ARCHIVUM MATHEMATICUM (BRNO)
Tomus 31 (1995), 259 – 262

INDEXED ANNIHILATORS IN LATTICES

Ivan Chajda

Abstract. The concept of annihilator in lattice was introduced by M. Mandelker. Although annihilators have some properties common with ideals, the set of all annihilators in L need not be a lattice. We give the concept of indexed annihilator which generalizes it and we show the basic properties of the lattice of indexed annihilators. Moreover, distributive and modular lattices can be characterized by using of indexed annihilators.

M. Mandelker [2] introduced the concept of annihilator in lattice as a natural generalization of the relative pseudocomplement $a * b$ of an element a of a lattice L relative to an element b:

Definition 1. The annihilator $\langle a, b\rangle$ for $a, b \in L$ is the set $\{x \in L; a \wedge x \leq b\}$.

Evidently, the greatest element of $\langle a, b\rangle$, if it exists, is the relative pseudocomplement $a * b$.

Although annihilators have some properties in common with ideals, there are also essential distinctions: the set of all annihilators in a lattice L need not be a lattice. This can be seen by means of the six element lattice $L = \{0, a, b, c, d, 1\}$ where the elements a, b, c, d are pairwise incomparable and all between 0 and 1. In this case,

$$\langle a, c\rangle = \{0, b, c, d\} \quad \text{and} \quad \langle b, c\rangle = \{0, a, c, d, \}$$

but $\langle a, c\rangle \cap \langle b, c\rangle = \{0, c, d\}$, which is not an annihilator in L. Moreover, there is no annihilator in L properly contained in $\langle a, c\rangle$ or in $\langle b, c\rangle$. This motivates us to look after a new concept generalizing that of annihilator.

Definition 2. Let L be a lattice, $\Gamma \neq \emptyset$ an index set and a_γ, $b_\gamma \in L$ for $\gamma \in \Gamma$. By an *indexed annihilator* is meant the set

$$\{x \in L;\ a_\gamma \wedge x \leq b_\gamma \text{ for each } \gamma \in \Gamma\}\,.$$

The following assertion is evident:

1991 *Mathematics Subject Classification*: 06B10, 06C05, 06D05.
Key words and phrases: lattice, distributive lattice, modular lattice, annihilator, ideal, indexed annihilator.
Received May 17, 1994.

Lemma 1. *Let L be a lattice, $\Gamma \neq \emptyset$ and $a_\gamma, b_\gamma \in L$ for $\gamma \in \Gamma$. Then*

$$\{x \in L;\ a_\gamma \wedge x \leq b_\gamma \text{ for each } \gamma \in \Gamma\} = \cap\{\langle a_\gamma, b_\gamma\rangle;\ \gamma \in \Gamma\}\,.$$

By Lemma 1, every indexed annihilator is intersection of annihilators. Therefore, we need not any huge notation for this concept and the indexed annihilator will be denoted simply by $\cap\{\langle a_\gamma, b_\gamma\rangle;\ \gamma \in \Gamma\}$.

Lemma 2. *Let L be a lattice. Then*
(a) *every annihilator in L is an indexed annihilator in L;*
(b) *if B is an indexed annihilator in L and $x \in B$ and $y \leq x$ then $y \in B$.*

Proof. Then first assertion follows immediately from Lemma 1 by using $\Gamma = \{1\}$. The second follows directly by Definition 2. □

If a lattice L has not the least element then clearly $\emptyset$ is the least indexed annihilator in L with respect to set inclusion since

$$\emptyset = \cap\{\langle a, b\rangle;\ a, b \in L\}\,.$$

If a lattice L has the least element 0, then $0 \in \langle a, b\rangle$ for every a, b of L, i.e. $0 \in \cap\{\langle a, b\rangle;\ a, b \in L\}$. On the other hand, $\{0\} = \cap\{\langle a, b\rangle;\ a, b \in L\}$, thus $\{0\}$ is the least annihilator in L with respect to set inclusion.

Denote by $IA_0(L)$ the set of all indexed annihilators in L and by $IA(L)$ the set of all non-void annihilators in L.

Theorem 1. *The set $IA_0(L)$ of all indexed annihilators in a lattice L forms a complete lattice with respect to set inclusion. The greatest element of $IA_0(L)$ is L and the least element is $\cap\{\langle a, b\rangle;\ a, b, \in L\}$. The operation meet coincides with set intersection, If L has the least element 0 then $IA_0(L) = IA(L)$ and $\{0\}$ is the least element of the lattice $IA(L)$.*

Proof. It follows directly by Lemma 1 and the foregoing remarks. Since $\langle a, a\rangle = L$ for each $a \in L$, L is the greatest element of $IA(L)$. □

Remark 1. If a lattice has the greatest element 1 then every principal ideal $J = (c]$ is an annihilator in L, namely $J = \langle 1, c\rangle$.

Let L be a lattice and M be a subset of L. Since $IA_0(L)$ is a complete lattice, there exists the least indexed annihilator in L containing M. Denote it by $A(M)$ and call an *indexed annihilator generated by M*. It is easy to show that $A(M) = \cap\{\langle a, b\rangle;\ m \in \langle a, b\rangle \text{ for each } m \in M\}$.

Theorem 2. *Every principal ideal of L is an indexed annihilator in L.*

Proof. Let $c \in L$ and $J = (c]$. By the definition of annihilator, we have

$$A(J) = \cap\{\langle a, b\rangle;\ j \in \langle a, b\rangle \text{ for each } \ j \in J\} = \cap\{\langle a, b\rangle;\ c \in \langle a, b\rangle\}, \text{ i.e.}$$
$$A(J) = \{x \in L;\ a \wedge c \leq b \Rightarrow a \wedge x \leq b \text{ for each } a, b, \in A\} = \{x \in L;\ x \leq c\} = J\,.$$

However, $J = A(J)$ implies that J is an indexed annihilator in L. □

Theorem 3. *Let L be a lattice with the least element 0. Then the lattice $IA(L)$ of all indexed annihilators is pseudocomplemented.*

Proof. Let $A \in IA(L)$ and put $B = \{\langle a, b\rangle;\ a \in A\}$. If $x \in A \cap B$ then $x \wedge a = 0$ for each $a \in A$, i.e. $A \cap B = \{0\}$. Moreover, if $A \cap C = \{0\}$ for some $c \in IA(L)$, suppose $y \in A \cap C$ and $y \notin B$. Then there exists an element $a \in A$ with $a \wedge y \neq 0$. By (b) of Lemma 2 we conclude

$$a \in A,\ y \in C \Longrightarrow a \wedge y \in A \cap C = \{0\}\,,$$

i.e. $a \wedge y = 0$, a contradiction. Hence $y \in B$ proving $C \subseteq B$. Thus B is a pseucomplement of A in $IA(L)$. □

Theorem 4. *Let L be a lattice. The following conditions are equivalent:*
(1) *L is distributive;*
(2) *every non-void indexed annihilator in L is an ideal of L.*

Proof. Let $J = \cap\{\langle a_\gamma, b_\gamma\rangle;\ \gamma \in L\} \in IA(L)$ and suppose $x, y \in J$, $a \in L$. Applying distributivity of L, we obtain

$$a_\gamma \wedge (x \vee y) = (a_\gamma \wedge x) \vee (a_\gamma \wedge y) \leq b_\gamma \vee b_\gamma = b_\gamma$$

for each $\gamma \in \Gamma$, thus $x \vee y \in J$. Moreover,

$$a_\gamma \wedge (x \wedge a) \leq a_\gamma \wedge x \leq b_\gamma$$

for each $\gamma \in \Gamma$ thus also $x \wedge a \in J$, i.e. J is an ideal of L.

Conversely, if every non-void indexed annihilator in L is an ideal of L then, by (a) of Lemma 2, also every annihilator in L is an ideal of L. By Theorem 1 in [2], we conclude that L is distributive. □

Corollary. *In every finite distributive lattice L, ideals and indexed annihilators coincide.*

Proof. If a lattice L is finite then L has 0, thus every indexed annihilator in L is non-void. Moreover, every ideal in a finite lattice is principal, thus, by Theorem 2, every ideal is an indexed annihilator in L. The converse statement follows by Theorem 4. □

Remark 2. The assertion of Corollary need not be valid if L is an infinite lattice, see the following:

Example. Let S be an infinite set and consider the lattice $(\operatorname{Exp} S, \subseteq)$ of all subsets of S. Trivially, $(\operatorname{Exp} S, \subseteq)$ is a complete and distributive lattice. Denote by J the set of all finite subsets of S. Clearly, J is an ideal of $(\operatorname{Exp} S, \subseteq)$ which is not principal. However, for any $A \in IA_0(\operatorname{Exp} S)$, $A \neq S$ there exists a finite subset $F \subseteq S - A$, thus $A(J) = S \neq J$. Hence J is not an indexed annihilator in $(\operatorname{Exp} S, \subseteq)$. □

Annihilators in modular lattices were treated by B. Davey and J. Nieminen [1]. For indexed annihilators, we can state the following characterization of modular lattices:

Theorem 5. *For a lattice L, the following conditions are equivalent*

(1) *L is modular;*

(2) *for each $J = \cap\langle a_\gamma, b_\gamma\rangle;\ \gamma \in \Gamma\} \in IA(L)$ with $b_\gamma \leq a_\gamma$ for each $\gamma \in \Gamma$ it holds:*

$$\textit{if } \ x \in \cap\{(b_\gamma];\ \gamma \in \Gamma\} \quad \textit{and} \quad y \in J \quad \textit{then} \quad x \vee y \in J\,.$$

Proof. Let L be a modular lattice and $b_\gamma \leq a_\gamma$ for each $\gamma \in \Gamma$. Let $J = \cap\{a_\gamma, b_\gamma\rangle;\ \gamma \in \Gamma\} \in IA(L)$. Suppose $x \in \cap\{(b_\gamma];\ \gamma \in \Gamma\}$ and $\gamma \in J$. Then

$$a_\gamma \wedge y \leq b_\gamma \quad \text{and} \quad x \leq b_\gamma \leq a_\gamma \quad \text{for each} \quad \gamma \in \Gamma\,.$$

Applying modularity of L, we obtain

$$a_\gamma \wedge (x \vee y) = x \wedge (a_\gamma \wedge y) \leq b_\gamma$$

for each $\gamma \in \Gamma$, which yields $x \vee y \in J$.

Conversely, let L satisfies (2) and x, y, $z \in L$ with $x \leq z$. Then $x \vee (z \wedge y) \leq z$ and $x \in (p]$ for $p = x \vee (z \wedge y)$ and $z \wedge y \leq x \vee (z \wedge y)$. It implies

$$y \in \langle z, x \vee (z \wedge y)\rangle\,.$$

By Lemma 2, $\langle z, x \vee (z \wedge y)\rangle$ is an indexed annihilator in L. Applying (2) we conclude

$$x \vee y \in \langle z, x \vee (z \wedge y)\rangle\,,$$

which gives $z \wedge (x \vee y) \leq x \vee (z \wedge y)$. Hence, L is modular. □

References

[1] Davey, B., Nieminen, J., *Annihilators in modular lattices*, Algebra Univ. **22** (1986), 154-158.
[2] Mandelker, M., *Relative annihilators in lattices*, Duke Math. J. **40** (1970), 377-386.

Ivan Chajda
Palacký University Olomouc
Tomkova 40
779 00 Olomouc, CZECH REPUBLIC