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ON CONNECTEDNESS OF GRAPHS ON DIRECT PRODUCT OF WEYL GROUPS

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Abstract. In this paper, we have studied the connectedness of the graphs on the direct product of the Weyl groups. We have shown that the number of the connected components of the graph on the direct product of the Weyl groups is equal to the product of the numbers of the connected components of the graphs on the factors of the direct product. In particular, we show that the graph on the direct product of the Weyl groups is connected iff the graph on each factor of the direct product is connected.

1. Introduction.

In this paper, the connectedness of the graphs on the direct product of the Weyl groups is investigated. It is shown that the number of the connected components of the graph on the direct product of the Weyl groups is equal to the product of the numbers of the connected components of the graphs on the factors of the direct product. From this we deduce that the graph on the direct product of the Weyl groups is connected iff the graph on each factor of the direct product is connected. The graph on Weyl groups has been defined and studied in [1]. The relevant definitions and the results on the Weyl groups can be found in [2]. We have used the notations given in [3]. We briefly summarize below the required results and the notations.

Let E be a fixed euclidean space i.e., E is a finite dimensional vector space over real numbers and has a positive definite symmetric bilinear form $(,)$. Let dimension of E be n. Given any vector $\alpha \in E$ we can define a reflection R_α in E given by $xR_\alpha = x - (x, \alpha^\vee)\alpha$ where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ for $x \in E$. The reflection R_α is an invertible linear transformation which leaves the plane $P_\alpha = \{y \in E | (\alpha, y) = 0\}$ invariant and any nonzero vector parallel to α is sent to its negative. Also R_α preserves the inner product $(,)$ on E i.e., it is an orthogonal linear transformation. A finite subset Δ of nonzero vectors of E is called a root system in E if the following holds : (1) Δ spans E and $\alpha \in \Delta$ implies $k\alpha \in \Delta$ only if $k = \pm 1$. (2) If

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$\alpha \in \Delta$ then the reflection R_α leaves Δ invariant i.e., vectors of Δ are transformed by R_α into vectors of Δ . (3) If $\alpha, \beta \in \Delta$ then $(\beta, \alpha^\vee)$ is an integer.

If $\alpha, \beta \in \Delta$ then the condition (3) restricts the values of $(\alpha, \beta^\vee)(\beta, \alpha^\vee)$ to 0, 1, 2, and 3 only. The hyperplane $P_\alpha, \alpha \in \Delta$ partitions E into finitely many regions. The connected components of $E - \bigcup_{\alpha \in \Delta} P_\alpha$ are called the Weyl chambers of E.

Let Δ be a root system in E. The group generated by the reflections R_α for $\alpha \in \Delta$ is called the Weyl group $W(\Delta)$ of Δ. Since $W(\Delta)$ permutes the vectors in Δ, by the condition (3) on Δ, we can identify the Weyl group as the subgroup of the permutation group on Δ. In particular, the Weyl group $W(\Delta)$ is a finite group.

It may be recalled that if Δ is a root system in E of dimension n then it is possible to choose the set of simple roots $\alpha_1, \alpha_2, ..., \alpha_n$ i.e., these roots form a basis of E and any root β in Δ can be written as a linear combination of $\alpha_1, \alpha_2, ..., \alpha_n$ with integral coefficients all nonnegative or all nonpositive. Then the Weyl group $W(\Delta)$ is generated by the reflections $R_{\alpha_i}, i = 1, 2, ...n$. We write $R_{\alpha_i} = R_i, i = 1, 2, ..., n$.

A root system Δ is called irreducible if it cannot be written as a union of two proper subsets Δ_1 and Δ_2 such that each root in Δ_1 is orthogonal to each root in Δ_2. Otherwise Δ is called reducible. Therefore , if Δ is reducible then $\Delta = \Delta_1 \cup \Delta_2$ such that each root in Δ_1 is orthogonal to each root in Δ_2. Further, if Δ is reducible then the simple roots of Δ can also be partitioned into the two sets S_1 and S_2 such that a simple root in S_1 is orthogonal to every simple root in S_2. Also the Weyl group $W(\Delta) = W(\Delta_1) \times W(\Delta_2)$ and each $W(\Delta_i)$ is generated by the simple roots in Δ_i i.e., S_i.

We know that if Δ is a root system then for $\alpha, \beta \in \Delta$, $(\alpha, \beta^\vee)(\beta, \alpha^\vee)$ takes the values 0, 1, 2, or 3. We define a Coxeter graph of Δ to be a graph which has n vertices and for $i \neq j$, ith vertex is joined to the jth vertex by $(\alpha_i, \alpha_j^\vee)(\alpha_j, \alpha_i^\vee)$ number of edges. It is obvious that the Coxeter graph is connected iff Δ is an irreducible root system. The order of the element $R_i R_j$ of $W(\Delta)$ is 2, 3, 4, or 6 according as $(\alpha_i, \alpha_j^\vee)(\alpha_j, \alpha_i^\vee)$ takes the values 0 , 1, 2 or 3 respectively. Now the lengths of the simple roots may not be equal. Therefore, in Coxeter graph we add an arrow to an edge which points to the shorter root. This resulting graph is called the Dynkin diagram of Δ . The Dynkin diagram of Δ also determines the Weyl group $W(\Delta)$ completely.

The classification theorem of irreducible root systems shows that if Δ is irreducible then its Dynkin diagram is one of the following types :

A_n for $n \geq 1$, B_n for $n \geq 2$, C_n for $n \geq 3$, D_n for $n \geq 4$, E_6, E_7, E_8, F_4 and G_2 .

The type of the irreducible root system Δ is defined to be same as the type of its Dynkin diagram. If $\alpha_1, \alpha_2, ..., \alpha_n$ are the simple roots of Δ we define the fundamental weights $\lambda_1, \lambda_2, ..., \lambda_n$ of Δ by $(\lambda_i, \alpha_j^\vee) = \delta_{i,j}$ (Kronecker delta). We have $\lambda_i R_j = \lambda_i - \delta_{i,j}\alpha_j$ for $i, j = 1, 2, ..., n$. Let $\sigma \in W$. Then σ can be written as a product of the generators $R_1, R_2, ..., R_n$. There is more than one way of writing σ as a product of the generators. Suppose $\sigma = R_{i_1} R_{i_2} ... R_{i_k}$. The minimum value of k is called the length $\ell(\sigma)$ of σ. There is a unique element $\sigma_0 \in W$ which has maximum length. For $\sigma \in W$ we define $I_\sigma = \{i | 1 \leq i \leq n, \ell(\sigma R_i) < \ell(\sigma)\}$. Let $\delta_\sigma = \sum\limits_{i \in I_\sigma} \lambda_i$. Define $\epsilon_\sigma = \delta_\sigma \sigma^{-1}$. Finally, let $D(\lambda), \lambda \in E$ be the Weyl's dimension polynomial. Then it is known that

$$D(\lambda) = \frac{\prod\limits_{\alpha \in \Delta^+} (\lambda, \alpha^\vee)}{\prod\limits_{\alpha \in \Delta^+} (\delta, \alpha^\vee)}$$

where Δ^+ is the set of positive roots of Δ and $\delta = \sum\limits_{i=1}^{n} \lambda_i$.

We define the graph $\Gamma(W(\Delta))$ on the Weyl group $W(\Delta)$ whose vertices are elements of the Weyl group. We define the edges of this graph, with the help of the underlying root system Δ , as described below. For convenience we write W for $W(\Delta)$. A point $\lambda \in E$ is called W-regular iff $D(\lambda) \neq 0$ which is equivalent to saying that λ lies in the interior of a Weyl chamber of Δ. Recall that σ_0 is the unique element of W with maximal length. First we define a relation $\longrightarrow$ on W. For $\sigma, \tau \in W$ define $\sigma \longrightarrow \tau$ iff $-\epsilon_{\sigma\sigma_0} + \epsilon_\tau$ is W-regular. It easily follows that $\sigma \longrightarrow \sigma$ for all $\sigma \in W$, since $-\epsilon_{\sigma\sigma_0} = (\delta - \delta_\sigma)\sigma^{-1}$ [4]. We construct the graph $\Gamma(W(\Delta))$ by using the relation $\longrightarrow$ on W. For $\sigma, \tau \in W$ with $\sigma \neq \tau$ an edge (σ, τ) is an unordered pair where either $\sigma \longrightarrow \tau$ or $\tau \longrightarrow \sigma$. It is proved in [4] that at most one of $\sigma \longrightarrow \tau$ or $\tau \longrightarrow \sigma$ holds for $\sigma \neq \tau$. Thus we get at most one edge joining distinct σ and τ in $\Gamma(W(\Delta))$. We write $\Gamma(W(\Delta))$ as $\Gamma(W)$ or $\Gamma(\Delta)$ depending on the context. It should be noted that this graph depends on the Δ. If the root system Δ is of type X , we write $W(\Delta)$ as $W(X)$ and the graph $\Gamma(\Delta)$ as $\Gamma(X)$. For example $\Gamma(G_2)$ means the graph on the Weyl group $W(G_2)$ whose underlying root system is of type G_2. It is interesting to note that for $n \geq 3$ the graphs $\Gamma(B_n)$ and $\Gamma(C_n)$ are distinct although the Weyl groups $W(B_n)$ and $W(C_n)$ are isomorphic.

2. The connectedness of $\Gamma(W)$.

Let Δ be a union of two root systems Δ_1 and Δ_2. We write this as $\Delta = \Delta_1 \times \Delta_2$. In this case the Dynkin diagrams of Δ_1 and Δ_2 are disjoint. Also $W(\Delta) = W(\Delta_1) \times W(\Delta_2)$, the direct product. Let $W = W(\Delta), W_1 = W(\Delta_1)$ and $W_2 = W(\Delta_2)$. If $\rho \in W$ then $\rho = \sigma\tau$ with unique σ, τ and $\sigma \in W_1, \tau \in W_2$. From the definition of I_ρ it easily follows that $I_\rho = I_\sigma \cup I_\tau$ (disjoint union) and $\delta_\rho = \delta_\sigma \oplus \delta_\tau$ (direct sum), which gives $\epsilon_\rho = \epsilon_\sigma \oplus \epsilon_\tau$. Therefore $\epsilon_{\sigma\tau} = \epsilon_\sigma \oplus \epsilon_\tau$ for $\sigma \in W_1$ and $\tau \in W_2$. If δ, δ_1 and δ_2 are the sums of the fundamental weights of Δ, Δ_1 and Δ_2 respectively then $\delta = \delta_1 \oplus \delta_2$. If $\sigma_0^{'}$ and $\sigma_0^{''}$ are the unique elements of maximal length in W_1 and W_2 respectively, then $\sigma_0 = \sigma_0^{'}\sigma_0^{''}$. These results can be generalized to the case when Δ is union of more than two root systems. With above notations we have the following result.

Lemma 1. *Let $\sigma_1, \sigma_2 \in W_1$ and $\tau_1, \tau_2 \in W_2$. Then $\sigma_1 \longrightarrow \sigma_2$ in W_1 and $\tau_1 \longrightarrow \tau_2$ in W_2 iff $\sigma_1\tau_1 \longrightarrow \sigma_2\tau_2$ in W.*

Proof. We have the following equalities.
$-\epsilon_{\sigma_1\tau_1\sigma_0}+\epsilon_{\sigma_2\tau_2}=(\delta-\delta_{\sigma_1\tau_1})(\sigma_1\tau_1)^{-1}+\epsilon_{\sigma_2\tau_2}=(\delta_1+\delta_2-\delta_{\sigma_1}-\delta_{\tau_1})\sigma_1^{-1}\tau_1^{-1}+(\epsilon_{\sigma_2}\oplus\epsilon_{\tau_2})=((\delta_1-\delta_{\sigma_1})\sigma_1^{-1}\oplus(\delta_2-\delta_{\tau_1})\tau_1^{-1})+(\epsilon_{\sigma_2}\oplus\epsilon_{\tau_2})=(-\epsilon_{\sigma_1\sigma_0'}+\epsilon_{\sigma_2})\oplus(-\epsilon_{\tau_1\sigma_0''}+\epsilon_{\tau_2})$ since $\lambda_i R_j=\lambda_i$ for $j\neq i$, and $\epsilon_{\sigma\sigma_0}=-(\delta-\delta_\sigma)\sigma^{-1}$, [4]. This shows that $-\epsilon_{\sigma_1\tau_1\sigma_0}+\epsilon_{\sigma_2\tau_2}$ is in the interior of a Weyl chamber of Δ iff $-\epsilon_{\sigma_1\sigma_0'}+\epsilon_{\sigma_2}$ and $-\epsilon_{\tau_1\sigma_0''}+\epsilon_{\tau_2}$ are in the interior of some Weyl chamber of Δ_1 and Δ_2 respectively. In other words $\sigma_1\longrightarrow\sigma_2$ in W_1 and $\tau_1\longrightarrow\tau_2$ in W_2 iff $\sigma_1\tau_1\longrightarrow\sigma_2\tau_2$ in W. $\square$

Remark. We can easily generalize the above result when $W=W_1\times W_2\times\cdots\times W_k$.

Let C be a subset of the Weyl group W. We write $\Gamma(C)$ for the induced subgraph on C . It easily follows that if Γ_1 is a connected component of $\Gamma(W)$ then $\Gamma_1=\Gamma_1(C_1)$ for a unique subset C_1 of W.

Theorem 1. *Let Γ_1 and Γ_2 be connected components of $\Gamma(W_1)$ and $\Gamma(W_2)$ respectively. Suppose $\Gamma_1=\Gamma_1(C_1)$ and $\Gamma_2=\Gamma_2(C_2)$ for (unique) subsets C_1 of W_1 and C_2 of W_2. Suppose $C_1\times C_2=\{\sigma\tau|\sigma\in C_1,\tau\in C_2\}$. Then $\Gamma(C_1\times C_2)$ is a connected component of $\Gamma(W_1\times W_2)$.*

Proof. Suppose $\rho_1,\rho_2\in C_1\times C_2$. We show that ρ_1 is connected to ρ_2. Now $\rho_1=\sigma_1\tau_1$ and $\rho_2=\sigma_2\tau_2$ where $\sigma_1,\sigma_2\in C_1$ and $\tau_1,\tau_2\in C_2$. Since $\sigma_1,\sigma_2\in C_1$ they are connected in $\Gamma_1(C_1)$. Similarly, τ_1,τ_2 are connected in $\Gamma_2(C_2)$. Therefore,

$$\sigma_1\longrightarrow\sigma_2'\longrightarrow\cdots\longrightarrow\sigma_m'\longrightarrow\sigma_2 \tag{1}$$

and

$$\tau_1\longrightarrow\tau_2'\longrightarrow\cdots\longrightarrow\tau_r'\longrightarrow\tau_2 \tag{2}$$

for some $\sigma_2',\ldots,\sigma_m'\in C_1$ and $\tau_2',\ldots,\tau_r'\in C_2$. By the repeated application of the lemma, Eqn.(1) gives

$$\sigma_1\tau_1\longrightarrow\sigma_2'\tau_1\longrightarrow\cdots\longrightarrow\sigma_m'\tau_1\longrightarrow\sigma_2\tau_1$$

and Eqn.(2) gives

$$\sigma_2\tau_1\longrightarrow\sigma_2\tau_2'\longrightarrow\cdots\longrightarrow\sigma_2\tau_r'\longrightarrow\sigma_2\tau_2$$

which implies that $\rho_1=\sigma_1\tau_1$ is connected to $\rho_2=\sigma_2\tau_2$.

Suppose $\rho\in C_1\times C_2$ is connected to $\rho'\in W_1\times W_2$. We show that $\rho'\in C_1\times C_2$. Suppose $\rho=\sigma_1\tau_1$ and $\rho'=\sigma'\tau'$ where $\sigma_1\in C_1,\tau_1\in C_2,\sigma'\in W_1$ and $\tau'\in W_2$. Since ρ is connected to ρ', we have

$$\rho\longrightarrow\rho_1'\longrightarrow\rho_2'\longrightarrow\cdots\longrightarrow\rho_m'\longrightarrow\rho' \tag{3}$$

where $\rho_1',\rho_2',\ldots,\rho_m'\in W_1\times W_2$. Suppose, for $i=1,\ldots,m$, $\rho_i'=\sigma_i'\tau_i'$ where $\sigma_i'\in W_1,\tau_i'\in W_2$. Now Eqn.(3) implies that

$$\sigma\tau\longrightarrow\sigma_1'\tau_1'\longrightarrow\sigma_2'\tau_2'\longrightarrow\cdots\longrightarrow\sigma_m'\tau_m'\longrightarrow\sigma'\tau' \tag{4}$$

By the repeated application of the lemma, Eqn.(4) gives

$$\sigma \longrightarrow \sigma_1' \longrightarrow \cdots \longrightarrow \sigma_m' \longrightarrow \sigma' \quad \text{and} \quad \tau \longrightarrow \tau_1' \longrightarrow \cdots \longrightarrow \tau_m' \longrightarrow \tau'.$$

This proves that σ is connected to σ' in $\Gamma(W_1)$ and τ is connected to τ' in $\Gamma(W_2)$. But $\sigma \in C_1$ and $\tau \in C_2$ implies that $\sigma' \in C_1$ and $\tau' \in C_2$ since $\Gamma_1(C_1)$ and $\Gamma_2(C_2)$ are connected components of $\Gamma(W_1)$ and $\Gamma(W_2)$ respectively. Therefore, $\rho' = \sigma'\tau' \in C_1 \times C_2$. This completes the proof. $\square$

Corollary 1. *If $\Gamma(W_1)$ has p components and $\Gamma(W_2)$ has q components then $\Gamma(W_1 \times W_2)$ has pq components.* $\square$

Theorem 2. *Let Γ' be a connected component of $\Gamma(W)$ where $W = W_1 \times W_2$, the direct product of Weyl groups W_1 and W_2. Let C be the (unique) subset of W for which $\Gamma = \Gamma(C)$. Suppose $C_1 = \{\sigma \in W | \sigma\tau_1 \in C$ for some $\tau_1 \in W_2\}$ and $C_2 = \{\tau \in W_2 | \sigma_1\tau \in C$ for some $\sigma_1 \in W_1\}$. Then $\Gamma(C_1)$ and $\Gamma(C_2)$ are connected components of $\Gamma(W_1)$ and $\Gamma(W_2)$ respectively. Further $\Gamma' = \Gamma(C_1 \times C_2)$.*

Proof. First we show that C_1 is a connected component of $\Gamma(W_1)$. Let $\sigma_1, \sigma_2 \in C_1$. Then $\sigma_1\tau_1 \in C$ and $\sigma_2\tau_2 \in C$ for some $\tau_1, \tau_2 \in W_2$. Since C is a connected component of $\Gamma(W), \sigma_1\tau_1$ is connected to $\sigma_2\tau_2$ in $\Gamma(C)$. Therefore,

$$\sigma_1\tau_1 \longrightarrow \sigma_2'\tau_2' \longrightarrow \cdots \longrightarrow \sigma_m'\tau_m' \longrightarrow \sigma_2\tau_2$$

for some $\sigma_i'\tau_i' \in C$ i.e., $\sigma_i' \in C_1$ and $\tau_i' \in C_2$ for $i = 2, \ldots, m$. By the lemma, $\sigma_1 \longrightarrow \sigma_2' \longrightarrow \cdots \longrightarrow \sigma_m' \longrightarrow \sigma_2$ i.e., σ_1 is connected to σ_2 in C_1.

Let $\sigma \in C_1$ and σ be connected to $\sigma' \in W_1$. We show that $\sigma' \in C_1$. Now $\sigma \in C_1$ implies that $\sigma\tau \in C$ for some $\tau \in W_2$. Also σ connected to σ' in W_1 gives $\sigma \longrightarrow \sigma_1'' \longrightarrow \sigma_2'' \longrightarrow \cdots \longrightarrow \sigma_m'' \longrightarrow \sigma'$ where $\sigma_i'' \in W_1$ for $i = 1, \ldots, m$. By the lemma, $\sigma\tau \longrightarrow \sigma_1''\tau \longrightarrow \cdots \longrightarrow \sigma_m''\tau \longrightarrow \sigma'\tau$. Therefore, $\sigma\tau$ is connected to $\sigma'\tau$ in $\Gamma(W)$. Since $\sigma\tau \in C$ and C is a connected component of $\Gamma(W), \sigma'\tau \in C$ and therefore, $\sigma' \in C_1$. This shows that C_1 is a connected component of $\Gamma(W_1)$. Similarly we can show that C_2 is a connected component of $\Gamma(W_2)$. It trivially follows that $\Gamma' = \Gamma(C_1 \times C_2)$.

¿From theorem 1 and theorem 2 we can easily prove the following.

Theorem 3. *Let W_1 and W_2 be the Weyl groups. Then $\Gamma(W_1)$ and $\Gamma(W_2)$ are connected iff $\Gamma(W_1 \times W_2)$ is connected.* $\square$

We have the following information about $\Gamma(\Delta)$ when Δ is an irreducible root system of low rank. The graphs $\Gamma(A_1)$ and $\Gamma(A_2)$ are totally disconnected with 2 and 6 vertices respectively. $\Gamma(B_2)$ has 4 disjoint edges and $\Gamma(A_3)$ has 8 disconnected vertices and 8 disjoint edges. $\Gamma(G_2)$ is a connected graph. The graphs $\Gamma(B_3), \Gamma(B_4), \Gamma(C_3), \Gamma(C_4)$ and $\Gamma(D_4)$ are connected. Note that the groups $W(B_4)$, $W(C_4)$ and $W(D_4)$ are of order 384, 384 and 192 respectively. In all these graphs we have used the " Fusion Method " to determine the connectivity [5]. We have also shown [6] that $\Gamma(A_n), n \geq 4$ is a connected graph. This strongly suggests the following

Conjecture. If Δ is an irreducible root system which is not of type A_1, A_2, A_3 or B_2 then $\Gamma(\Delta)$ is a connected graph.

Assuming the truth of the conjecture, from the theorem 3 we have

Theorem 4. *If $\Delta = \Delta_1 \times \Delta_2 \times \cdots \times \Delta_k$ where Δ_i are irreducible root systems which are not of the type A_1, A_2, A_3 or B_2 then $\Gamma(\Delta)$ is a connected graph.* $\square$

Remark. If Δ has components of type A_1, A_2, A_3 or B_2 then one can easily write the number of components of $\Gamma(\Delta)$ by using the corollary of theorem 1.

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