ARCHIVUM MATHEMATICUM (BRNO)
Tomus 31 (1995), 263 – 277

CLASSIFICATION OF NONOSCILLATORY SOLUTIONS OF HIGHER ORDER NEUTRAL TYPE DIFFERENCE EQUATIONS

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Abstract. The authors consider the difference equation

$$(*) \qquad \Delta^m[y_n - p_n y_{n-k}] + \delta q_n y_{\sigma(n+m-1)} = 0$$

where $m \geq 2$, $\delta = \pm 1$, $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, $\Delta y_n = y_{n+1} - y_n$, $q_n > 0$, and $\{\sigma(n)\}$ is a sequence of integers with $\sigma(n) \leq n$ and $\lim_{n\to\infty} \sigma(n) = \infty$. They obtain results on the classification of the set of nonoscillatory solutions of $(*)$ and use a fixed point method to show the existence of solutions having certain types of asymptotic behavior. Examples illustrating the results are included.

1. Introduction

This paper is concerned with the asymptotic behavior of nonoscillatory solutions of neutral linear difference equations of the type

$$\text{(E)} \qquad \Delta^m[y_n - p_n y_{n-k}] + \delta q_n y_{\sigma(n+m-1)} = 0$$

where $m \geq 2$, $\delta = \pm 1$, $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, and Δ denotes the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$ and $\Delta^i y_n = \Delta(\Delta^{i-1} y_n)$, $1 \leq i \leq m$. The following conditions are assumed to hold throughout the remainder of this paper. There is an $n_0 \in \mathbb{N}_0$ such that:

(c_1) $\{p_n\}$ is a real sequence satisfying $|p_n| \leq \lambda < 1$ for all $n \geq n_0$;
(c_2) $\{\sigma(n)\}$ is a sequence of integers, $\sigma(n) \leq n$ for $n \geq n_0$, and $\lim_{n\to\infty} \sigma(n) = \infty$;
(c_3) $\{q_n\}$ is a real sequence and $q_n > 0$ for all $n \geq n_0$.

By a solution of equation (E), we mean a sequence $\{y_n\}$ of real numbers defined for $n \geq n_0 - m + 1 - \max\{k, \min_{i\in\mathbb{N}_0}\{\sigma(i + m - 1)\}\}$ and which satisfies (E) for

1991 *Mathematics Subject Classification*: 39A10, 39A11.
Key words and phrases: difference equations, nonlinear, asymptotic behavior, nonoscillatory solutions.
*Research supported by the Mississippi State University Biological and Physical Sciences Research Institute.
Received August 1, 1994.

all $n \in \mathbb{N}_0$. A solution of (E) is said to be *nonoscillatory* if it is either eventually positive or eventually negative. Otherwise, it is called *oscillatory*.

In recent years there has been an increasing interest in oscillation theory of difference equations of neutral type; see, for example, [1–17] and the references contained therein. Most of the literature, however, is focused on first and second order equations with relatively few results available for higher order equations. The purpose of this paper is to classify the possible nonoscillatory solutions of (E) according to their asymptotic behavior as $n \to \infty$ and to give necessary conditions for the existence of nonoscillatory solutions $\{y_n\}$ having the following types of asymptotic behavior:

$$(\mathrm{I}_j) \qquad \lim_{n\to\infty} \frac{y_n - p_n y_{n-k}}{n^{(j)}} = \text{ constant} \neq 0 \text{ for some } j \in \{0, 1, \ldots, m-1\},$$

or

$$(\mathrm{II}_\ell) \qquad \lim_{n\to\infty} \frac{y_n - p_n y_{n-k}}{n^{(\ell)}} = 0, \quad \lim_{n\to\infty} \frac{y_n - p_n y_{n-k}}{n^{(\ell-1)}} = \pm\infty$$

for some $\ell \in \{1, 2, \ldots, m-1\}$ where $n^{(j)}$ is the usual generalized factorial notation. In addition, using a fixed point technique, we are able to give sufficient conditions for the existence of a nonoscillatory solution of the types I_j and II_ℓ.

2. Classification of Nonoscillatory Solutions

We begin by classifying the asymptotic behavior of nonoscillatory solutions of (E) on the basis of a discrete analogue of Kiguradze's Lemma [17] (also see [1; Theorem 1.7.11]).

Lemma 2.1. *Let $\{x_n\}$ be a sequence of real numbers and let x_n and $\Delta^m x_n$ be of constant sign with $\Delta^m x_n$ not eventually identically zero. If*

$$\delta x_n \Delta^m x_n < 0, \tag{2.1}$$

then there exist integers $\ell \in \{0, 1, 2, \ldots, m\}$ and $N > 0$ such that $(-1)^{m-\ell-1}\delta = 1$ and

$$\begin{aligned} &x_n \Delta^j x_n > 0 \text{ for } j = 0, 1, \ldots, \ell, \\ &(-1)^{j-\ell} x_n \Delta^j x_n > 0 \text{ for } j = \ell+1, \ldots, m \end{aligned} \tag{2.2}$$

for $n \geq N$.

A sequence $\{x_n\}$ satisfying (2.2) is called a sequence of (Kiguradze) degree ℓ. The possible asymptotic behaviors of a sequence of degree ℓ are as follows.

(i) If $\ell = 0$ (which is possible only when $\delta = 1$ and m is odd or $\delta = -1$ and m is even), then either

$$\lim_{n\to\infty} x_n = \text{ constant } \neq 0 \text{ or } \lim_{n\to\infty} x_n = 0.$$

(ii) If $1 \le \ell \le m-1$, then one of the following three cases holds:

$$\lim_{n\to\infty} \frac{x_n}{n^{(\ell)}} = \text{ constant } \neq 0$$

$$\lim_{n\to\infty} \frac{x_n}{n^{(\ell-1)}} = \text{ constant } \neq 0$$

$$\lim_{n\to\infty} \frac{x_n}{n^{(\ell)}} = 0 \text{ and } \lim_{n\to\infty} \frac{x_n}{n^{(\ell-1)}} = \pm\infty.$$

(iii) If $\ell = m$ (which is possible only when $\delta = -1$), then

$$\lim_{n\to\infty} \frac{x_n}{n^{(m-1)}} = \pm\infty.$$

Let $\{y_n\}$ be a nonoscillatory solution of equation (E). Clearly, $y_n - p_n y_{n-k}$ is eventually of one sign, so either

$$y_n(y_n - p_n y_{n-k}) > 0 \tag{2.3}$$

or

$$y_n(y_n - p_n y_{n-k}) < 0 \tag{2.4}$$

for all sufficiently large n. If (2.3) holds, then the sequence $x_n = y_n - p_n y_{n-k}$ satisfies (2.1) for all large n, so by Lemma 2.1, $\{x_n\}$ is a sequence of Kiguradze degree ℓ for some $\ell \in \{0,1,\ldots,m\}$ and $(-1)^{m-\ell-1}\delta = 1$. Let $\mathcal{N}_\ell^+$ denote the set of solutions $\{y_n\}$ of (E) satisfying (2.3) and for which $y_n - p_n y_{n-k}$ is of degree ℓ. On the other hand, if (2.4) holds, then $x_n = p_n y_{n-k} - y_n$ satisfies (2.1) (with δ repalced by $-\delta$) for all large n. However, the degree of $\{x_n\}$ must be zero. In fact, from (2.4) we have $|y_n| \le |p_n y_{n-k}| \le \lambda |y_{n-k}|$, and hence $|y_{n+jk}| \le \lambda^j |y_n|$, $j = 1, 2, \ldots$, which in turn implies $\lim_{n\to\infty} y_n = 0$. The set of all solutions $\{y_n\}$ of (E) satisfying (2.4) will be denoted by $\mathcal{N}_0^-$. It is clear that the class $\mathcal{N}_0^-$ is empty if $(-1)^{m-1}\delta = 1$, that is, if $\delta = 1$ and m is odd or $\delta = -1$ and m is even. From the above observations, we have the following classification of the set $\mathcal{N}$ of all nonoscillatory solutions of (E):

$$\begin{array}{ll} \mathcal{N} = \mathcal{N}_1^+ \cup \mathcal{N}_3^+ \cup \cdots \cup \mathcal{N}_{m-1}^+ \cup \mathcal{N}_0^- & \text{for } \delta = 1 \text{ and } m \text{ even;} \\ \mathcal{N} = \mathcal{N}_0^+ \cup \mathcal{N}_2^+ \cup \cdots \cup \mathcal{N}_{m-1}^+ & \text{for } \delta = 1 \text{ and } m \text{ odd;} \\ \mathcal{N} = \mathcal{N}_0^+ \cup \mathcal{N}_2^+ \cup \cdots \cup \mathcal{N}_{m}^+ & \text{for } \delta = -1 \text{ and } m \text{ even;} \\ \mathcal{N} = \mathcal{N}_1^+ \cup \mathcal{N}_3^+ \cup \cdots \cup \mathcal{N}_{m}^+ \cup \mathcal{N}_0^- & \text{for } \delta = -1 \text{ and } m \text{ odd.} \end{array} \tag{2.5}$$

We note that if $\{p_n\}$ is either oscillatory or eventually negative, then (E) cannot possess a nonoscillatory solution $\{y_n\}$ satisfying (2.4), so in this case the class $\mathcal{N}_0^-$ should be removed from (2.5).

From the above discussion, it follows that a nonoscillatory solution $\{y_n\}$ of (E) falls into one of the following four cases:

(I) $\lim_{n\to\infty} \dfrac{y_n - p_n y_{n-k}}{n^{(j)}} = \text{constant} \neq 0$ for some $j \in \{0, 1, 2, \ldots, m-1\}$;

(II) $\lim_{n\to\infty} \dfrac{y_n - p_n y_{n-k}}{n^{(l)}} = 0$, $\lim_{n\to\infty} \dfrac{y_n - p_n y_{n-k}}{n^{(l-1)}} = \pm\infty$ for some $l \in \{1, 2, \ldots, m-1\}$ with $(-1)^{m-l-1}\delta = 1$;

(III) $\lim_{n\to\infty} \dfrac{y_n - p_n y_{n-k}}{n^{(m-1)}} = \pm\infty$;

(IV) $\lim_{n\to\infty}[y_n - p_n y_{n-k}] = 0$.

Next, we will see how the asymptotic behavior of $y_n - p_n y_{n-k}$ affects the behavior of the solution $\{y_n\}$ itself. It is enough to consider only the solutions $\{y_n\}$ of (E) satisfying (2.3). Let $\{y_n\}$ be such a solution for $n \geq n_1$. Then $x_n = y_n - p_n y_{n-k}$ satisfies (2.2) for some $\ell \in \{0, 1, \ldots, m\}$ with $(-1)^{m-\ell-1}\ \delta = 1$. Let $n_2 > n_1$ be such that $n - k \geq n_1$ for $n \geq n_2$. Using the relation

$$y_n = x_n + p_n y_{n-k} \tag{2.6}$$

repeatedly, we have

$$y_n = \sum_{i=0}^{j-1} H_i(n) x_{n-ik} + H_j(n) y_{n-jk}, \quad n \geq n_2 \tag{2.7}$$

where j denotes the least positive integer such that $n_1 < n - jk \leq n_2$ and $H_j(n)$, $j = 0, 1, 2, \ldots$, are defined by

$$H_0(n) = 1, \quad H_j(n) = \prod_{i=0}^{j-1} p_{n-ik}, \quad j = 1, 2, \ldots \tag{2.8}$$

From (2.7) and the fact that $|H_j(n)| \leq \lambda^j$, it follows that

$$|y_n| \leq \frac{|x_n|}{1-\lambda} + \eta, \quad n \geq n_2 \tag{2.9}$$

if $\ell \geq 1$, and

$$|y_n| \leq \frac{|x_{n_1}|}{1-\lambda} + \eta, \quad n \geq n_2 \tag{2.10}$$

if $\ell = 0$, where $\eta > 0$ is a constant.

If $\{p_n\}$ is eventually positive, then we have

$$|y_n| \geq |x_n| \text{ for all large } n. \tag{2.11}$$

On the other hand, using (2.6) we obtain

$$y_n = x_n + p_n x_{n-k} + p_n p_{n-k} y_{n-2k},$$

which shows that if

$$p_n p_{n-k} \geq 0 \text{ for all large } n \tag{2.12}$$

and if the Kiguradze degree ℓ of $\{x_n\}$ is positive, then

$$|y_n| \geq (1-\lambda)|x_n| \text{ for all large } n. \tag{2.13}$$

In view of (2.9), (2.10), (2.11) and (2.13), we conclude that under the hypothesis (2.12), the following four types of asymptotic behavior are possible for nonoscillatory solutions $\{y_n\}$ of equation (E):

(A) $0 < \liminf_{n\to\infty} \dfrac{|y_n|}{n^{(j)}} \leq \limsup_{n\to\infty} \dfrac{|y_n|}{n^{(j)}} < \infty$ for some $j \in \{0, 1, \ldots, m-1\}$;

(B) $\lim_{n\to\infty} \dfrac{y_n}{n^{(l)}} = 0$ and $\lim_{n\to\infty} \dfrac{|y_n|}{n^{(l-1)}} = \infty$ for some $l \in \{1, 2, \ldots, m-1\}$ with $(-1)^{m-l-1}\delta = 1$;

(C) $\lim_{n\to\infty} \dfrac{|y_n|}{n^{(m-1)}} = \infty$;

(D) $\lim_{n\to\infty} y_n = 0$.

3. Existence of Nonoscillatory Solutions

The purpose of this section is to obtain criteria for equation (E) to have certain kinds of nonoscillatory solutions. In addition to the fact that our results apply to equations of order m greater than just 1 and 2, the results here differ from previously known work in that we give some necessary and some sufficient conditions for equation (E) to have nonoscillatory solutions with a prescribed asymptotic behavior. By contrast, most other known results are either criteria for all solutions to oscillate or for nonoscillatory solutions $\{y_n\}$ to satisfy broad asymptotic properties such as $y_n \to 0$ or $|y_n| \to \infty$ as $n \to \infty$ (see, for example, Erbe and Zhang [2], Georgiou et al. [3, 4], Lalli et al. [5 – 9], Thandapani et al. [12 – 16], and Zafer and Dahiya [17]). We begin with a necessary condition for the existence of Type I solutions.

Theorem 3.1. *Suppose that (2.12) holds. If equation (E) has a nonoscillatory solution $\{y_n\}$ satisfying (2.3) and*

$$\lim_{n\to\infty} \frac{y_n - p_n y_{n-k}}{n^{(j)}} = \text{constant} \neq 0 \tag{3.1}$$

for some $j \in \{0, 1, \ldots, m-1\}$, i.e., $\{y_n\}$ is a Type I solution, then

$$\sum_{n=n_0}^{\infty} n^{(m-j-1)}(\sigma(n+m-1))^{(j)} q_n < \infty. \tag{3.2}$$

Proof. Let $\{y_n\}$ be a solution of (E) satisfying (2.3) and (3.1). Observe that

$$\lim_{n\to\infty} \Delta^i[y_n - p_n y_{n-k}] = 0, \quad j+1 \leq i \leq m-1.$$

If $j < m-1$, a repeated summation of (E) shows that

$$\sum_{n=N}^{\infty} n^{(m-j-1)} q_n |y_{\sigma(n+m-1)}| < \infty \tag{3.3}$$

provided $N > n_0$ is large enough. If $j = m-1$, a summation of (E) implies that (3.3) holds. On the other hand, from (2.12) and (3.1), we have

$$\liminf_{n\to\infty} \frac{|y_{\sigma(n+m-1)}|}{(\sigma(n+m-1))^{(j)}} > 0. \tag{3.4}$$

Inequality (3.2) then follows from (3.3) and (3.4).

The method of proof to be used for the next theorem involves an application of the Knaster-Tarski fixed point theorem (see, for example, Moore [11]) and the contraction mapping principle. This technique requires that an appropriate operator be defined on the proper function space. We give sufficient conditions for the existence of Type I solutions in case either

$$(-1)^{m-j-1}\delta = 1 \text{ and condition (2.12) holds,} \tag{3.5}$$

or

$$(-1)^{m-j-1}\delta = -1 \text{ and } p_n \geq 0 \text{ for all } n \geq n_0. \tag{3.6}$$

Theorem 3.2. *Suppose that (3.5) or (3.6) holds. Equation (E) has a nonoscillatory solution $\{y_n\}$ satisfying (2.3) and (3.1) for some $j \in \{0, 1, \ldots, m-1\}$ if*

$$\sum_{n=n_0}^{\infty} n^{(m-j-1)} (\sigma(n+m-1))^{(j)} q_n < \infty. \tag{3.7}$$

Proof. Suppose that (3.5) holds. Choose $N > n_0$ so large that

$$N_0 = \min\{N-k, \inf_{i \geq N} \sigma(i+m-1)\} \geq n_0 \tag{3.8}$$

and

$$\sum_{n=N}^{\infty} n^{(m-j-1)} (\sigma(n+m-1))^{(j)} q_n \leq (1-\lambda)^2. \tag{3.9}$$

Consider the Banach space $\ell_\infty^{N_0}$ of all bounded real sequences $Y = \{y_n\}_{n\geq N_0}$ with norm $||Y|| = \sup_{n\geq N_0}(|y_n/\rho_n|)$ where $\rho_n = \frac{(n-N)^{(j)}}{j!}$, $n \geq N_0$. We define a closed bounded subset S_j of $\ell_\infty^{N_0}$ by

(3.10)
$$S_j = \{Y \in \ell_\infty^{N_0} : c \leq y_n/\rho_n \leq \frac{c}{\lambda} \text{ for } n \geq N+1 \text{ and } y_n = y_N \text{ for } N_0 \leq n \leq N\}$$

where $c > 0$ is an arbitrary but fixed constant. We define a partial order on $\ell_\infty^{N_0}$ in the usual way. Thus, if for any $X = \{x_n\}$, $Y = \{y_n\} \in \ell_\infty^{N_0}$, $x_n = y_n$ for all sufficiently large n, we will consider such sequences to be the same. Then, for every subset A of S_j both $\inf A$ and $\sup A$ exist and belong to S_j. With each $\{y_n\} \in S_j$, we associate a real sequence $\{\bar{y}_n\}$ defined by

$$\bar{y}_n = \begin{cases} \displaystyle\sum_{s=0}^{i-1} H_s(n) y_{n-sk} + \frac{y_N}{1-p_N} H_i(n), & n \geq N+1 \\ \displaystyle\frac{y_N}{1-p_N}, & N_0 \leq n \leq N \end{cases} \tag{3.11}$$

where i is the least positive integer such that $N_0 < n - ik \leq N$ and $H_s(n)$, $s = 0, 1, 2, \ldots$ are given by (2.8). It is easy to verify that $\{\bar{y}_n\}$ is positive and satisfies the equation

$$\bar{y}_n - p_n \bar{y}_{n-k} = y_n, \quad n \geq N. \tag{3.12}$$

Now define the mapping $T : S_j \to \ell_\infty^{N_0}$ as follows: if $j \geq 1$, then

(3.13)

$$(Ty)_n = \begin{cases} \displaystyle c\rho_n + \sum_{r=N}^{n-j} \frac{(n-r-1)^{(j-1)}}{(j-1)!} \sum_{i=r}^{\infty} \frac{(i-r+m-j-1)^{(m-j-1)}}{(m-j-1)!} q_i \bar{y}_{\sigma(i+m-1)}, \\ \qquad n \geq N \\ 0, \qquad N_0 \leq n \leq N; \end{cases}$$

if $j = 0$, then

$$(Ty)_n = \begin{cases} \displaystyle c + \sum_{i=n}^{\infty} \frac{(i+m-1-n)^{(m-1)}}{(m-1)!} q_i \bar{y}_{\sigma(i+m-1)}, & n \geq N \\ \displaystyle c + \sum_{i=N}^{\infty} \frac{(i+m-1-N)^{(m-1)}}{(m-1)!} q_i \bar{y}_{\sigma(i+m-1)}, & N_0 \leq n \leq N. \end{cases} \tag{3.14}$$

Letting $y_n \in S_j$ and using (3.11), we have

$$0 \leq \bar{y}_{\sigma(n+m-1)} \leq \sum_{s=0}^{i-1} \lambda^s c \frac{(\sigma(s+m-1))^{(j)}}{\lambda j!} + \frac{\lambda^i y_N}{1-\lambda},$$

which implies that for all $n \geq N$

$$0 \leq \bar{y}_{\sigma(n+m-1)} \leq \frac{c}{\lambda(1-\lambda)} (\sigma(n+m-1))^{(j)}$$

if $j \geq 1$, and

$$0 \leq \bar{y}_{\sigma(n+m-1)} \leq \frac{c}{\lambda(1-\lambda)}$$

if $j = 0$. From the above inequalities and (3.9), we see that for $n \geq N$, if $j \geq 1$, then

$$\begin{aligned}
0 &\leq \sum_{r=N}^{n-j} \frac{(n-r-1)^{(j-1)}}{(j-1)!} \sum_{i=r}^{\infty} \frac{(i-r+m-j-1)^{(m-j-1)}}{(m-j-1)!} q_i \bar{y}_{\sigma(i+m-1)} \\
&\leq \sum_{r=N}^{n-j} \frac{(n-r-1)^{(j-1)}}{(j-1)!} \sum_{i=N}^{\infty} i^{(m-j-1)} q_i \frac{c}{\lambda(1-\lambda)} (\sigma(i+m-1))^{(j)} \\
&\leq \frac{(1-\lambda)c}{\lambda} \rho_n,
\end{aligned}$$

and if $j = 0$, then

$$0 \leq \sum_{i=n}^{\infty} \frac{(i+m-1-n)^{(m-1)}}{(m-1)!} q_i \bar{y}_{\sigma(i+m-1)} \leq \sum_{i=N}^{\infty} i^{(m-1)} \frac{c}{\lambda(1-\lambda)} q_i \leq \frac{c(1-\lambda)}{\lambda}.$$

Using these inequalities in (3.13) and (3.14), we conclude that $T(S_j) \subset S_j$ and that T is an increasing mapping. By the Knaster-Tarski fixed point theorem [11], there exists $\{y_n^*\} \in S_j$ such that $(Ty^*)_n = y_n^*$. That is,

$$y_n^* = \begin{cases} c\rho_n + \displaystyle\sum_{r=N}^{n-j} \frac{(n-r-1)^{(j-1)}}{(j-1)!} \sum_{i=r}^{\infty} \frac{(i-r+m-j-1)^{(m-j-1)}}{(m-j-1)!} q_i \bar{y}_{\sigma(i+m-1)}, \\ \qquad j \geq 1 \\ c + \displaystyle\sum_{i=n}^{\infty} \frac{(i-n+m-1)^{(m-1)}}{(m-1)!} q_i \bar{y}^*_{\sigma(i+m-1)}, \\ \qquad j = 0 \end{cases}$$

for $n \geq N$. From (3.12)–(3.14) we obtain

$$\begin{aligned}
&\bar{y}_n^* - p_n \bar{y}_{n-k}^* \\
&\quad = c\rho_n + \sum_{r=N}^{n-j} \frac{(n-r-1)^{(j-1)}}{(j-1)!} \sum_{i=r}^{\infty} \frac{(i-r+m-j-1)^{(m-j-1)}}{(m-j-1)!} q_i \bar{y}_{\sigma(i+m-1)}
\end{aligned}$$

if $j \geq 1$, and

$$\bar{y}_n^* - p_n \bar{y}_{n-k}^* = c + \sum_{i=n}^{\infty} \frac{(i-n+m-1)^{(m-1)}}{(m-1)!} q_i \bar{y}^*_{\sigma(i+m-1)}$$

if $j = 0$, and we see that $\{\bar{y}_n^*\}$ is a positive solution of (E) satisfying (2.3) and (3.1).

Now suppose (3.6) holds. In the above proof, instead of (3.13) and (3.14), for $j \geq 1$ we define

$$(Ty)_n = \begin{cases} \dfrac{c}{\lambda}\rho_n - \displaystyle\sum_{r=N}^{n-j} \frac{(n-r-1)^{(j-1)}}{(j-1)!} \sum_{i=r}^{\infty} \frac{(i-r+m-j-1)^{(m-j-1)}}{(m-j-1)!} q_i \bar{y}_{\sigma(i+m-1)}, \\ \qquad n \geq N \\ 0, \qquad N_0 \leq n \leq N, \end{cases}$$

and for $j = 0$

$$(Ty)_n = \begin{cases} \dfrac{c}{\lambda} - \displaystyle\sum_{i=n}^{\infty} \frac{(i+m-1-n)^{(m-1)}}{(m-1)!} q_i \bar{y}_{\sigma(i+m-1)}, & n \geq N \\ \dfrac{c}{\lambda} - \displaystyle\sum_{i=N}^{\infty} \frac{(i+m-1-N)^{(m-1)}}{(m-1)!} q_i \bar{y}_{\sigma(i+m-1)}, & N_0 \leq n \leq N. \end{cases}$$

Then, as above, $T(S_j) \subset S_j$. Next, we show that the operator T is a contraction on S_j. First note that from (3.12) we have $\bar{y}_n \geq y_n$. Now, for $Y = \{y_n\}$, $X = \{x_n\} \in S_j$, we have

$$\begin{aligned} \frac{1}{\rho_n}|(Ty)_n - (Tx)_n| &\leq \frac{1}{\rho_n} \sum_{r=N}^{n-j} \frac{(n-r-1)^{(j-1)}}{(j-1)!} \sum_{i=r}^{\infty} \frac{(i-r+m-j-1)^{(m-j-1)}}{(m-j-1)!} \\ &\qquad \times q_i |y_{\sigma(i+m-1)} - x_{\sigma(i+m-1)}| \\ &\leq \frac{1}{\rho_n} \sum_{r=N}^{n-j} \frac{(n-r-1)^{(j-1)}}{(j-1)!} \sum_{i=r}^{\infty} \frac{(i-r+m-j-1)^{(m-j-1)}}{(m-j-1)!} \\ &\qquad \times q_i (\sigma(i+m-1))^{(j)} \left| \frac{y_{\sigma(i+m-1)}}{\rho_{\sigma(i+m-1)}} - \frac{x_{\sigma(i+m-1)}}{\rho_{\sigma(i+m-1)}} \right| \\ &\qquad \leq (1-\lambda)^2 ||Y - X||, \end{aligned}$$

so

$$||TY - TX|| \leq (1-\lambda)^2 ||Y - X||.$$

That is, T is a contraction on S_j for $j \geq 1$. Similarly, we can prove that T is a contraction on S_0. Thus, for $j \geq 0$, T has a unique fixed point in S_j. That is, there exists $\{y_n^*\} \in S_j$ such that $(Ty^*)_n = y_n^*$. Hence, for $n \geq N$,

$$y_n^* = \begin{cases} \dfrac{c}{\lambda}\rho_n - \displaystyle\sum_{r=N}^{n-j} \frac{(n-r-1)^{(j-1)}}{(j-1)!} \sum_{i=r}^{\infty} \frac{(i-r+m-j-1)^{(m-j-1)}}{(m-j-1)!} \; q_i \bar{y}^*_{\sigma(i+m-1)}, \\ \qquad j \geq 1 \\ \dfrac{c}{\lambda} - \displaystyle\sum_{i=n}^{\infty} \frac{(i-n+m-1)^{(m-1)}}{(m-1)!} \; q_i \bar{y}^*_{\sigma(i+m-1)}, \\ \qquad j = 0. \end{cases}$$

From (3.12) with $y = y^*$, we obtain

$$\begin{aligned}\bar{y}_n^* &- p_n \bar{y}_{n-k}^* \\ &= \frac{c}{\lambda}\rho_n - \sum_{r=N}^{n-j} \frac{(n-r-1)^{(j-1)}}{(j-1)!} \sum_{i=r}^{\infty} \frac{(i-r+m-j-1)^{(m-j-1)}}{(m-j-1)!} q_i \bar{y}_{\sigma(i+m-1)}^*\end{aligned}$$

for $j \geq 1$, and

$$\bar{y}_n^* - p_n \bar{y}_{n-k}^* = \frac{c}{\lambda} - \sum_{i=n}^{\infty} \frac{(i-n+m-1)^{(m-1)}}{(m-1)!} q_i \bar{y}_{\sigma(i+m-1)}^*$$

for $j = 0$. Once again, $\{\bar{y}_n^*\}$ is a positive solution of (E) satisfying (2.3) and (3.1). This completes the proof of the theorem.

Remark. To this point in time, we have been unable to obtain sufficient conditions for the existence of Type I solutions when $(-1)^{m-j-1}\delta = -1$ and $\{p_n\}$ is eventually negative. Such a result would be of interest.

We now consider nonoscillatory solutions of Type II, that is, those solutions $\{y_n\}$ which satisfy (2.3) and

$$\lim_{n\to\infty} \frac{y_n - p_n y_{n-k}}{n^{(\ell)}} = 0, \quad \lim_{n\to\infty} \frac{y_n - p_n y_{n-k}}{n^{(\ell-1)}} = \pm\infty \tag{3.15}$$

for some $\ell \in \{1, 2, \ldots, m-1\}$ such that $(-1)^{m-\ell-1}\delta = 1$. If $\{y_n\}$ is a positive such solution of (E), then a summation of (E) yields

$$\sum_{n=N}^{\infty} n^{(m-\ell-1)} q_n y_{\sigma(n+m-1)} < \infty$$

and

$$\sum_{n=N}^{\infty} n^{(m-\ell)} q_n y_{\sigma(n+m-1)} = \infty$$

for some sufficiently large $N > n_0$. Suppose that (2.12) holds. Now (2.9), (2.12) and (3.15) imply that there exist positive constants α and β such that

$$|y_n| \geq \alpha n^{(\ell-1)} \text{ and } |y_n| \leq \beta n^{(\ell)}$$

for $n \geq N$. It then follows that

$$\sum_{n=N}^{\infty} n^{(m-\ell-1)} (\sigma(n+m-1))^{(\ell-1)} q_n < \infty \tag{3.16}$$

and

$$\sum_{n=N}^{\infty} n^{(m-\ell)} (\sigma(n+m-1))^{(\ell)} q_n = \infty. \tag{3.17}$$

Thus, under condition (2.12), (3.16) and (3.17) are necessary for the existence of a solution $\{y_n\}$ satisfying (2.3) and (3.15). The following result summarizes these observations.

Theorem 3.3. *Suppose condition (2.12) holds. Then in order for equation (E) to have a nonoscillatory solution $\{y_n\}$ satisfying (2.3) and (3.15), i.e., a Type II solution, it is necessary that (3.16) and (3.17) hold.*

Our final theorem provides sufficient conditions for the existence of a Type II solution of (E) in the case where $\{p_n\}$ is eventually nonnegative. Such a result in the case where $\{p_n\}$ is eventually nonpositive would, of course, also be of interest.

Theorem 3.4. *Let $p_n \geq 0$ and $\sigma(n) < n$ for $n \geq n_0$, and let $\ell \in \{1, 2, \ldots, m-1\}$ satisfy $(-1)^{m-\ell-1}\delta = 1$. Equation (E) has a nonoscillatory solution $\{y_n\}$ satisfying (2.3) and (3.15) if*

$$\sum_{n=N}^{\infty} n^{(m-\ell-1)}(\sigma(n+m-1))^{(\ell)} q_n < \infty \tag{3.18}$$

and

$$\sum_{n=N}^{\infty} n^{(m-\ell)}(\sigma(n+m-1))^{(\ell-1)} q_n = \infty. \tag{3.19}$$

Proof. Choose $N > n_0$ so large that (3.8) holds and

$$\sum_{s=N}^{\infty} s^{(m-\ell-1)}(\sigma(s+m-1))^{(\ell)} q_s \leq \frac{(1-\lambda)}{2}.$$

Let $c > 0$ be fixed and consider the subset S_ℓ of $\ell_\infty^{N_0}$ given by

$$S_\ell = \{Y \in \ell_\infty^{N_0} : \frac{c(n-N)^{(\ell-1)}}{(\ell-1)!} \leq y_n \leq \frac{c(n-N)^{(\ell-1)}}{(\ell-1)!} + \frac{c(n-N)^{(\ell)}}{\ell!} \text{ for } n \geq N$$
$$\text{and } y_n = y_N \text{ for } N_0 \leq n \leq N\}.$$

We define a partial order on $\ell_\infty^{N_0}$ in the usual way, and we will avoid introducing equivalence classes in $\ell_\infty^{N_0}$. Thus, if for any $X = \{x_n\}$, $Y = \{y_n\} \in \ell_\infty^{N_0}$, $x_n = y_n$ for all $n >> 1$, we will consider such sequences to be the same. Then, for every subset A of S_ℓ both $\inf A$ and $\sup A$ exist and belong to S_ℓ. If $\{y_n\} \in S_\ell$, then since $y_n \leq 2cn^{(\ell)}/(\ell-1)!$ for $n \geq N$, the sequence $\{\bar{y}_n\}$ defined by (3.11) satisfies

$$\bar{y}_{\sigma(n+m-1)} \leq \frac{2c}{1-\lambda}(\sigma(n+m-1))^{(\ell)}$$

for $n \geq N$, and so the mapping T defined by

$$(Ty)_n = \begin{cases} \dfrac{c(n-N)^{(\ell-1)}}{(\ell-1)!} + \displaystyle\sum_{r=N}^{n-\ell} \frac{(n-r-1)^{(\ell-1)}}{(\ell-1)!} & \\ \qquad\qquad \times \sum_{s=r}^{\infty} \frac{(s-r+m-\ell-1)^{(m-\ell-1)}}{(m-\ell-1)!} q_s \bar{y}_{\sigma(s+m-1)}, & n \geq N \\ 0, \qquad N_0 \leq n \leq N \text{ for } \ell \geq 2 & \\ c, \qquad N_0 \leq n \leq N \text{ for } \ell = 1 & \end{cases}$$

maps S_ℓ onto itself and is increasing. By the Knaster-Tarski fixed point theorem [11], there exists an element $\{y_n^*\} \in S_\ell$ such that $y_n^* = (Ty^*)_n$. As in the proof of Theorem 3.2, the $\{\bar{y}_n^*\}$ associated with $\{y_n^*\}$ via (3.11) satisfies the equation

$$(3.20)\quad \bar{y_n}^* - p_n \bar{y}_{n-k}^* = \frac{c(n-N)^{(\ell-1)}}{(\ell-1)!} + \sum_{r=N}^{n-\ell} \frac{(n-r-1)^{(\ell-1)}}{(\ell-1)!} \sum_{s=r}^{\infty} \frac{(s-r+m-\ell-1)^{(m-\ell-1)}}{(m-\ell-1)!} q_s y_{\sigma(s+m-1)}^*, \quad n \geq N.$$

Clearly, $\{\bar{y}_n^*\}$ is also a solution of equation (E). To show that $\{\bar{y}_n^*\}$ has the desired asymptotic behavior, we note that

$$(3.21)\quad \Delta^{\ell-1}[\bar{y}_n^* - p_n \bar{y}_{n-k}^*] = c + \sum_{r=N}^{n-1} \sum_{s=r}^{\infty} \frac{(s-r+m-\ell-1)^{(m-\ell-1)}}{(m-\ell-1)!} q_s \bar{y}_{\sigma(s+m-1)}^*$$

and

$$(3.22)\qquad \Delta^{\ell}[\bar{y}_n^* - p_n \bar{y}_{n-k}^*] = \sum_{s=n}^{\infty} \frac{(s-n+m-\ell-1)^{(m-\ell-1)}}{(m-\ell-1)!} q_s \bar{y}_{\sigma(s+m-1)}^*$$

for $n \geq N$. In view of (3.12) with $y = y^*$ and (2.13), we have

$$(3.23)\qquad \bar{y}_n^* = y_n^* + p_n \bar{y}_{n-k}^* \geq (1-\lambda)\bar{y}_n^* \geq (1-\lambda)\frac{c(n-N)^{(\ell-1)}}{(\ell-1)!}$$

for all large n. Combining (3.23) with the inequality

$$\Delta^{\ell-1}[\bar{y}_n^* - p_n \bar{y}_{n-k}^*] \geq c + \sum_{s=N}^{n-1} \frac{(s-N+m-\ell)^{(m-\ell)}}{(m-\ell)!} q_s \bar{y}_{\sigma(s+m-1)}^*,$$

which is a consequence of (3.21), condition (3.19) then implies

$$\lim_{n\to\infty} \Delta^{\ell-1}[\bar{y}_n^* - p_n \bar{y}_{n-k}^*] = \infty.$$

On the other hand, from (3.22) we have

$$\lim_{n\to\infty} \Delta^{\ell}[\bar{y}_n^* - p_n \bar{y}_{n-k}^*] = 0.$$

Thus, $\{\bar{y}_n^*\}$ satisfies (2.3) and (3.15). This completes the proof of the theorem.

4. Examples

We present some examples to illustrate the results obtained in the previous section.

Example 4.1. Consider the equation

$$\Delta^2[y_n - \lambda y_{n-k}] + (\lambda e^k - 1)\left(\frac{e-1}{e}\right)^2 y_n = 0, \quad n \geq 1 \tag{4.1}$$

where $0 < \lambda < 1$ and $k \geq 1$.

(i) Suppose that $\lambda e^k > 1$. From (2.5) we have $\mathcal{N} = \mathcal{N}_1^+ \cup \mathcal{N}_0^-$ for (4.1). Note that $\mathcal{N}_0^- \neq \phi$ since (4.1) has a solution $\{y_n\} = \{e^{-n}\}$ belonging to this class. The possible asymptotic behaviors of the solutions $\{y_n\}$ in $\mathcal{N}_1^+$ are

$$\lim_{n\to\infty} \frac{y_n - \lambda y_{n-k}}{n} = \text{ constant } \neq 0, \tag{4.2}$$

$$\lim_{n\to\infty} [y_n - \lambda y_{n-k}] = \text{ constant } \neq 0, \tag{4.3}$$

or

$$\lim_{n\to\infty} \frac{y_n - \lambda y_{n-k}}{n} = 0, \quad \lim_{n\to\infty} [y_n - \lambda y_{n-k}] = \pm\infty. \tag{4.4}$$

Since condition (3.7) does not hold ($m = 2$), equation (4.1) has neither a solution satisfying (4.2) nor a solution satisfying (4.3) (see Theorem 3.1).

(ii) Suppose that $\lambda e^k < 1$. The classification (2.5) then reduces to $\mathcal{N} = \mathcal{N}_0^+ \cup \mathcal{N}_2^+$ and the possible types of asymptotic behavior of the nonoscillatory solutions $\{y_n\}$ of (4.1) are (4.2), (4.3),

$$\lim_{n\to\infty} [y_n - \lambda y_{n-k}] = 0, \tag{4.5}$$

or

$$\lim_{n\to\infty} \frac{y_n - \lambda y_{n-k}}{n} = \pm\infty. \tag{4.6}$$

Exactly the same statements as in (i) hold for solutions which satisfy (4.2) and (4.3). Equation (4.1) has a solution $\{y_n\} = \{e^{-n}\}$ satisfying (4.5). No information can be drawn about the solutions of (4.1) satisfying (4.6).

Example 4.2. Consider the difference equation

$$\Delta^2[y_n - \lambda y_{n-k}] - (1 - \lambda e^{-k})(e-1)^2 y_n = 0, \quad n \geq 1 \tag{4.7}$$

where $0 < \lambda < 1$ and $k \geq 2$. The classification and the asymptotic behavior of nonoscillatory solutions of (4.7) are the same as in (ii) of Example 4.1. This equation has a solution $\{y_n\} = \{e^n\}$ satisfying (4.6). It is not known if there is a solution of (4.7) satisfying (4.5).

Example 4.3. The equation

$$\Delta^m[y_n - p_n y_{n-k}] + \delta n^\alpha y_{\sigma(n+m-1)} = 0, \quad n \geq 2, \tag{4.8}$$

where $p_n \geq 0$ and $\sigma(n)$ is the greatest integer function of $n^{\frac{1}{2}}$, satisfies the hypotheses of Theorem 3.4 provided ℓ and α satisfy

$$(-1)^{m-\ell-1}\delta = 1 \text{ and } \ell/2 - m - 1/2 \leq \alpha < \ell/2 - m.$$

Thus, equation (4.8) will have a nonoscillatory solution $\{y_n\}$ with

$$\lim_{n\to\infty} \frac{y_n - p_n y_{n-k}}{n^{(\ell)}} = 0 \text{ and } \lim_{n\to\infty} \frac{y_n - p_n y_{n-k}}{n^{(\ell-1)}} = \pm\infty.$$

In conclusion, note that the results of this paper can be easily extended to equations of the form

$$\Delta^m[y_n - p_n y_{n-k}] + \delta \sum_{i=1}^{M} q_{i_n} y_{\sigma_i(n+m-1)} = 0,$$

where $m \geq 2$, $\delta = \pm 1$, $\{p_n\}$ and k are the same as before, $\{q_{i_n}\}$ are non-negative sequences of real numbers, and $\{\sigma_i(n)\}$ are sequences of integers such that $\sigma_i(n) \leq n$ and $\lim_{n\to\infty} \sigma_i(n) = \infty$, $1 \leq i \leq M$.

Acknowledgement. The authors would like to thank Professor O. Došlý of Masaryk University in Brno, Czech Republic for making several valuable suggestions concerning the results in this paper.

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