,x)x\}, \end{aligned} \tag{3.3}$$

for any $x, y \in T_pM$ and any $p \in M$. For any unit v orthogonal to ξ, let $\{v, \phi v, \xi\}$ be an adapted orthonormal basis of $T_pM(p \in M)$. Then from (3.3) we get $g((\bar\nabla_x R)(v,x)x, v) = 0$, $g((\bar\nabla_x R)(\phi v,x)x,\phi v) = 0$ and $g((\bar\nabla_x R)(\xi,x)x,\xi) = 0$, and summing up these three equalities, we have

$$(\bar\nabla_x \rho)(x,x) = 0. \tag{3.4}$$

Also, from (3.3) we get $(\bar\nabla_v R)(\phi v, v)v = 0$, $(\bar\nabla_v R)(\xi, v)v = 0$ and thus we have

$$(\bar\nabla_v \rho)(\phi v, \phi v) = (\bar\nabla_v \rho)(\xi,\xi) \tag{3.5}$$

and

$$(\bar\nabla_v \rho)(\phi v, \xi) = 0. \tag{3.6}$$

Taking account of (3.1), we see that

$$T_x y = \eta(y)(\phi x + \phi h x) - \eta(x)\phi y - g(\phi x + \phi h x, y)\xi \tag{3.7}$$

for $x, y \in T_pM$ and $p \in M$. From (2.11) and (3.7) we have the formulas (3.8),(3.9) and (3.10) which are equivalent to (3.4),(3.5) and (3.6), respectively:

$$(\nabla_x \rho)(x,x) = 2\{\eta(x)\rho(\phi h x, x) - g(\phi h x, x)\rho(\xi, x)\}, \tag{3.8}$$

$$(\nabla_v \rho)(\xi,\xi) - (\nabla_v \rho)(\phi v, \phi v) = 2\{(2 + g(hv,v))\rho(\xi,\phi v) + \rho(\phi h v, \xi)\}, \tag{3.9}$$

$$(\nabla_v \rho)(\phi v, \xi) = \rho(\phi v, \phi v) + \rho(\phi v, \phi h v) - \{1 + g(hv,v)\}\rho(\xi,\xi) \tag{3.10}$$

for any unit $x \in T_pM$ and unit vector v orthogonal to ξ.

Let W be the subset of M on which the number of distinct eigenvalues of h is constant. Then W is an open and dense subset of M. We fix any point q in W. Then from (2.3) there exists a C^∞ function λ such that $he_1 = \lambda e_1$, $he_2 = -\lambda e_2$, $h\xi = 0$ where $\{e_1, e_2 = \phi e_1, e_3 = \xi\}$ is a local orthonormal frame field on a neighborhood $N_q(\subset W)$ containing q. We denote $\Gamma_{ijk} = g(\nabla_{e_i} e_j, e_k)$,

$\rho_{ij} = \rho(e_i, e_j)$, $\nabla_i \rho_{jk} = (\nabla_{e_i}\rho)(e_j, e_k)$ and $\nabla_h R_{ijkl} = g((\nabla_h R)(e_i, e_j)e_k, e_l)$ for $h, i, j, k, l = 1, 2, 3$. Then from (2.4) we get

$$\Gamma_{132} = -(1+\lambda), \quad \Gamma_{231} = 1-\lambda \tag{3.11}$$

and

$$\Gamma_{131} = \Gamma_{232} = 0. \tag{3.12}$$

Also, from (2.6) and taking account of (2.5) and (3.2), we have

$$\xi\lambda = \rho_{12} \tag{3.13}$$

and

$$4\lambda\Gamma_{312} = \rho_{22} - \rho_{11}. \tag{3.14}$$

Moreover, from (3.8) we get

$$\nabla_1\rho_{11} = 0, \quad \nabla_2\rho_{22} = 0 \tag{3.15}$$

and

$$\nabla_3\rho_{33} = 0. \tag{3.16}$$

Substituting $x = \frac{1}{\sqrt{2}}(e_1+e_2)$ and $x = \frac{1}{\sqrt{2}}(e_1-e_2)$, respectively in (3.8) and taking account of (3.15), we have

$$2\nabla_1\rho_{12} + 2\nabla_2\rho_{12} + \nabla_1\rho_{22} + \nabla_2\rho_{11} = -4\lambda(\rho_{31}+\rho_{32})$$

and

$$-2\nabla_1\rho_{12} + 2\nabla_2\rho_{12} + \nabla_1\rho_{22} - \nabla_2\rho_{11} = 4\lambda(\rho_{31}-\rho_{32}).$$

By summing these two equalities, we have

$$\nabla_1\rho_{22} + 2\nabla_2\rho_{12} = -4\lambda\rho_{23} \tag{3.17}$$

and substracting (3.17) from the preceding one, we have

$$\nabla_2\rho_{11} + 2\nabla_1\rho_{12} = -4\lambda\rho_{13}. \tag{3.18}$$

Also, substituting $x = \frac{1}{\sqrt{2}}(e_1+e_3)$ and $x = \frac{1}{\sqrt{2}}(e_1-e_3)$, respectively in (3.8) and taking account of (3.16), we have

$$2\nabla_1\rho_{13} + 2\nabla_3\rho_{31} + \nabla_1\rho_{33} + \nabla_3\rho_{11} = 2\lambda\rho_{23}$$

and

$$-2\nabla_1\rho_{13} + 2\nabla_3\rho_{31} + \nabla_1\rho_{33} - \nabla_3\rho_{11} = 2\lambda\rho_{23}.$$

Summing these two equalities we have

$$\nabla_1\rho_{33} + 2\nabla_3\rho_{13} = 2\lambda\rho_{23} \tag{3.19}$$

and substracting (3.19) from the preceding one, we have

$$\nabla_3\rho_{11} + 2\nabla_1\rho_{31} = 0. \tag{3.20}$$

A similar calculation for $x = \frac{1}{\sqrt{2}}(e_2 + e_3)$ and $x = \frac{1}{\sqrt{2}}(e_2 - e_3)$ gives

$$\nabla_2\rho_{33} + 2\nabla_3\rho_{23} = 2\lambda\rho_{13} \tag{3.21}$$

and

$$\nabla_3\rho_{22} + 2\nabla_2\rho_{32} = 0. \tag{3.22}$$

On the one hand, from the second Bianchi identity, we have

$$2\nabla_2\rho_{12} + 2\nabla_3\rho_{13} - \nabla_1\rho_{22} - \nabla_1\rho_{33} = 0. \tag{3.23}$$

$$2\nabla_1\rho_{21} + 2\nabla_3\rho_{23} - \nabla_2\rho_{11} - \nabla_2\rho_{33} = 0. \tag{3.24}$$

From (3.17), (3.19) and (3.23) (resp.(3.18), (3.21) and (3.24)), we have (3.25) (resp.(3.26)):

$$\nabla_1\rho_{22} + \nabla_1\rho_{33} = -\lambda\rho_{23}, \tag{3.25}$$

$$\nabla_2\rho_{11} + \nabla_2\rho_{33} = -\lambda\rho_{13}. \tag{3.26}$$

On the other hand, from (3.5) we have

$$\nabla_1\rho_{33} - \nabla_1\rho_{22} = 4(\lambda + 1)\rho_{23} \tag{3.27}$$

and

$$\nabla_2\rho_{33} - \nabla_2\rho_{11} = 4(\lambda - 1)\rho_{13}. \tag{3.28}$$

Thus, from (3.25)-(3.28) we have

$$\nabla_1\rho_{33} = \frac{1}{2}(3\lambda + 4)\rho_{23}, \quad \nabla_2\rho_{33} = \frac{1}{2}(3\lambda - 4)\rho_{13} \tag{3.29}$$

and

$$\nabla_1\rho_{22} = -\frac{1}{2}(5\lambda + 4)\rho_{23}, \quad \nabla_2\rho_{11} = -\frac{1}{2}(5\lambda - 4)\rho_{13}. \tag{3.30}$$

Also, from (3.17),(3.18) and (3.30), we have

$$\nabla_1\rho_{12} = -\frac{1}{4}(3\lambda + 4)\rho_{13} \quad \text{and} \quad \nabla_2\rho_{21} = -\frac{1}{4}(3\lambda - 4)\rho_{23}. \tag{3.31}$$

Lemma 3.2. *$\rho_{ij} = 0$ on $N_q(\subset W)$, where $i \neq j$, $i, j = 1, 2, 3$.*

Proof. . Differentiating (2.5) in the direction ξ and taking account of (3.16) we have $\xi\lambda = 0$. Thus from (3.13) we have $\rho_{12} = 0$ on N_q.

Now we prove $\rho_{13} = 0$ and $\rho_{23} = 0$ on N_q. Differentiating (2.5) in the directions e_1 and e_2 and taking account of (3.11), (3.12) and (3.29) we have

$$\rho_{23} = 8(e_1\lambda) \tag{3.32}$$

and

$$\rho_{13} = 8(e_2\lambda), \tag{3.33}$$

respectively.

Also, differentiating $\rho_{12} = 0$ in the direction ξ, we have

$$\nabla_3\rho_{12} = \Gamma_{312}(\rho_{11} - \rho_{22}). \tag{3.34}$$

Substituting $x = \xi$ in (3.3), we get $\bar{\nabla}_3\rho_{12} = 0$, and from (3.7) we get $\bar{\nabla}_3\rho_{12} = \nabla_3\rho_{12} + \rho_{22} - \rho_{11}$. Thus we see that

$$\nabla_3\rho_{12} = \rho_{11} - \rho_{22}. \tag{3.35}$$

At first, if there exists a point in $N_q(\subset W)$ such that $\rho_{11} = \rho_{22}$, then $\rho_{13} = \rho_{23} = 0$ at that point. In fact, differentiating $\rho_{12} = 0$ in the direction e_1 and e_2, then from the assumption and (3.11) we have $\nabla_1\rho_{12} = -(1+\lambda)\rho_{13}$ and $\nabla_2\rho_{21} = (1-\lambda)\rho_{23}$, respectively. Thus taking account of (3.31) we have $\rho_{13} = \rho_{23} = 0$ at the point in N_q. Next, suppose there exists a point m such that $\rho_{11}(m) \neq \rho_{22}(m)$. Then we see that $\rho_{11} \neq \rho_{22}$ on a sufficiently small neighborhood $U(m)$ of m. From (3.34) and (3.35) we get $\Gamma_{312} = 1$ on $U(m)$. Thus (3.14) becomes $4\lambda = \rho_{22} - \rho_{11}$ on $U(m)$. Differentiating this equation in the directions e_1 and e_2 and taking account of (3.11), (3.12), (3.32) and (3.33) we have $\nabla_1\rho_{22} = -\frac{1}{2}(4\lambda+3)\rho_{23}$ and $\nabla_2\rho_{11} = -\frac{1}{2}(4\lambda-3)\rho_{13}$. Thus taking account of (3.30) we have

$$(\lambda+1)\rho_{23} = 0 \tag{3.36}$$

and

$$(\lambda-1)\rho_{13} = 0 \tag{3.37}$$

on $U(m)$. Differentiating (3.36)(resp.(3.37)) in the direction e_1(resp.e_2) and taking account of (3.32) and (3.33), we have

$$\frac{1}{8}\rho_{23}^2 + (\lambda+1)(e_1\rho_{23}) = 0 \tag{3.38}$$

and

$$\frac{1}{8}\rho_{13}^2 + (\lambda - 1)(e_2\rho_{13}) = 0 \tag{3.39}$$

on $U(m)$. If there exists a point n in $U(m)$ such that $\rho_{13}(n) \neq 0$, then from (3.37) we get $\lambda(n) = 1$, and from (3.39) we get $\rho_{13}(n) = 0$, a contradiction. Also, if there exist a point n in $U(m)$ such that $\rho_{23}(n) \neq 0$, then from (3.36) we get $\lambda(n) = -1$, and from (3.38) we get $\rho_{23}(n) = 0$, a contradiction. Thus we have $\rho_{13} = \rho_{23} = 0$ on $U(m)$. At last, we conclude that $\rho_{13} = \rho_{23} = 0$ also on N_q. □

From Lemma 3.2, we see that λ is locally constant on $N_q(\subset W)$. Since $\rho_{13} = \rho_{23} = 0$, from (3.29)-(3.31), we get

$$\begin{aligned} &\nabla_1\rho_{12} = 0,\ \nabla_1\rho_{22} = 0,\ \nabla_1\rho_{33} = 0,\\ &\nabla_2\rho_{12} = 0,\ \nabla_2\rho_{11} = 0,\ \nabla_2\rho_{33} = 0. \end{aligned} \tag{3.40}$$

Also, taking account of (3.12), we have

$$\nabla_1\rho_{13} = 0 \quad \text{and} \quad \nabla_2\rho_{23} = 0. \tag{3.41}$$

The equations (3.19)-(3.22), together with (3.40) and (3.41), yield

$$\nabla_3\rho_{11} = 0,\ \nabla_3\rho_{13} = 0,\ \nabla_3\rho_{22} = 0,\ \ \nabla_3\rho_{23} = 0. \tag{3.42}$$

From (3.15), (3.16), (3.40) and (3,42), we see that the scalar curvature τ is constant. Returning to the condition (C), from (3.3), by using polarization, we have

(3.43)

$$\begin{aligned} 0 =& S_{x,z,w}\big[(\nabla_x\rho)(z,w)y + \eta(Qz)g(\phi x + \phi hx, w)y - \eta(x)g(\phi Qz, w)y\\ &- g(\phi x + \phi hx, Qz)\eta(w)y - \eta(z)g(Q\phi x + Q\phi hx, w)y + \eta(x)g(Q\phi z, w)y\\ &+ g(\phi x + \phi hx, z)\eta(Qw)y - (\nabla_x\rho)(y,z)w - \eta(Qw)g(\phi x + \phi hx, y)z\\ &+ \eta(x)g(\phi Qw, y)z + g(\phi x + \phi hx, Qw)\eta(y)z + \eta(w)g(Q\phi x + Q\phi hx, y)z\\ &- \eta(x)g(Q\phi w, y)z - g(\phi x + \phi hx, w)\eta(Qy)z + g(x,z)\{(\nabla_w Q)y\\ &+ \eta(Qy)(\phi w + \phi hw) - \eta(w)\phi Qy - g(\phi w + \phi hw, Qy)\xi\\ &- \eta(y)(Q\phi w + Q\phi hw) + \eta(w)Q\phi y + g(\phi w + \phi hw, y)Q\xi\} - g(y,x)\{(\nabla_z Q)w\\ &+ \eta(Qw)(\phi z + \phi hz) - \eta(z)\phi Qw - g(\phi z + \phi hz, Qw)\xi\\ &- \eta(w)(Q\phi z + Q\phi hz) + \eta(z)Q\phi w + g(\phi z + \phi hz, w)Q\xi\}\big] \end{aligned}$$

for any $x, y, z, w \in T_qM$, where $S_{x,z,w}$ denotes the cyclic sum for tangent vectors x, z, w. First, substitute $y = e_1$, $x = e_1$, $z = e_2$, $w = e_3$ into (3.43). Then taking account of (3.40) and (3.41) we have

$$\begin{aligned} &\nabla_1\rho_{23} + \nabla_3\rho_{12} - \nabla_2\rho_{31}\\ &- \lambda\rho_{22} + (3\lambda - 1)\rho_{33} - (2\lambda - 1)\rho_{11} = 0. \end{aligned} \tag{3.44}$$

Next, substitute $y = e_2$, $x = e_1$, $z = e_2$, $w = e_3$ into (3.43). Then taking account of (3.41) and (3.42) we have

$$\begin{aligned} &\nabla_1\rho_{23} + \nabla_2\rho_{31} - \nabla_3\rho_{12} \\ &- (\lambda - 1)\rho_{11} - (\lambda + 1)\rho_{22} - 2\lambda\rho_{33} = 0. \end{aligned} \tag{3.45}$$

Finally, substitute $y = e_3$, $x = e_1$, $z = e_2$, $w = e_3$ into (3.43). Then taking account of (3.40) we have

$$\begin{aligned} &\nabla_2\rho_{31} + \nabla_3\rho_{12} - \nabla_1\rho_{23} \\ &+ (\lambda - 1)\rho_{33} + \rho_{22} - \lambda\rho_{11} = 0. \end{aligned} \tag{3.46}$$

From (3.44), (3.45) and (3.46), we have

$$\begin{aligned} 2\nabla_2\rho_{31} &= (2\lambda - 1)\rho_{11} + \lambda\rho_{22} - (3\lambda - 1)\rho_{33}, \\ 2\nabla_3\rho_{12} &= (3\lambda - 1)\rho_{11} + (\lambda - 1)\rho_{22} - (4\lambda - 2)\rho_{33}, \\ 2\nabla_1\rho_{23} &= (3\lambda - 2)\rho_{11} + (2\lambda + 1)\rho_{22} - (5\lambda - 1)\rho_{33}. \end{aligned} \tag{3.47}$$

Now suppose there exists a point $m \in N_q$ such that $\rho_{11}(m) \neq \rho_{22}(m)$. Then we see that $\Gamma_{312}(m) = 1$ in the proof of Lemma 3.2, and from (3.10) and (3.14) we obtain

$$\begin{aligned} \nabla_2\rho_{31} &= (\lambda - 1)(\rho_{11} - \rho_{33}), \\ \nabla_3\rho_{12} &= \rho_{11} - \rho_{22}, \\ \nabla_1\rho_{23} &= (\lambda + 1)(\rho_{22} - \rho_{33}) \end{aligned} \tag{3.48}$$

at m. Thus from (3.47) and (3.48) we have

$$\begin{aligned} &\rho_{11} + \lambda\rho_{22} - (\lambda + 1)\rho_{33} = 0, \\ &3(\lambda - 1)\rho_{11} + (\lambda + 1)\rho_{22} - 2(2\lambda - 1)\rho_{33} = 0, \\ &(3\lambda - 2)\rho_{11} - \rho_{22} - 3(\lambda - 1)\rho_{33} = 0 \end{aligned} \tag{3.49}$$

at m. Since $\rho_{22}(m) - \rho_{11}(m) = 4\lambda(m)$ from (3.14), the above (3.49) gives

$$\begin{aligned} &(\lambda + 1)(\rho_{11} - \rho_{33}) = -4\lambda^2, \\ &2(2\lambda - 1)(\rho_{11} - \rho_{33}) = -4\lambda(\lambda + 1), \\ &3(\lambda - 1)(\rho_{11} - \rho_{33}) = 4\lambda \end{aligned}$$

which yields $\lambda(m) = 0$. Since λ is locally constant on N_q, we see that $\lambda = 0$. Now, we consider $||h||^2$. Then $||h||^2 = 2\lambda^2$ is a function on M, and by the continuity argument we observe that $h = 0$ on M. Thus by Lemma 3.1 we see that M is Sasakian, and by Theorem 3.1 we see that M is locally ϕ-symmetric.

Or suppose $\rho_{11} = \rho_{22}$ on N_q. Then from (3.10) and (3.14) we have

$$\begin{aligned} \nabla_2 \rho_{31} &= (\lambda - 1)(\rho_{11} - \rho_{33}), \\ \nabla_3 \rho_{12} &= 0, \\ \nabla_1 \rho_{23} &= (\lambda + 1)(\rho_{22} - \rho_{33}). \end{aligned} \tag{3.50}$$

From (3.47) and (3.50), we see that

$$\begin{aligned} &(\lambda + 1)(\rho_{11} - \rho_{33}) = 0, \\ &2(2\lambda - 1)(\rho_{11} - \rho_{33}) = 0, \\ &3(\lambda - 1)(\rho_{11} - \rho_{33}) = 0 \end{aligned}$$

which yields $\rho_{11} = \rho_{33}$ on N_q. In this case, taking account of Lemma 3.2, we see that M is a Einstein manifold and hence, of constant sectional curvature. At last, we have our conclusion. □

4. A contact metric manifold with ξ belonging to the k-nullity distribution

In the present section we prove Theorem C. The following Lemma is known (cf. p. 446-447 in [13] or p. 251 in [10]).

Lemma 4.1. *Let $M = (M^{2n+1}, \phi, \xi, \eta, g)$ be a contact metric manifold with ξ belonging to the k-nullity distribution. Then*

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX). \tag{4.1}$$

Proof of Theorem C. Let $M^{2n+1} (n \geq 2)$ be a contact metric manifold with ξ belonging to the k-nullity distribution, i.e.,

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y), \tag{4.2}$$

where k is a real number. From (4.2) we see that

$$(\bar{\nabla}_\xi R)(Y, X)\xi = 0$$

for all vector fields X and Y on M. Thus, by virtue of Theorem 2.5, it only remains to examine $g((\bar{\nabla}_\xi R)(Y, V)V, X) = 0$ for all vector fields $V, X, Y \in \mathfrak{D}$. From (2.9) and (4.1) we get

$$T_X Y = \eta(Y)(\phi X + \phi hX) - \eta(X)\phi Y - g(\phi X + \phi hX, Y)\xi. \tag{4.3}$$

Then it follows from (2.11) and (4.3), together with (2.1) and (2.2), that

(4.4)
$$\begin{aligned} g((\bar{\nabla}_\xi R)(Y, V)V, X) =& (\nabla_\xi R)(Y, V)V, X) - g(\phi R(Y, V)V, X) + g(R(\phi Y, V)V, X) \\ &+ g(R(X, \phi V)V, Y) + g(R(X, V)\phi V, Y) \end{aligned}$$

for all vector fields $V, X, Y \in \mathfrak{D}$. On the other hand, from (4.2) and the second Bianchi identity we obtain

(4.5)
$$\begin{aligned} g((\nabla_\xi R)(Y,V)V,X) =& k\{g(\phi Y,V)g(V,X) + g(\phi hY,V)g(V,X) \\ & - g(\phi Y,X)g(V,V) - g(\phi hY,X)g(V,V)\} \\ & - g(\phi hV,V)g(Y,X) + g(\phi V,X)g(V,Y) \\ & + g(\phi hV,X)g(V,Y)\} \\ & + g(R(V,X)\phi Y,V) + g(R(V,X)\phi hY,V) \\ & - g(R(V,X)\phi V,Y) - g(R(V,X)\phi hV,Y), \end{aligned}$$

where $X, Y \in \mathfrak{D}$. From the definition of the curvature tensor, taking account of (2.4) and (4.1), we obtain

(4.6)
$$\begin{aligned} & g(R(Y,X)\phi Z,W) - g(\phi R(Y,X)Z,W) \\ & \quad = g(\phi Y + \phi hY,Z)g(X + hX,W) - g(X + hX,Z)g(\phi Y + \phi hY,W) \\ & \quad\ - g(\phi X + \phi hX,Z)g(Y + hY,W) + g(Y + hY,Z)g(\phi X + \phi hX,W), \end{aligned}$$

where $X, Y, Z, W \in \mathfrak{D}$. Since $g((\bar\nabla_\xi R)(Y,V)V,X) = 0$, from (4.4), (4.5) and (4.6), we have

(4.7)
$$\begin{aligned} & (k-1)\{g(\phi Y,V)g(X,V) - g(\phi Y,X)g(V,V) + g(\phi V,X)g(V,Y)\} \\ & \quad + (k+3)\{g(\phi hY,V)g(X,V) - g(\phi hY,X)g(V,V) - g(\phi hV,V)g(X,Y) \\ & \quad + g(\phi hV,X)g(V,Y)\} \\ & \ = g(\phi Y,V)g(hX,V) - g(\phi Y,X)g(hV,V) + g(\phi V,X)g(hV,Y) \\ & \quad - 3\{g(\phi hY,V)g(hX,V) - g(\phi hY,X)g(hV,V) - g(\phi hV,V)g(hX,Y) \\ & \quad + g(\phi hV,X)g(hV,Y)\} + g(R(V,X)\phi hV,Y) - g(R(V,X)\phi hY,V), \end{aligned}$$

for all vector fields $V, X, Y \in \mathfrak{D}$. Since h is symmetric operator and $2n+1 \geq 5$, we assume that $hY = \lambda Y$ and $hV = \lambda V$, where Y and V are unit and mutually orthogonal. Then from (4.7) we obtain

(4.8)
$$\begin{aligned} & (k-1)g(Y,\phi X) + (k+3)\lambda g(Y,\phi X) \\ =& \lambda g(Y,\phi X) - 3\lambda^2 g(Y,\phi X) \\ & + \lambda g(\phi R(V,X)Y,V) - \lambda g(R(V,X)\phi Y,V). \end{aligned}$$

Also, from (4.6) we have

(4.9)
$$g(\phi R(V,X)Y,V) - g(\phi R(V,X)\phi Y,V) = (1-\lambda^2)g(Y,\phi X).$$

The equations (4.8) and (4.9), together with $\lambda = \sqrt{1-k}$ (by Proposition 2.4), yield

$$\sqrt{1-k} - (1-k) = 0,$$

which yields $k = 0$ or $k = 1$. Thus we see that M is Sasakian (when $k = 1$) or M is a contact metric manifold whose structure vector ξ belongs to the 0-nullity distribution. Therefore by virtue of Theorems 2.3 and 2.6, we have our conclusion. □

References

[1] Berndt, J. and Vanhecke, L., *Two natural generalizations of locally symmetric spaces*, Diff. Geom. Appl. **2** (1992), 57-80.

[2] Blair, D.E., *Contact manifolds in Riemannian geometry*, Lecture Notes in Math. Springer-Verlag, Berlin-Heidelberg-New-York. **509** (1976).

[3] Blair, D.E., Koufogiorgos, T., and Sharma, R., *A classification of 3-dimensional contact metric manifolds with* $Q\phi = \phi Q$, Kodai Math.J. **13** (1990), 391-401.

[4] Blair, D.E. and Sharma, R., *Three-dimensional locally symmetric contact metric manifolds*, to appear in Boll.Un.Mat.Ital..

[5] Blair, D.E. and Vanhecke, L., *Symmetries and ϕ-symmetric spaces*, Tôhoku Math.J. **39** (1987), 373-383.

[6] Cartan, E., *Lecons sur la géométrie des espaces de Riemann, 2nd éd.*, Gauthier-Villars, Paris (1946).

[7] Cho, J.T., *On some classes of almost contact metric manifolds*, Tsukuba J. Math. **19** (1995), 201-217.

[8] Cho, J.T., *On some classes of contact metric manifolds*, Rend.Circ.Mat. Palermo **XLIII** (1994), 141–160.

[9] Cho, J.T., *Generalizations of locally symmetric spaces and locally ϕ-symmetric spaces*, Nigata Univ. Doctorial Thesis (1994).

[10] Olszak, Z., *On contact metric manifolds*, Tôhoku Math. J. **31** (1979).

[11] Takahashi, T., *Sasakian ϕ-symmetric spaces*, Tôhoku Math. J. **29** (1977), 91-113.

[12] Tanaka, N., *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan J. Math. **2** (1976), 131-190.

[13] Tanno, S., *Ricci curvature of contact Riemannian manifolds*, Tôhoku Math. J. **40** (1988), 441-448.

[14] ———, *Variational problems on contact Riemannian manifolds,*, Trans. Amer. Math. Soc. **314** (1989), 349-379.

[15] Tricerri, F. and Vanhecke, L., *Homogeneous structures on Riemannian manifolds*, London Math. Soc. Lecture Note Ser. 83, Cambridge University Press, London (1983).

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