ARCHIVUM MATHEMATICUM (BRNO)
Tomus 31 (1995), 283 – 289

IMPROVEMENT OF INEQUALITIES FOR THE (r, q)–STRUCTURES AND SOME GEOMETRICAL CONNECTIONS

VOJTECH BÁLINT AND PHILIPPE LAURON

ABSTRACT. The main results are the inequalities (1) and (6) for the minimal number of (r,q)-structure classes,which improve the ones from [3], and also some geometrical connections, especially the inequality (13).

1. INEQUALITIES

1.1. Definition. Let m, n, r, q be natural numbers such that $n \geq 3$ and $r \leq n$. Let M be a set which contains at least $n+q-1$ elements. Let $A = \{a_1, ..., a_n\} \subset M$. Let $P(M)$ be the set of all the subsets of M. Let the set $B = \{B_1, ..., B_m\} \subset P(M)$ fulfil the following three conditions :

(i) Each element $B_k \in B$ for $k = 1, ..., m$ contains at least r distinct elements $a_{i_1}, a_{i_2}, ..., a_{i_r} \in A$;

(ii) If $a_{i_1}, a_{i_2}, ..., a_{i_r}$ are r distinct elements of the set A, then there exist exactly q distinct elements $B_{j_1}, B_{j_2}, ..., B_{j_q} \in B$ such that for $p = 1, 2, ..., q$ we have: $a_{i_s} \in B_{j_p}$ for each $s \in \{1, 2, ..., r\}$;

(iii) For every $r+1$ distinct elements $a_{i_1}, a_{i_2}, ..., a_{i_{r+1}} \in A$ there are at most one element $B_k \in B$ such that $a_{i_s} \in B_k$ for each $s \in \{1, 2, ..., r+1\}$.

Then the ordered triplet (M, A, B) is called (r,q)-*structure.*

The elements of B are called *classes* and the elements of A are called *points.* An element $B_j \in B$ is called *class of order* k when B_j contains exactly k distinct points of A. A point $a_i \in A$ is called *point of degree* k, iff there exist exactly k distinct classes of B, which contain this point. The (r,q) -structure (M, A, B) is called *ordinary* iff all the classes of B are ordinary, i.e. contain exactly r points. We'll say that the class $B_i \in B$ is *bigger* than the class B_j iff $|B_i| > |B_j|$, and that the class $B_i \in B$ is the *biggest* iff $|B_i| \geq |B_j|$ for every $j \in \{1, 2, ..., m\}$.

1991 *Mathematics Subject Classification*: 05B30, 51E30.
Key words and phrases: structure, line, circle, horocycle.
Received January 24, 1995.

1.2. Example. Put $M = E_2$, where E_2 is the Euclidean plane.Let A be a set of n points in E_2. Let B be the set of all the straight lines determined by the points of A. Then (M, A, B) is a $(2, 1)$ -structure.

Consider the Poincaré model of the *hyperbolic* plane H_2. The points of the hyperbolic plane are interpreted as inner points of the Euclidean upper half-plane determined by the x-axis.

Horocycles (i.e. curves of identically one curvature) are either circles of the upper half-plane touching the x-axis or straight lines parallel to the x-axis. So every couple of points determines two horocycles.

1.3. Example. . Put $M = H_2$. Let A be a set of n points in H_2. Let B be the set of all the horocycles determined by the points of A. Then (M, A, B) is a $(2, 2)$ -structure.

1.4. Example. Take $M = E_2$. Let A be a set of n points in E_2 such that $\mathrm{diam} A < 2$. Let B be the set of all the unit circles containing at least two points of A. Then (M, A, B) is a $(2, 2)$ -structure.

1.5. Example. Take $M = E_2$. Let A be a set of n points in E_2, no three collinear. Let B be the set of all the circles determined by the points of A. Then (M, A, B) is a $(3, 1)$-structure.

1.6. Example. Let $M = E_3$ and $A = \{a_1, ..., a_n\} \subset M$ such that no four points from A are coplanar. Every triplet of points $a_i, a_j, a_n \in A$ uniquely determine a circle with a centre $S_{i,j,k}$. The number of such circles is finite, consequently there is a number g=the greatest of the radii. Take $G > g$ arbitrary. Now every triplet of points $a_i, a_j, a_k \in A$ determines exactly two spheres with a radius G, the centres of which are lying on the normal to that plane which is determined by the points a_i, a_j, a_k and passing through $S_{i,j,k}$. If we take just such spheres instead of B, then (M, A, B) is a $(3, 2)$-structure.

1.7. Remark. Obviously, for $q = 1$ the axiom (iii) is redundant, but for $q \geq 2$ is important and the examples 1.3 and 1.4 show its geometrical sense.

1.8. Theorem. *For any $(r, 2)$-structure, $r \geq 2$, it holds*

$$m \geq n. \tag{1}$$

Proof. For a given (r, q)-structure we denote p_k the number of points from A of degree k and t_k the number of classes from B of order k. Then

$$\sum_{k=r}^{n} k t_k = \sum_{k=q}^{m} k p_k \tag{2}$$

and

$$\sum_{k=r}^{n} \binom{k}{r} t_k = q \binom{n}{r}. \tag{3}$$

(For the proof see [3]). Hence for $r = 2$ we get

$$m(m-1) - n(n-1) = (m-1)\sum_{k=2}^{n} t_k - \sum_{k=2}^{n} \binom{k}{2} t_k = \sum_{k=2}^{n} \{m - 1 - \binom{k}{2}\} t_k.$$

Choose $k \in \{2, 3, ..., n\}$ arbitrarily. If B contains some classes of order k, then $m \geq \binom{k}{2} + 1$, and so

$$\{m - 1 - \binom{k}{2}\} t_k \geq 0. \tag{4}$$

If B contains no class of order k, then $t_k = 0$ and (4) holds, too. From this we get $m \geq n$ for $r = 2$. Let now $r \geq 3$ and choose a class $B^* \in B$ arbitrarily. If $|B^*| = n$ then $m \geq 1 + \binom{n}{r} \geq n$ and we have finished. Therefore, let $|B^*| \leq n - 1$. Take a point $a \in A$ such that $a \notin B^*$ and denote $B_a = \{B_i \in B; a \in B_i\}$. Now $A - a$,B_a give $(r-1, 2)$-structure and the number $m(a)$ of its classes satisfies $m(a) \leq m$. This induction argument (together with $m \geq n$ for $r = 2$) implies (1). □

1.9. Remark. The proof of inequality (1) for a (2,1)-*structure* one can find in [6].

1.10. Corollary. *The total number of horocycles determined by n points is at least n.*

1.11. Lemma. *Let (M, A, B) be a (r, q)-structure , $r \geq 2$, $q \leq r$. Let $a \in A$ and d is its degree. Then*

$$d \geq \binom{D-1}{r-1}(q-1) + 1 \tag{5}$$

where D is the order of the biggest class which contains a.

Proof. Let $a \in A$ be an arbitrary point; without loss of generality $a = a_n$. Denote by B the biggest class from B which contains a; w.l.o.g. we can fix $B = \{a_1, a_2, ..., a_{D-1}, a_n\}$. From (ii) we know that every r-tuple of points belongs exactly to q classes. So every $(r-1)$-tuple from the points $a_1, a_2, ..., a_{D-1}$ belongs - together with the point a_n - to exactly $(q-1)$ classes different from B. So we have $\binom{D-1}{r-1}(q-1)$ such classes and it is easy to see that they are distinct. □

1.12. Theorem. *For any (r,q)-structure , $r \geq 2$, $q \leq r$ it holds*

$$m \geq \sqrt{\frac{rq(q-1)}{n}\binom{n}{r}}. \tag{6}$$

Proof. Let's consider all the possible orders of classes of investigated (r,q)-structure (M,A,B) and let's write them in an increasing sequence of natural numbers $\{k_i\}_{i=0,1,\dots,s}$, precisely

$$r \leq k_0 < k_1 < k_2 < \cdots < k_s \leq n. \tag{7}$$

Let us consider now all the r-tuples of n points. They are $\binom{n}{r}$. Every r-tuple must belong to q classes. So we have to "put" in our classes $\binom{n}{r}q$ r-tuples. A class of order k_i for $i \in \{0,1,...,s\}$ contains $\binom{k_i}{r}$ r-tuples. From that we have

$$C = \binom{n}{r}q = \sum_{i=0}^{s} m_i \binom{k_i}{r} \leq m\binom{k_s}{r}, \tag{8}$$

where m_i is the number of classes of order k_i for $i \in \{0,1,...,s\}$. Further we'll denote $k = k_s$. From (8) we have

$$\binom{k}{r} \geq \frac{C}{m}. \tag{9}$$

Let $B^* \in B$ be a class of maximal order $k_s = k$. Let $a \in B^*$ an arbitrary point of this class and d the degree of this point. The order of the biggest class which contains a is of course k. Now from lemma 1.11 and inequality (9) we have

$$d \geq \binom{k-1}{r-1}(q-1)+1 > \frac{r}{k}\binom{k}{r}(q-1) \geq \frac{rC(q-1)}{km}.$$

So

$$d \geq \frac{\binom{n}{r}rq(q-1)}{km} \geq \frac{\binom{n}{r}rq(q-1)}{nm}$$

because $k \leq n$.

Trivially, $m \geq d$ and so

$$m \geq \frac{\binom{n}{r}rq(q-1)}{nm},$$

which yields the inequality (6). □

1.13. Remark. In [3] the authors proved

$$m \geq \frac{q-1}{r-1}n. \tag{10}$$

For $r = q = 2$ the estimate (10) is better than (6), but already for $r \geq 3$ it is not so and (6) is better than (1) for $r \geq 4$, too.

1.14. Remark. In [3] the authors presented the examples of $(2,2)$-structures for $n = 4, 7, 11$ and 16, which are showing that the estimates (1) and (10) are the best possible at least for the above-mentioned values of n.

2. Some geometrical connections

The definition of the abstract (r, q)-structure comprehends a large family of geometrical models according to certain common combinatorical features. But some concrete geometrical models are already a long time subject of interest for research. Naturally, the most investigated is the (2,1)-structure of points and straight lines in E_2 (example 1); selfevidently, in geometrical terms (see e.g.[18],[6],[10],[16],[12],[11],[5],[7]). But already the paper [6] gives the proof of inequality $m \geq n$ by means of combinatorical methods. Very natural is also the example 1.5 , i.e. the (3,1)-structure of points and circles in E_2;see [9],[2], [17]. About the (2,2)-structure from the example 4, i.e. points and unit circles it is possible to find pretty results for instance in [8], [13], [14].

Concerning the (2,2)-structure of points and horocycles Jucovič [15] asked the question, what is the minimal number $h(n)$ of horocycles determined by n points. Beck [4] proved

$$h(n) > c_5 n^2, \tag{11}$$

but his constant c_5 is extremely small. In this direction our conjecture is the following:

$$h(n) \geq \binom{n-1}{2} + 3. \tag{12}$$

Of course, it's surprising that - as we know - it has not been proved yet that every system of $n \geq 2$ points determines at least one ordinary horocycle. The reason of the troubles may be concealed in the following

2.1. Proposition. *Let A be the set of $n \geq 2$ points in a hyperbolic plane H_2. Then through every point $a_i \in A$ pass at least*

$$\frac{1+\sqrt{8n-7}}{2} \tag{13}$$

horocycles.

Proof. Consider the Poincaré model of the hyperbolic plane H_2. Choose the point $a_i \in A$ arbitrarily. Let K be any circle with centre a_i. We take the inversion with respect to K. Now the x-axis mapps into the (Euclidean) circle x' passing through a_i and every horocycle passing through a_i mapps into a straight line touching the circle x'. Let's denote the number of these touching lines by m. The intersection-points of those m lines must contain all the points of A' (perhaps with exception of a_i), therefore $\binom{m}{2} \geq n-1$. From this we obtain the asked inequality (13). □

If we take $n = \frac{j^2+j+2}{2}$, where $j = 1, 2, 3, \ldots$ then $\frac{1+\sqrt{8n-7}}{2}$ is integer. In this case from the proof of the previous proposition it's obvious to construct the point-set

A of n points and to choose the point $a_i \in A$ such that through a_i are passing $\frac{1+\sqrt{8n-7}}{2} \sim \sqrt{2n}$ horocycles. So the estimate (13) is the best possible.

Compare this result with the (3,1)-structure of points and circles in E_2. There it holds (see [2]) that at least $\frac{33(n-1)}{247}$ (i.e. linearly many) ordinary circles are passing through *every* point.That is an essential difference in comparison with (13). And this is perhaps a main reason why it's so difficult to obtain a good lower estimate for the number of ordinary horocycles determined by n points.

References

[1] Bálint, V., *On a certain class of incidence structures*, Práce a štúdie Vysokej Školy Dopravnej v Žiline **2** (1979), 97-106 (In Slovak; summary in English, German and Russian).

[2] Bálint, V., Bálintová, A., *On the number of circles determined by n points in Euclidean plane*, Acta Mathematica Hungarica **63 (3-4)** (1994), 283-289.

[3] Bálint, V., Lauron, Ph., *Some inequalities for the* (r,q)*-structures*, STUDIES OF UNIVERSITY OF TRANSPORT AND COMMUNICATIONS IN ŽILINA, Mathematical - Physical Series, Volume **10** (1995), 3-10.

[4] Beck, J., *On the lattice property of the plane and some problems of Dirac, Motzkin and Erdős in combinatorical geometry*, Combinatorica **3 (3-4)** (1983), 281-297.

[5] Borwein, P., Moser, W. O. J., *A survey of Sylvester's problem and its generalizations*, Aequa. Math. **40** (1990), 111-135.

[6] de Bruijn, N. G., Erdős, P., *On a combinatorical problem*, Nederl. Acad. Wetensch. **51** (1948), 1277-1279.

[7] Csima, J., Sawyer, E. T., *A short proof that there exist* $6n/13$ *ordinary points*, Discrete and Computational Geometry **9** (1993), no. 2, 187-202.

[8] Elekes, G., *n points in the plane can determine* $n^{\frac{3}{2}}$ *unit circles*, Combinatorica **4** (1984), 131.

[9] Elliott, P. D. T. A., *On the number of circles determined by n points*, Acta Math. Acad. Sci. Hung. **18 (3-4)** (1967), 181-188.

[10] Erdős, P., *Néhány geometriai problémáról*, Mat. Lapok **8** (1957), 86-92.

[11] Erdős, P., *On some metric and combinatorical geometric problems*, Discrete Math. **60** (1986), 147-153.

[12] Hansen, S., *Contributions to the Sylvester-Gallai-Theory*, Doctoral dissertation, University of Copenhagen, 1981.

[13] Harborth, H., Mengersen, I., *Point sets with many unit circles*, Discrete Math. **60** (1985), 193-197.

[14] Harborth, H., *Einheitskreise in ebenen Punktmengen*, 3.Kolloquium über Diskrete Geometrie, Institut für Mathematik der Universität Salzburg (1985), 163-168.

[15] Jucovič, E., *Problem* 24, Combinatorical Structures and their Applications, New York-London-Paris, Gordon and Breach, 1970.

[16] Kelly, L. M., Moser, W. O. J., *On the number of ordinary lines determined by n points*, Canad. J. Math. **10** (1958), 210-219.

[17] Klee, V., Wagon, S., *Old and New Unsolved Problems in Plane Geometry and Number Theory*, Mathematical Assoc. Amer., Washington, DC, 1991.

[18] Sylvester, J. J., *Mathematical Question* 11851, Educational Times **59** (1893), 98.

Vojtech Bálint
Department of Mathematics
University of Transport and Communications
Veľký Diel
010 26 Žilina, SLOVAKIA

Philippe Lauron
Gymnázium Tajovského 25
974 00 Banská Bystrica, SLOVAKIA