ARCHIVUM MATHEMATICUM (BRNO)
Tomus 31 (1995), 163 – 170

# ON CONNECTEDNESS OF GRAPHS ON WEYL GROUPS OF TYPE $A_n (n \geq 4)$

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Abstract. A graph structure is defined on the Weyl groups. We show that these graphs are connected for Weyl groups of type $A_n$ for $n \geq 4$.

## 1. Introduction.

We have defined a graph structure on Weyl groups through the root system associated with them. The planarity and the other properties of such graphs has been studied elsewhere ( [1] and [2] ). The motivation for these graphs come from the method employed in proving the truth of Verma's conjecture on Weyl's dimension polynomial [3] which arose in connection with the irreducible representations of algebraic Chevalley groups and their Lie algebras. A certain matrix was defined there which imposes a new partial order on Weyl groups. That matrix has been also explored by Chastkofsky [4]. The same matrix is the weighted incidence matrix for our definition of graphs on Weyl groups. We prove in this paper that such graphs are connected for Weyl groups of type $A_n$ for $n \geq 4$. The graphs on Weyl groups of type $A_1, A_2, A_3$ and $B_2$ are disconnected. In fact, they consist of isolated points and isolated edges. Except these graphs other graphs on Weyl groups seem to be connected as supported by the data on Weyl groups of type $B_3, C_3, B_4, C_4$ and $D_4$. We end up the paper with a conjecture on the connectedness of graphs on Weyl groups. We use the definitions, notations and the results given by [5], and also results from [6].

## 2. The Subgroup $\Omega_0$.

Let $\Delta$ be a real root system in a real vector space $E$ of dimension $n$ with a positive definite inner product (,). Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the simple roots and $W(\Delta)$ be the Weyl group associated to $\Delta$. Then $W(\Delta)$ is generated by $R_i, i = 1, 2, \dots, n$ where $R_i = R_{\alpha_i}, i = 1, 2, \dots, n$. We also write $W$ in place of $W(\Delta)$. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the fundamental weights of $\Delta$, i.e., $(\lambda_i, \alpha_j^\vee) = \delta_{ij}$ (Kronecker delta) where $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ for $\alpha \in \Delta$. Suppose $X = \sum Z\alpha_i$ and $X' = \sum Z\lambda_i$

1991 *Mathematics Subject Classification.* 20F55.
*Key words and phrases.* Weyl groups, root systems, connectivity of graphs.
Received March 7, 1994

are root lattice and weight lattice respectively and $Z$ is the set of integers. Here $X' \subset X$. Let $D$ be the group of translations $T_\lambda$ for $\lambda \in X$ acting on $E$ or $X$ defined by $xT_\lambda = x + \lambda$. Similarly, $D'$ be the group of translations $T_\lambda$ for $\lambda \in X'$. Suppose $\widetilde{W}$ and $\widetilde{W'}$ are the groups generated by $W, D$ and $W, D'$ respectively. $\widetilde{W'}$ is the affine Weyl group associated with $\Delta$. It is known that $\widetilde{W'}$ is generated by $R_0, R_1, \dots, R_n$ where $R_0 = R_{\alpha_0} T_{-\alpha_0}$ and $-\alpha_0$ is the short dominant root in $\Delta$. The fundamental domain of $\widetilde{W'}$ in $E$ is $E^\Delta$ where

$$E^\Delta = \{x \in E| \quad (x, \alpha_i^\vee) \geq 0, \quad (x, -\alpha_0^\vee) \leq 1, \quad 1 \leq i \leq n\},$$

We call $E^\Delta$ the fundamental simplex. $E^\Delta$ has the usual property: any $x \in E$ has a unique image in $E^\Delta$ under the action of the group $\widetilde{W'}$. Let $\sigma_0$ be the unique element of $W$ of maximal length. For fixed $j$, let $\Delta_j = \{\alpha_i | 1 \leq i \leq n, i \neq j\}$, then $W(\Delta_j)$ is the Weyl group for the root system $\Delta_j$. Let $\sigma_j$ be the unique element of maximal length in $W(\Delta_j)$. If $-\alpha_0^\vee = \sum_{i=1}^{n} n_i \alpha_i^\vee$ then write it as $\sum_{i=0}^{n} n_i \alpha_i^\vee = 0$ and define $J_0 = \{j | n_j = 1\}$. The group $\widetilde{W}$ acts as the permutation group on the set of simplices $\{(E^\Delta)^w | w \in \widetilde{W'}\}$. The stabilizer $\Omega$ of $E^\Delta$ is given by $\Omega = \{\gamma_j T_{\lambda_j} | j \in J_0\}$ where $\gamma_j = \sigma_0 \sigma_j, j \in J_0$. The group $\Omega$ is isomorphic to the subgroup $\Omega_0 = \{\gamma_j | j \in J_0\}$ of $W$. This subgroup $\Omega_0$ is important for our discussions. For $\sigma \in W$, let $I_\sigma = \{i | 1 \leq i \leq n, \ell(\sigma R_i) < \ell(\sigma)\}$ where $\ell(\sigma)$ is the length of $\sigma$. For $\sigma \in W$, define $\delta_\sigma = \sum_{i \in I_\sigma} \lambda_i$ and $\epsilon_\sigma = \delta_\sigma \sigma^{-1}$.

## 3. A graph $\Gamma(W)$ on $W$.

We define a graph $\Gamma(W)$ on $W$ whose vertices are elements of $W$. A point $x \in E$ is called $W$-regular if $x$ lies in the interior of a Weyl chamber. This is also equivalent to $D(x) \neq 0$ where $D(x)$ is the Weyl's dimension polynomial at $x \in E$. For $\sigma, \tau \in W$ we write $\sigma \longrightarrow \tau$ iff $-\epsilon_{\sigma\sigma_0} + \epsilon_\tau$ is $W$-regular. It is known that only one of $-\epsilon_{\sigma\sigma_0} + \epsilon_\tau$ and $-\epsilon_{\tau\sigma_0} + \epsilon_\sigma$ is $W$-regular [3]. For $\sigma, \tau \in W$ with $\sigma \neq \tau$ we define an unordered pair $(\sigma, \tau)$ to be an edge in the graph $\Gamma(W)$ on $W$ iff either $\sigma \longrightarrow \tau$ or $\tau \longrightarrow \sigma$. This definition of graph depends upon the root system. Therefore the correct notation for the graph is $\Gamma(W(\Delta))$ but we write $\Gamma(\Delta)$ or $\Gamma(W)$ depending upon the context. For $x \in E$ and $\sigma \in W$, let $x^{(\sigma)}$ be the unique image of $x\sigma$ in the fundamental domain of D. It can be easily shown that

$$x^{(\sigma\tau)} = (x^{(\sigma)})^{(\tau)} \tag{3.1}$$

for $\sigma, \tau \in W$. If $x \in E^\Delta$ then $x^{(\sigma)} = x\sigma + T_{\delta_\sigma}$. Further $x^{(\gamma)} \in E^\Delta$ for $\gamma \in \Omega_0$. From eqn.(3.1), we can easily obtain $\delta_{\gamma\sigma} = \delta_\gamma \sigma + \delta_\sigma$ for $\sigma \in W$ and $\gamma \in \Omega_0$ which simplifies to

$$\epsilon_{\gamma\sigma} = \epsilon_\gamma + \epsilon_\sigma \gamma^{-1} \tag{3.2}$$

This leads to the following

**Lemma 3.1.** *Let $\sigma, \tau \in W$ and $\gamma \in \Omega_0$. Then $\sigma \longrightarrow \tau$ iff $\gamma\sigma \longrightarrow \gamma\tau$. In particular, $(\sigma, \tau)$ is an edge in $\Gamma(W)$ iff $(\gamma\sigma, \gamma\tau)$ is an edge in $\Gamma(W)$.*

*Proof.* We have $-\epsilon_{\gamma\sigma\sigma_0} + \epsilon_{\gamma\tau} = (-\epsilon_{\sigma\sigma_0} + \epsilon_\tau)\gamma^{-1}$ from eqn. (3.2). This shows that $-\epsilon_{\sigma\sigma_0} + \epsilon_\tau$ is $W$-regular iff $-\epsilon_{\gamma\sigma\sigma_0} + \epsilon_{\gamma\tau}$ is $W$-regular, which proves the lemma. $\square$

## 4. Some results on $\Omega_0$.

First we prove a result for $\gamma_j \in \Omega_0$ which holds for all Weyl groups associated with the irreducible root system.

**Lemma 4.1.** *Let $\gamma_j = \sigma_0\sigma_j$. Then $\gamma_j$ is characterized by the following property: $\gamma_j$ is the unique element for which $\ell(\gamma_j) = \ell(\sigma_0) - \ell(\sigma_j), \ell(\gamma_j R_j) < \ell(\gamma_j)$ and $\ell(\gamma_j R_i) > \ell(\gamma_j)$ for $i \neq j$, holds.*

*Proof.* The relation $\gamma_j = \sigma_0\sigma_j$ easily gives the equality $\ell(\gamma_j) = \ell(\sigma_0) - \ell(\sigma_j)$ since $\ell(\sigma_0\sigma) = \ell(\sigma_0) - \ell(\sigma)$ holds for any element $\sigma \in W$. Suppose $\ell(\gamma_j R_i) < \ell(\gamma_j)$ for some $i \neq j$, then $\gamma_j = \gamma'_j R_i$ where $\ell(\gamma_j) = \ell(\gamma'_j) + 1$. Since $\sigma_j^{-1} = \sigma_j$, we have $\sigma_0 = \gamma_j\sigma_j$ and therefore $\ell(\sigma_0) = \ell(\gamma_j\sigma_j) = \ell(\gamma'_j R_i\sigma_j) \leq \ell(\gamma'_j) + \ell(R_i\sigma_j) \leq \ell(\gamma'_j) + \ell(\sigma_j) - 1 = \ell(\gamma_j) + \ell(\sigma_j) - 2 < \ell(\gamma_j) + \ell(\sigma_j) = \ell(\sigma_0)$, a contradiction. Hence $\ell(\gamma_j R_i) > \ell(\gamma_j)$ for $i \neq j$. If $\ell(\gamma_j R_j) > \ell(\gamma_j)$ also then $\gamma_j$ is identity, which is a contradiction. Therefore $\ell(\gamma_j R_j) < \ell(\gamma_j)$.

Now we prove the uniqueness. Suppose $\tau$ has the property $\ell(\tau) = \ell(\sigma_0) - \ell(\sigma_j), \ell(\tau R_i) > \ell(\tau)$ for $i \neq j$ and $\ell(\tau R_j) < \ell(\tau)$. We show that $\tau = \gamma_j$. Since $\ell(\tau R_j) < \ell(\tau)$, we must have $\tau \neq id$. We show that

$$\ell(\tau\sigma_j) = \ell(\tau) + \ell(\sigma_j) \tag{4.1.1}$$

If $\ell(\tau) = 1$ then $\tau = R_j$ and the equality follows easily, we assume $\ell(\tau) > 1$. Note that if we write $\tau = \tau'_2\tau'_1$ where $\ell(\tau) = \ell(\tau'_2) + \ell(\tau'_1)$ then $\ell(\tau'_1 R_j) < \ell(\tau'_1)$ and $\ell(\tau'_1 R_i) > \ell(\tau'_1)$ for $i \neq j$, otherwise we get a contradiction for the condition on $\tau$. Suppose

$$\ell(\tau\sigma_j) < \ell(\tau) + \ell(\sigma_j) \tag{4.1.2}$$

Since $\ell(\tau) > 1$ we can write $\tau = \tau_2 R_k \tau_1$ where

$$\ell(\tau) = \ell(\tau_2) + \ell(\tau_1) + 1, \quad \ell(\tau_2) \geq 0 \tag{4.1.3}$$

Choose $\tau_1$ such that

$$\ell(\tau_1\sigma_j) = \ell(\tau_1) + \ell(\sigma_j), \quad \ell(R_k\tau_1\sigma_j) < \ell(\tau_1\sigma_j) \tag{4.1.4}$$

This is possible because of the assumptions in eqn.(4.1.2). The choice of $\tau_1$ shows that

$$\ell(R_k\tau_1\sigma_j) = \ell(\tau) + \ell(\sigma_j) - 1 \tag{4.1.5}$$

By writing any reduced expression for $\tau_1, \sigma_j$ and applying "exchange condition" to $\tau_1\sigma_j$ we conclude from eqn.(4.1.4) that $R_k\tau_1\sigma_j$ is equal to either $\tau'_1\sigma_j$ or $\tau_1\sigma'_j$ where $\tau'_1$ has one generator less than that of expression for $\tau_1$ and similarly for $\sigma'_j$, and therefore $\ell(\tau'_1) < \ell(\tau_1)$. In fact, $\ell(\tau'_1) = \ell(\tau_1) - 1$ and $\ell(\sigma'_j) = \ell(\sigma_j) - 1$. This can be proved as follows. If $R_k\tau_1\sigma_j = \tau'_1\sigma_j$ then $\ell(\tau'_1\sigma_j) \leq \ell(\tau'_1) + \ell(\sigma_j)$ and also

$\ell(\tau_1'\sigma_j) = \ell(\tau_1)+\ell(\sigma_j)-1$ from eqn.(4.1.5) which gives $\ell(\tau_1)-1 \le \ell(\tau_1') < \ell(\tau_1)$ and we get the result for $\tau_1'$. Similarly the claim for $\sigma_j'$ can be proved. If $R_k\tau_1\sigma_j = \tau_1'\sigma_j$ holds then $R_k\tau_1 = \tau_1'$ where $\ell(R_k\tau_1) < \ell(\tau_1)$, a contradiction to eqn.(4.1.3). If $R_k\tau_1\sigma_j = \tau_1\sigma_j'$ then $R_k\tau_1 = \tau_1\sigma_j'\sigma_j = \tau_1 R_\ell$ for some $\ell \ne j$, since $\ell(\sigma_j'\sigma_j) = 1$ because $\sigma_j$ is the unique element of maximal length in $W(\Delta_j)$ and $\sigma_j' \in W(\Delta_j)$. In this case we have $\tau = \tau_2 R_k\tau_1 = \tau_2\tau_1 R_\ell$ for some $\ell \ne j$, and $\ell(\tau R_\ell) = \ell(\tau_2\tau_1) \le \ell(\tau_2)+\ell(\tau_1) < \ell(\tau)$ from eqn.(4.1.3). Again a contradiction to the assumptions on $\tau$. This proves eqn.(4.1.1). Therefore, we have $\ell(\tau\sigma_j) = \ell(\tau)+\ell(\sigma_j) = \ell(\sigma_0)$ from the condition on $\tau$. Since $\sigma_0$ is the unique element of maximal length in $W$, we must have $\tau\sigma_j = \sigma_0$ i.e. $\tau = \sigma_0\sigma_j = \gamma_j$ since $\sigma_j = \sigma_j^{-1}$. This completes the proof of the lemma 4.1. ☐

From now on we restrict our discussions to Weyl groups of type $A_n$. Let $\bar{A}_{n-1}$ denote the Dynkin diagram obtained by omitting the $n$th node in the Dynkin diagram of $A_n$. Therefore, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the simple roots corresponding to $A_n$ then $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the simple roots for $\bar{A}_{n-1}$. Let $W(A_n)$ and $W(\bar{A}_{n-1})$ denote the corresponding Weyl groups. We have the following

**Lemma 4.2.** *The subgroup $\Omega_0$ of $W(A_n)$ is a cyclic group of order $(n+1)$ generated by $\gamma_1 = R_nR_{n-1}\ldots R_2R_1$. Further all the nonidentity elements of $\Omega_0$ do not lie in $W(\bar{A}_{n-1})$.*

*Proof.* We have following correspondence from [5], page 661:

$$\lambda_i + \lambda_j = \lambda_k (mod X') \quad \text{iff} \quad \gamma_i\gamma_j = \gamma_k \tag{4.2.1}$$

where $\gamma_i = \sigma_0\sigma_i$ and $\sigma_i$ is the maximal element in the group $W(\Delta_i)$ where $\Delta_i = \{\alpha_j | j \ne i\}$. It is known that for $A_n$, $X/X'$ is a cyclic group of order $(n+1)$ generated by $\lambda_1 + X'$. Therefore, by the correspondence in eqn. (4.2.1), $\Omega_0$ is generated by $\gamma_1 = \sigma_0\sigma_1$. It is easy to see that $\ell(\gamma_1) = n$, since the length of the maximal element in $W(A_n)$ is $n(n+1)/2$. Further, $\ell(\gamma_1 R_1) < \ell(\gamma_1)$ and $\ell(\gamma_1 R_i) > \ell(\gamma_1)$ for $i \ne 1$. One can easily verify that the element $R_nR_{n-1}\ldots R_1$ of $W(A_n)$ satisfies the conditions. Then by Lemma 4.1, we must have $\gamma_1 = R_nR_{n-1}\ldots R_2R_1$. The element $\gamma = \gamma_1^{-1} = R_1R_2\ldots R_n$ is also a generator of $\Omega_0$. We have $R_{i+1}\gamma = \gamma R_i$ for $i = 1, 2, \ldots, n-1$. This gives $R_n\gamma^i = \gamma^i R_{n-1}$ for $i = 1, 2, \ldots, n-1$. Therefore, we have $R_n = \gamma^i R_{n-1}\gamma^{-i}$ for $i = 1, 2, \ldots, n-1$. Now $R_{n-i}$ for $i = 1, 2, \ldots, n-1$ lies in $W(\bar{A}_{n-1})$ and if any one of $\gamma^i$ for $i = 1, 2, \ldots, n-1$ lies in $W(\bar{A}_{n-1})$ then $R_n \in W(\bar{A}_{n-1})$. A contradiction. Therefore $\gamma^i$ for $i = 1, 2, \ldots, n-1$ and $\gamma^n = \gamma^{-1}$ do not lie in $W(\bar{A}_{n-1})$. ☐

The above lemma 4.2 leads to the decomposition of $W(A_n)$ given below.

**Lemma 4.3.** *Let $\gamma = R_1R_2 \ \ldots \ R_n$. Then $W(A_n)$ is disjoint union of $W(\bar{A}_{n-1})$, $\gamma\, W(\bar{A}_{n-1})$, $\ldots$, $\gamma^n W(\bar{A}_{n-1})$.*

*Proof.* Recall that $|W(A_n)| = (n+1)\,!$. Consider $\Omega_0\sigma$ for $\sigma \in W(\bar{A}_{n-1})$. Then $\Omega_0\sigma = \{\sigma, \gamma\sigma, \ldots, \gamma^n\sigma\}$. Now none of $\gamma^i\sigma$ for $i = 1, 2, \ldots, n$ can lie in $W(\bar{A}_{n-1})$ since $\gamma^i\sigma = \tau \in W(\bar{A}_{n-1})$ implies $\gamma^i = \tau\sigma^{-1} \in W(\bar{A}_{n-1})$ which contradicts lemma 4.2. Also $|\Omega_0\sigma| = n+1$. The cosets $\{\Omega_0\sigma | \sigma \in W(\bar{A}_{n-1})\}$ are $n\,!$ in number and they make up for $n\,!\,(n+1)$, i.e., $(n+1)\,!$ elements. Therefore $W(A_n) = \{\Omega_0\sigma | \sigma \in$

$W(\bar{A}_{n-1})\}$. This shows that $W(A_n) = \{\gamma^i \sigma | i = 0, 1, \ldots, n, \sigma \in W(\bar{A}_{n-1})\}$ which can be written as $W(A_n) = \{\gamma^i W(\bar{A}_{n-1}) | i = 0, 1, \ldots, n\}$ and this completes the proof. □

**Corollary 4.3.** Every coset of $\Omega_0$ in $W(A_n)$ contains a unique element of $W(\bar{A}_{n-1})$.

**Remark.** We can consider the Dynkin diagram obtained by deleting the first node in the Dynkin diagram of $A_n$ which we can denote by $\tilde{A}_{n-1}$. Then all the results mentioned in lemma 4.2 and lemma 4.3 are valid with $\bar{A}_{n-1}$ replaced by $\tilde{A}_{n-1}$ and $\gamma$ replaced by $\gamma_1$.

Now we come to our crucial lemma for proving the connectedness of $\Gamma(A_n)$.

**Lemma 4.4.** *In $W(A_n)$, for $n > 4$ we have $R_{n-3}R_{n-4} \longrightarrow R_n R_{n-1} R_{n-2} R_{n-1}$. In other words, for $n > 4$, $(R_{n-3}R_{n-4}, R_n R_{n-1} R_{n-2} R_{n-1})$ is an edge in $\Gamma(A_n)$.*

*Proof.* We have from [6] (pages 205-206) ,

$$\lambda_i = e_1 + \cdots + e_i - (i/(n+1))(e_1 + \cdots + e_{n+1}), \qquad \alpha_i = e_i - e_{i+1},$$

for $i = 1, 2, \ldots .n$ where $e_i$ are the orthonormal basis of the Euclidean space of dimension $(n+1)$. From this it easily follows that for $n > 4$,

$$\lambda_{n-1} + \lambda_{n-2} - \lambda_{n-4} = \alpha_{n-3} + 2\alpha_{n-2} + 2\alpha_{n-1} + \alpha_n. \tag{4.4.1}$$

This gives

$$(\lambda_{n-1} + \lambda_{n-2}) R_{n-1} R_{n-2} R_{n-1} R_n + (\delta - \lambda_{n-4}) R_{n-4} R_{n-3} = \delta R_n \tag{4.4.2}$$

In fact it is easy to verify that eqn.(4.4.2) gives eqn. (4.4.1) after using $\lambda_i R_j = \lambda_i - \delta_{ij}\alpha_j$, $\alpha_i R_{i+1} = \alpha_i + \alpha_{i+1}$, $\alpha_i R_{i-1} = \alpha_i + \alpha_{i-1}$, and $\alpha_i R_j = \alpha_i$ for $j \neq i+1, i-1$. Now eqn.(4.4.2) implies $R_{n-3}R_{n-4} \longrightarrow R_n R_{n-1} R_{n-2} R_{n-1}$ since $\delta R_n$ is $W$-regular as it is in the interior of a Weyl chamber. □

## 5. Connectedness of $\Gamma(A_n)$.

We state our main result in the following

**Theorem 5.1.** *The graph $\Gamma(A_n)$ for $n \geq 4$ is connected.*

*Proof.* The proof is by induction on n. The graph $\Gamma(A_4)$ is connected as it can be easily verified by the fusion method described in [7] applied to the claws of $\Gamma(A_4)$ given in the appendix. Suppose $\Gamma(A_{n-1})$ for $n > 4$ is connected. We show that $\Gamma(A_n)$ is connected. The subgraph $\Gamma(\bar{A}_{n-1})$ of $\Gamma(A_n)$ can be taken as $\Gamma(A_{n-1})$ as $\bar{A}_{n-1}$ is of same type as $A_{n-1}$. Suppose $C$ is a connected component of $\Gamma(A_n)$ containing the connected subgraph $\Gamma(\bar{A}_{n-1})$. From lemma 3.1, it easily follows that $\gamma' C$ is again a connected component for $\gamma' \in \Omega_0$.

From lemma 4.4, for $n > 4$, we have $(R_{n-3}R_{n-4}, R_n R_{n-1} R_{n-2} R_{n-1})$ is an edge in $\Gamma(A_n)$. This edge lies in $C$ as $R_{n-3}R_{n-4} \in W(\bar{A}_{n-1})$. Now the edge $(\gamma R_{n-3}R_{n-4}, \gamma R_n R_{n-1} R_{n-2} R_{n-1})$ which is same as $(R_1 R_2 \ldots R_n R_{n-3} R_{n-4}, R_1 R_2 \ldots R_{n-3} R_{n-1})$ lies in $\gamma C$. But the vertex $R_1 R_2 \ldots R_{n-3} R_{n-1}$ lies in $C$ as it is an element of $W(\bar{A}_{n-1})$. In other words, the vertex $R_1 R_2 \ldots R_{n-3} R_{n-1} \in C \cap \gamma C$.

This implies $C = \gamma C$, which in turn gives $C = \gamma^i C$ for $i = 1, 2, \ldots, n$. Therefore $C$ is stable under the group $\Omega_0$, since $\gamma$ generates $\Omega_0$. We conclude that $C$ has all the elements of $\gamma^i W(\bar{A}_{n-1})$ for $i = 0, 1, \ldots, n$, as vertices. By lemma 4.3, $W(A_n)$ is union of $\gamma^i W(\bar{A}_{n-1})$ for $i = 0, 1, 2, \ldots, n$. Therefore, $C$ contains all the elements of $W(A_n)$ as vertices. This proves $\Gamma(A_n)$ is connected for $n \geq 4$. $\square$

The theorem suggests the following conjecture on the connectivity of $\Gamma(W)$ for any Weyl group $W$.

**Conjecture.** If $\Delta$ is an irreducible root system then $\Gamma(W(\Delta))$ is a connected graph except when $\Delta$ is of type $A_1, A_2, A_3$ and $B_2$.

This conjecture is strongly supported by the graphs of the Weyl groups of type $G_2, B_3, C_3, B_4, C_4$ and $D_4$ for which we have the complete data with us.

## Appendix

A pair of vertices $u, v$ of a graph $\Gamma$ are said to be fused if the two vertices $u, v$ are replaced by a new vertex $w$ such that every edge incident on either $u$ or $v$ or on both is incident on $w$. Take any vertex $v_o$ of a graph and fuse all the vertices adjacent to it. Take this fused vertex and fuse it with all the vertices adjacent to it. Repeat this process till it is impossible to fuse the vertices any more. This method gives a connected component of the graph $\Gamma$ containing the vertex $v_0$. In this way we can find all the connected components of the graph $\Gamma$. In particular, if all the vertices of $\Gamma$ are fused into a single vertex then $\Gamma$ is connected. We have applied this method to the graph $\Gamma(A_4)$ to conclude that it is connected.

Let $\Gamma_1$ be a subgraph with vertices $v_0, v_1, \ldots, v_n$ and edges $(v_0, v_1), (v_0, v_2), \ldots, (v_0, v_n)$. We call $\Gamma_1$ a claw with centre $v_0$. We write this claw as $(\underline{v_0}, v_1, \ldots, v_n)$ where the centre $v_0$ is underlined. In general $(u_1, u_2, \ldots, \underline{u_i}, \ldots, u_n)$ denotes a claw with centre $u_i$.

Our computation of edges in $\Gamma(W)$ naturally gives the claws whose number is less than the order of $W$. In general, the values of $-\epsilon_{\sigma\sigma_0} + \epsilon_\tau$ with $\sigma$ fixed and $\tau$ varying gives a claw with centre $\sigma$. Listed below are the claws in $\Gamma(A_4)$. By applying the "fusion" method to these claws we find that $\Gamma(A_4)$ is connected.

The generators of $W(A_4)$ are $R_1, R_2, R_3$ and $R_4$ with relations $R_1^2 = R_2^2 = R_3^2 = R_4^2 = id, (R_1R_2)^3 = (R_2R_3)^3 = (R_3R_4)^3 = id, R_1R_3 = R_3R_1, R_1R_4 = R_4R_1$ and $R_2R_4 = R_4R_2$ where $id$ is the identity element of $W(A_4)$. If $R_{i_1}R_{i_2}\ldots R_{i_m}$ is an element of $W(A_4)$ then we write it as $i_1i_2\ldots i_m$. The claw $(3, \underline{3423}, 3121, 34121)$ means the subgraph of $\Gamma(A_4)$ with edges $(3423, 3), (3423, 3121)$ and $(3423, 34121)$ where $3, 3423, 3121$ and $34121$ are elements of $W(A_4)$.

**Claws of $\Gamma(A_4)$.**

($\underline{\mathrm{id}}$, 213, 3214, 2314, 41232, 324), ($\underline{234}$, 1213, 43213, 34213, 4232, 323413), ($\underline{1234}$, 213, 431213, 341213, 41232, 3123413), (423, 121, 4121, $\underline{3413}$, 4323413, 234232), (3, $\underline{34123}$, 43121, 13, 434121, 23123413), (23, $\underline{234123}$, 423121, 213, 4234121, 3123413), ($\underline{312}$, 343, 123121, 4121, 1234121, 3413), (2, $\underline{42312}$, 24, 12343, 34234121, 123413), (32, $\underline{342312}$, 324, 312343, 4234121, 3123413), ($\underline{321}$, 1232, 4324, 23124, 12324, 234121), ($\underline{4321}$, 41232, 324, 423124, 412324, 4234121), (124, $\underline{4123124}$, 123121, 234123121, 4121, 1213214321), ($\underline{23214}$, 21232, 34232, 123121, 234232, 1213214321), (14, $\underline{4123214}$, 3421232, 2341232, 34123121, 21234232), (413, $\underline{3431213}$, 3413, 234232, 321234232, 1213214321), ($\underline{34}$, 3213, 34213, 232), ($\underline{4123}$, 43121, 43123413, 1234232), (3, $\underline{3423}$, 3121, 34121), (2, $\underline{2312}$, 2343, 23413), ($\underline{4312}$, 12343, 4123121, 41234121), ($\underline{21}$, 232, 2324, 23124), (3124, 12324, $\underline{34123124}$, 34121), ($\underline{123214}$, 341232, 3421232, 1234232), ($\underline{423214}$, 421232, 2341232, 4123121), (4213, 43213, $\underline{23431213}$, 23413), ($\underline{4}$, 213, 4213), ($\underline{23}$, 121, 323413), ($\underline{123}$, 3121, 3123413), ($\underline{32}$, 343, 234121), ($\underline{432}$, 2343, 4234121), ($\underline{1}$, 324, 3124), ($\underline{32343}$, 34121, 234121), ($\underline{312343}$, 434121, 1234121), ($\underline{2312343}$, 4234121, 41234121), (24, $\underline{42324}$, 4123121), ($\underline{43124}$, 4123121, 34123121), (124, $\underline{412324}$, 34123121), (24, $\underline{423124}$, 234123121), (324, 2324, $\underline{3423124}$), ($\underline{314}$, 232, 1232), ($\underline{214}$, 232, 4232), ($\underline{2314}$, 4232, 21232), ($\underline{3214}$, 1232, 34232), ($\underline{12314}$, 41232, 421232), ($\underline{43214}$, 412, 341232), ($\underline{41213}$, 1234232, 21234232), (13, $\underline{31213}$, 1234232), (13, $\underline{341213}$, 321234232), (413, $\underline{431213}$, 21234232), (213, 3213, $\underline{2341213}$), ($\underline{23121}$, 23413, 323413), ($\underline{423121}$, 123413, 4323413), ($\underline{3423121}$, 3123413, 43123413), (23413, 123413, $\underline{423123413}$), (34121, 434121, $\underline{341234121}$), ($\underline{43}$, 23413), ($\underline{12}$, 34121), ($\underline{2343}$, 4121), ($\underline{2324}$, 123121), ($\underline{3124}$, 123121), ($\underline{123124}$, 4123121), ($\underline{341232}$, 1213214321), ($\underline{421232}$, 1213214321), (232, $\underline{23421232}$), ($\underline{4213}$, 234232), ($\underline{3213}$, 234232), ($\underline{343213}$, 1234232), ($\underline{3121}$, 3413), (3413, $\underline{43123413}$), (4121, $\underline{41234121}$).

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