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TOTAL CONNECTIONS IN LIE GROUPOIDS

George Virsik

Abstract. A total connection of order r in a Lie groupoid Φ over M is defined as a first order connections in the $(r-1)$-st jet prolongations of Φ. A connection in the groupoid Φ together with a linear connection on its base, ie. in the groupoid $\Pi(M)$, give rise to a total connection of order r, which is called simple. It is shown that this simple connection is curvature-free iff the generating connections are. Also, an r-th order total connection in Φ defines a total reduction of the r-th prolongation of Φ to $\Phi \times \Pi(M)$. It is shown that when $r > 2$ then this total reduction of a simple connection is holonomic iff the generating connections are curvature free and the one on M also torsion-free.

The concept of higher order connections in differential geometry was introduced by Ehresmann who used Lie groupoids rather than principal bundles to study geometric structures (c.f. [Ehresmann 56]). It is a well known fact that the category of Lie groupoids and that of principal bundles are equivalent in the sense that any concept or result obtained in the "language" of one of them is readily translated into the "language" of the order. Roughly speaking, if P is a principal bundle, we can associate with it the Lie groupoid PP^{-1}, and conversely, if Φ is a Lie groupoid and a the source map, then $\Phi_x = \{\theta \in \Phi : a\theta = x\}$ for any element x of the base, is a principal bundle. Many results on higher order connections and associated geometric objects have been obtained since the pioneering work by Ehresmann, some using Lie groupoids, some using principal bundles and their prolongations as the basic geometric structure on which to study higher order connections. Recent applications in gauge theory, including the higher order case, have also contributed to a renewed interest in these studies.

The present paper uses the Lie groupoid approach to r-th order total connections defined as first order connections in the $(r-1)$-st prolongation of the given Lie groupoid. Though it is true that an r-th order total connection induces an r-th order connection in the sense of Ehresmann but in general not vice-versa, it turns out that some results about total connections can be obtained parallelly with those known for higher order connections in the sense of Ehresmann (c.f.

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Proposition 3 below). Kolář was the first to study explicitly the relation between these two approaches to higher order connections, and [Kolář 74], written in the Lie groupoid language, was also the starting point for the present paper. The basic method used is that of "transporting connections along functors" including those which arise from a connection as reductions of some prolongations of Lie groupoids. These reductions associated with higher order connections were first introduced and studied in [Que 67]. We generalise some results from [Kolář 74, 75], notably about problems concerning integrability of higher order connections obtained from a first order connection in the groupoid together with a linear connection on its base. These results are applied to the special case of the Lie groupoid associated with the frame bundle of a given manifold, and compared with those already obtained, for instance in [Kolář, to appear].

Manifolds and maps shall always be smooth – ie. C^∞ – and finite dimensional. Following [Kolář-Michor-Slovák] $\mathcal{M}f$ denotes the category of such manifolds, $\mathcal{M}f_m$ the subcategory of $\mathcal{M}f$ consisting of m-dimensional manifolds and local diffeomorphisms between them. $\mathcal{FM}$ will be the category of fibred manifolds $p : E \to M$ (ie. surjections of maximal rank) and fibre preserving maps, and $\mathcal{FM}(M) \subset \mathcal{FM}$ for a fixed manifold M will denote the subcategory of fibred manifolds over a fixed base M with morphisms as maps which induce the identity on the base. As usual, the fibre $p^{-1}(x) \subset E$ will also be denoted by E_x. If $\mathcal{C}$ is any of these categories, we shall use the term "$\mathcal{C}$-morphism" or "$\mathcal{C}$-equivalence" when referring to $\mathcal{C}$. We shall work mainly with the category of Lie groupoids over a fixed base M and smooth functors over the identity on M (c.f. [Que 67]).

Let $\mathbf{F}$ and $\mathbf{G}$ be two functors $\mathcal{FM}(M) \to \mathcal{FM}(M)$. For a given object $p : E \to M$ of $\mathcal{FM}(M)$ we shall denote by $\mathbf{F}(E) \to M$ and $\mathbf{G}(E) \to M$ its images under $\mathbf{F}$ and $\mathbf{G}$ respectively. $\mathbf{F}$ is said to be a *subfunctor* of $\mathbf{G}$, written $\mathbf{F} \subset \mathbf{G}$, if for each $p : E \to M$ the fibred manifold $\mathbf{F}(E) \to M$ is a fibred submanifold of $\mathbf{G}(E) \to M$ [ie. $\mathbf{F}(E) \subset \mathbf{G}(E)$ is a submanifold and the projection $\mathbf{F}(E)$ is the restriction of $\mathbf{G}(E) \to M$], and for each morphism $h : E \to E'$ the morphism $\mathbf{F}(h) : \mathbf{F}(E) \to \mathbf{F}(E')$ is the restriction of $\mathbf{G}(h) : \mathbf{G}(E) \to \mathbf{G}(E')$. In other words, $\mathbf{F}$ is a subfunctor of $\mathbf{G}$, if there is a natural transformation from $\mathbf{F}$ to $\mathbf{G}$ which is a regular embedding for each object of $\mathcal{FM}(M)$. Also, if $\mathbf{F}$ is a subfunctor of $\mathbf{G}$, $\mathbf{F'}$ a subfunctor of $\mathbf{G'}$ then a natural transformation $\Pi : \mathbf{F} \to \mathbf{G}$ is said to *preserve* these subfunctors if it is at the same time a natural transformation $\Pi : \mathbf{F'} \to \mathbf{G'}$.

For two manifolds M and N, denote as usual by $J^r(M,N)$ the manifold of all holonomic r-jets ($r \geq 1$) from M to N, and write $J(M,N)$ instead of $J^1(M,N)$. Denote also by $\alpha : J^r(M,N) \to M$ and $\beta : J^r(M,N) \to N$ the source and target maps respectively, by $J^r_x(M,N) \subset J^r(M,N)$ the submanifold of jets with source x, and by $J^r_x(M,N)_y \subset J^r(M,N)$ the submanifold of jets with source x and target y. Similarly for the manifolds $\widetilde{J}^r(M,N)$ and $\bar{J}^r(M,N)$ of non-holonomic and semi-holonomic jets respectively (c.f. [Ehresmann 54]). We shall use the symbol $\circ$ to denote composition of jets, ie. if $Z = j^r_x f \in J^r(M,N)$ and $Y = j^r_y g \in J^r(N,P)$, $y = f(x)$, then $Y \circ Z = j^r_x(g \circ f) \in J^r(M,P)$ with an appropriate extension to non-holonomic and semi-holonomic jets. Also, $j^r_x(t \mapsto f(t))$ will sometimes stand for $j^r_x f$, and we shall use the abbreviated notation $j^r_x = j^r_x(t \mapsto t)$ and $j^r_x[c] = j^r_x(t \mapsto c)$

for the jets of the identity map and the constant map respectively. Recall that in [KoláŠ, Michor, Slovák] J^r is regarded as a functor from the product category $\mathcal{M}f_m \times \mathcal{M}f$ into $\mathcal{FM}$. We shall define the functor $\mathbf{J}^r : \mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}$ recurrently as $\mathbf{J}^1 = \mathbf{J} = J^1$ and $\mathbf{J}^r(M,N) = J^1(M, \mathbf{J}^{r-1}(M,N))$, $\mathbf{J}^r(f,g) = J^1(f, \mathbf{J}^{r-1}(f,g))$ for $r > 1$ and call elements of $\mathbf{J}^r(M,N)$ *iterated jets* (of order r). Of special interest is the case when $N = E$ is the total space of a fibred manifold $p : E \to M$. In this case we shall write $\mathbf{J}(E) = \mathbf{J}(M,E)$ and regard it as the fibred manifold $\mathbf{J}(M,E) \to M$ where the projection is the source map $\alpha : J^1(M,E) \to M$. In this sense $\mathbf{J}$ can be seen as an endofunctor on $\mathcal{FM}(M)$ and similarly $\mathbf{J}^r : \mathcal{FM}(M) \to \mathcal{FM}(M)$ if it is defined as $\mathbf{J}^r(E) = \mathbf{J}^r(M,E) \to M$. Moreover, in the context of fibred manifolds, $\mathbf{J}^r : \mathcal{FM}(M) \to \mathcal{FM}(M)$ can be seen as an iteration of $\mathbf{J}$, explicitly $\mathbf{J}^r = \mathbf{J} \circ \mathbf{J}^{r-1}$ for $r > 1$.

Thus for a fixed manifold M we have a functor $\mathbf{J} : \mathcal{FM}(M) \to \mathcal{FM}(M)$ which assigns to the fibred manifold $p : E \to M$ the fibred manifold $\alpha : \mathbf{J}(E) \equiv J(M,E) \to M$, and to $h : E \to E'$ the map $\mathbf{J}(h) \equiv J(M,h) : J(M,E) \to J(M,E')$ given by the composition of one-jets $J(M,h)Z = j^1_{\beta Z} h \circ Z$. Denoting by $\mathbf{I}$ the identity functor on $\mathcal{FM}(M)$, by $\mathbf{O}$ the zero functor on $\mathcal{FM}(M)$ [which assigns to $p : E \to M$ the "collapsed" fibred manifold $\mathrm{id} : M \to M$], we can regard p, α and β as natural transformations $p : \mathbf{I} \to \mathbf{O}$, $\beta : \mathbf{J} \to \mathbf{I}$ and $\alpha = p \circ \beta : \mathbf{J} \to \mathbf{O}$.

We have also $\mathbf{J}^r$, the r-th iteration of $\mathbf{J}$, and we shall write also $\mathbf{J}^0 = \mathbf{I}$ and $\mathbf{J}^{-1} = \mathbf{O}$. For each $r \geq 0$ the target map β defines a natural transformation $\pi^r_{r-1} : \mathbf{J}^r \to \mathbf{J}^{r-1}$, and by iteration $\pi^r_s : \mathbf{J}^r \to \mathbf{J}^s$, for any $0 \leq s \leq r$ with $\pi^r_r = \mathrm{id}$. This can be further extended to $\pi^r_s : \mathbf{J}^r \to \mathbf{J}^s$, for any $-1 \leq s \leq r$ by defining $\pi^r_{-1} : \mathbf{J}^r \to \mathbf{O}$ as $\alpha^r \equiv p \circ \pi^r_0$. It is not hard to verify that

$$\pi^r_s = \pi^q_s \circ \pi^r_q \qquad \text{for} \qquad -1 \leq s \leq q \leq r\,. \tag{1}$$

Observe that $\mathbf{J}^r \circ \mathbf{J}^s = \mathbf{J}^s \circ \mathbf{J}^r = \mathbf{J}^{r+s}$ only for $0 \leq s \leq r$, whereas $\mathbf{J}^{-1} \circ \mathbf{J}^s = \mathbf{J}^{-1} = \mathbf{O}$ and $\mathbf{J}^r \circ \mathbf{J}^{-1} = \mathbf{O}^r$ is the constant functor which assigns to $E \to M$ the fibred manifold $\mathbf{J}^r(M) \to M$ and to any morphism of $\mathcal{FM}(M)$ the identity on $\mathbf{J}^r(M) \to M$. Elements of $\mathbf{J}^r(E)$ will be called *iterated jets* of $E \to M$.

The fact that π^r_s is a natural transformation is expressed by the formula

$$\pi^r_s \mathbf{J}^r(h) = \mathbf{J}^s(h)\pi^r_s \quad \text{for} \quad -1 \leq s \leq r\,, \text{ and any morphism } h \in \mathcal{FM}(M)\,. \tag{2}$$

Substituting here $h = \pi^k_l$, $0 \leq l \leq k$ for a fixed fibred manifold $E \to M$, and observing that now $\mathbf{J}^r(h) : \mathbf{J}^{r+k}(E) \to \mathbf{J}^{r+l}(E)$ for $0 \leq r$, similarly $\mathbf{J}^s(h)$, we obtain

$$\pi^{r+l}_{s+l} \mathbf{J}^r(\pi^k_l) = \mathbf{J}^s(\pi^k_l)\pi^{r+k}_{s+k} \quad \text{for} \quad 0 \leq s \leq r \quad \text{and} \quad 0 \leq l \leq k\,. \tag{3}$$

In case of $h = \pi^k_{-1}(E) : \pi^k_{-1} = \alpha^k : \mathbf{J}^k(E) \to \mathbf{J}^{-1}(E)$, $k \geq 0$, we have $\mathbf{J}^r(\alpha^k) : \mathbf{J}^{r+k}(E) \to \mathbf{O}^r(E)$ and similarly $\mathbf{J}^s(\alpha^k) : \mathbf{J}^{s+k}(E) \to \mathbf{O}^s(E)$ unless $s = -1$ in which case $\mathbf{J}^s(\alpha^k) = \mathbf{O}(E) \to \mathbf{O}(E)$. This leads to

$$\pi^r_s \mathbf{J}^r(\alpha^k) = \mathbf{J}^s(\alpha^k)\pi^{r+k}_{s+k} \qquad \text{for} \quad 0 \leq s \leq r, \text{ and } 0 \leq k\,, \tag{4}$$

$$\alpha^r \mathbf{J}^r(\alpha^k) = \alpha^{r+k} \qquad \text{for} \quad 0 \leq r, \text{ and } 0 \leq k \tag{5}$$

Similarly, taking $s=-1$ and $0\le l\le k$ in (3), one obtains

$$\alpha^{r+l}\mathbf{J}^r(\pi^k_l)=\alpha^{r+k} \qquad \text{for} \quad 0\le r\,, \text{and } 0\le l\le k\,. \tag{6}$$

For a fibred manifold $p:E\to M$ denote by $J_1E\subset\mathbf{J}(E)$ the space of one-jets of local sections of $p:E\to M$, ie. of one-jets j^1_xs where $ps(u)=u$ for u in a neighbourhood of x. Thus $Z\in\mathbf{J}(E)_x\equiv J_x(M,E)$ belongs to J_1E iff $\mathbf{J}(p)Z=j^1_x$. Hence J_1 is a subfunctor of $\mathbf{J}$ and $Z\in\mathbf{J}^r(E)_x$ belongs to $\widetilde{J}_r(E)$ iff $\mathbf{J}^r(p)Z=j^r_x$. Its iteration gives the subfunctor $\widetilde{J}_r$ of $\mathbf{J}^r$, explicitly $\widetilde{J}_r=J_1\circ\widetilde{J}_{r-1}$ and the elements of $\widetilde{J}_r(E)$ are called non-holonomic r-jets of local sections of $E\to M$; we let again $\widetilde{J}_0$ stand for the identity functor.

For each fibred manifold $E\to M$ and each $r\ge 0$ consider the subset $\mathbf{S}^r(E)=\{Z\in\mathbf{J}^r(E):\pi^r_s(Z)=\mathbf{J}^k(\pi^{r-k}_{s-k})(Z)$ whenever $0\le k\le s<r\}$. We obtain again subfunctors $\mathbf{S}^r\subset\mathbf{J}^r$ preserved by $\pi^r_s:\mathbf{S}^r\to\mathbf{S}^s$. The elements of $\bar{J}_r(E)=\widetilde{J}_r(E)\cap\mathbf{S}^r(E)$ are the semi-holonomic r-jets of local sections of $E\to M$. Explicitly, $Z\in\bar{J}_r(E)$ iff $Z\in\ge\widetilde{J}_r(E)$ and either

$$\begin{aligned} &Z=j^1_xs \quad \text{for some local section } s \text{ of } \bar{J}_{r-1}(E)\to M\\ &\qquad\qquad \text{and it satisfies } \mathbf{J}^1(\pi^{r-1}_{r-2})Z=\pi^r_{r-1}Z \end{aligned} \tag{7}$$

or

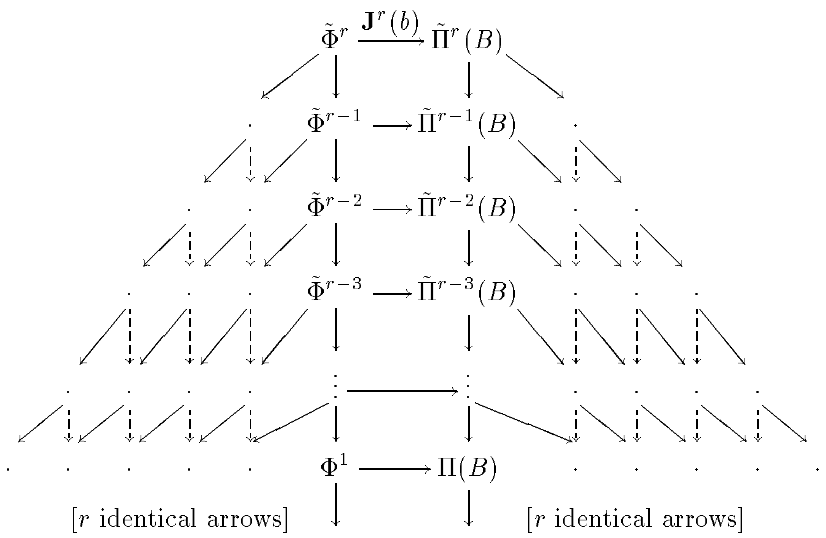
$$\pi^r_s(Z)=\mathbf{J}^k(\pi^{r-k}_{s-k})(Z) \quad \text{whenever } 0\le k\le s<r \tag{8}$$

or

$$\pi^r_s(Z)=\mathbf{J}(\pi^{k-1}_{s-1})\pi^r_k(Z) \text{ whenever } 1\le s<k\le r \tag{9}$$

Finally, $J_r(E)\subset\bar{J}_r(E)$ is the subbundle of holonomic r-jets j^r_xs of local sections s of $E\to M$. Thus for each $r>0$ we have a sequence of subfunctors $J_r\subset\bar{J}_r\subset\widetilde{J}_r\subset\mathbf{J}^r$ which are preserved by the natural transformations $\pi^r_s:\mathbf{J}^r\to\mathbf{J}^s$. In what follows, higher order jets, prolongations, connections etc. are understood to be non-holonomic unless otherwise stated.

Let Φ be a fixed Lie groupoid over M, a, $b:\Phi\to M$ the source and target surjections, $\sim$: $M\to\Phi$ the injection of units. Given an integer $r>0$ let $\widetilde{\Phi}^r=\{Z\in\widetilde{J}^r(M,\Phi):\mathbf{J}^r(a)Z=j^r_{\alpha(Z)},\ \mathbf{J}^r(b)Z\in\widetilde{\Pi}^r(M)\}$ be the r-th prolongation of Φ, and let $\widetilde{\Pi}^r(M)$ denote the groupoid of invertible r-jets from M to M, which is the r-th prolongation of the trivial groupoid $M\times M$. We can associate with Φ the diagram of functors



The vertical arrows in (10) stand for direct projections π^s_{s-1} and the slant ones for their "lifts" $\mathbf{J}(\pi^s_{s-1})$. In general, there are $\binom{r}{s}$ functors $\widetilde{\Phi}^r \to \widetilde{\Phi}^s$ (all suitable combinations of direct and lifted projections). However, because of (3), especially $\pi^{s-1}_{s-2} \circ \mathbf{J}(\pi^{s-1}_{s-2}) = \pi^s_{s-1}$, we need to consider in (10) as projections $\widetilde{\Phi}^r \to \widetilde{\Phi}^s$, $0 < s < r$ only "paths" consisting of a sequence of vertical arrows followed by a sequence of slant arrows. For a fixed pairs $s < r$ there are $r - s + 1$ of such paths, and they are

$$\pi^{r\to i}_s = \mathbf{J}(\pi^{i-1}_{s-1}) \circ \pi^r_i : \widetilde{\Phi}^r \to \widetilde{\Phi}^s, \quad i = s,\ s+1, \dots, r \tag{11}$$

or

$$\pi^{r\to i}_s = \mathbf{J}(\pi^{i-1}_{s-1}) \circ \pi^r_i : \widetilde{\Pi}^r(M) \to \widetilde{\Pi}^s(M), \quad i = s,\ s+1, \dots, r\,. \tag{12}$$

Observe that these are exactly the functors listed in (9), ie. an element of $\widetilde{\Phi}^r$ (or $\widetilde{\Pi}^r(M)$) is semiholonomic exactly when all these $r - s + 1$ functors $\widetilde{\Phi}^r \to \widetilde{\Pi}^s$ (or $\widetilde{\Pi}^r(M) \to \widetilde{\Pi}^s(M)$) coincide for any $s < r$.

In particular, we obtain r-projections $\pi^{r\to i}_1 : \widetilde{\Pi}^r(M) \to \Pi(M)$, $i = 1, \dots, r$, hence also

$$\Pi^r = (\pi^{r\to 1}_1, \pi^{r\to 2}_1, \dots \pi^{r\to r}_1) : \widetilde{\Pi}^r(M) \to \Pi(M) \times \cdots \times \Pi(M)_{[r\ \text{times}]}\,. \tag{13}$$

Composing $\mathrm{id}_\Phi \times \Pi^r$ with $(\pi^r_0, \mathbf{J}^r(b)) : \widetilde{\Phi}^r \to \Phi \times \widetilde{\Pi}^r(M)$ – and observing that $\pi^{r\to i}_1 \circ \mathbf{J}^r(b) = \mathbf{J}(b) \circ \pi^{r\to i}_1$ for $i = 1, \dots, r$ – we obtain the natural functor

(14)

$$\begin{aligned}\boldsymbol{\Phi}\Pi^r &= (\mathrm{id}_\Phi \times \Pi^r) \circ (\pi^r_0, \mathbf{J}^r(b)) = \\ &(\pi^r_0, \mathbf{J}(b) \circ \pi^{r\to 1}_1, \dots \mathbf{J}(b) \circ \pi^{r\to r}_1) : \widetilde{\Phi}^r \to \Phi \times \Pi(M) \times \cdots \times \Pi(M)_{[r\ \text{times}]}\,.\end{aligned}$$

Let $\widetilde{Q}^k(\Phi) = \{Z \in \widetilde{J}^k(M,\Phi) : \mathbf{J}^k(a)Z = j^k_x[x],\ \mathbf{J}^k(b)Z \in j^k_x,\ \pi^k_0 Z = \sim (x),$ $(\alpha(Z) = x)\}$. Then $\alpha : \widetilde{Q}^k(\Phi) \to M$ is a fibred manifold and its sections are called k-th order connections in Φ (c.f. [Ehresmann 56]). Recall that $\widetilde{Q}^k$ can be extended to a functor $\widetilde{Q}^k$ from the category of Lie groupoids over M to $\mathcal{FM}(M)$ and that $\widetilde{\Phi}^k$ acts on $\widetilde{Q}^k(\Phi)$ via

$$\begin{aligned} &\Xi \in \widetilde{Q}^k, \zeta \in \widetilde{Q}^k(\Phi) \Rightarrow \\ &(\Xi,\zeta) \mapsto \Xi \vartriangle \zeta = (\Xi \cdot \zeta \cdot j^k_x[\pi^k_0(\Xi^{-1})] \circ \{\mathbf{J}^k(b)\Xi\}^{-1}\,, \end{aligned} \tag{15}$$

where $\cdot$ denotes the prolongation of composition in Φ to $J^k(M,\widetilde{Q}^r)$.

A k-th order connection in $\widetilde{\Phi}^r$, $(k > 0$ and $r \geq 0)$, ie. a section $\Gamma : M \to \widetilde{Q}^k(\widetilde{\Phi}^r)$ will be called an *(r,k)-connection in Φ*. It defines a reduction of $\widetilde{\Phi}^{r+k}$, ie. an injective functor $[\Gamma] : \widetilde{\Phi}^r \times \widetilde{\Pi}^k(M) \to \widetilde{\Phi}^{r+k}$ which is a right inverse of the corresponding canonical projections via (10). The functor $[\Gamma] : \widetilde{\Phi}^r \times \widetilde{\Pi}^k(M) \to \widetilde{\Phi}^{r+k}$ is constructed as follows (c.f. [Que 67]). It is uniquely determined by the requirement that it be a groupoid isomorphism of $\widetilde{\Phi}^r \times \widetilde{\Pi}^k(M)$ onto the subgroupoid $\Phi_\Gamma = \{\Xi \in \widetilde{\Phi}^{r+k} :$ $\Xi \vartriangle \Gamma(x) = \Gamma(y)$ where $x = \alpha\Xi = \alpha\Gamma(x)$ and $y = b \circ \pi^{r+k}_0 \Xi = \alpha\Gamma(y)\}$. Here $\vartriangle$ denotes the action of $\widetilde{\Phi}^{r+k} = (\widetilde{\Phi}^r)^k$ on $\widetilde{Q}^k(\widetilde{\Phi}^r)$ defined by (15), ie.

$$\begin{aligned} &\Xi \in \widetilde{Q}^{r+k}, \zeta \in \widetilde{Q}^k(\widetilde{\Phi}^r) \Rightarrow \\ &(\Xi,\zeta) \mapsto \Xi \vartriangle \zeta = (\Xi \cdot \zeta \cdot j^k_x[\pi^{r+k}_r(\Xi^{-1})]) \circ \{\mathbf{J}^k(b \circ \pi^r_0)\Xi\}^{-1}\,, \end{aligned} \tag{15a}$$

where $\cdot$ again denotes the prolongation of composition in Φ to $J^k(M,\widetilde{\Phi}^r)$ and $b \circ \pi^r_0 : \widetilde{\Phi}^r \to M$ is the target surjection in the groupoid $\widetilde{\Phi}^r$ (c.f [Virsik 69]). It is a matter of straightforward verification to see that the required groupoid isomorphism $\widetilde{\Phi}^r \times \widetilde{\Pi}^k(M) \to \Phi_\Gamma \subset \widetilde{\Phi}^r$ is given by

$$\begin{aligned} &[\Gamma] : \widetilde{\Phi}^r \times \widetilde{\Pi}^k(M) \to \widetilde{\Phi}^{r+k} \\ &(Z,X) \mapsto (\Gamma(y) \circ X) \cdot j^1_x[Z] \cdot \Gamma(x)^{-1} \end{aligned} \tag{16}$$

where $\cdot$ denotes again the prolongation of composition in $\widetilde{\Phi}^r$ and $x = \alpha X = \alpha^r Z$, $y = \pi^k_0 X = b \circ \pi^r_0 Z$.

A connection $\Gamma : M \to \widetilde{Q}^k(\widetilde{\Phi}^r)$ can be a holonomic or semi-holonomic k-th order connection in the r-th holonomic or semi-holonomic prolongation of Φ, altogether nine possibilities. For instance, if Γ is a holonomic k-th order connection in the r-th order semi-holonomic prolongation of Φ, ie. if $\Gamma : M \to Q^k(\bar{\Phi}^r)$ then also $[\Gamma] : \bar{\Phi}^r \times \Pi^k(M) \to (\bar{\Phi}^r)^k$. In this case we shall say that Γ is a connection of type (rS, kH), or briefly, an (rS, kH)-connection.

If Ψ is another Lie groupoid over M and $\varphi : \Phi \to \Psi$ a smooth functor over the identity on M, then φ will assign to each (r,k)-connection in Φ an (r,k)-connection in Ψ of the same type, explicitly, $\varphi\langle\Gamma\rangle : x \mapsto \mathbf{J}^{r+k}(\varphi)\Gamma(x)$. The diagram (10) of functors can thus generate all sorts of connections of order k from a given (r,k)-connection. On the other hand, a given (r,k)-connection Γ itself can serve as a generator of (s,l)-connections in $\widetilde{\Phi}^{r+k}$ from those in $\widetilde{\Phi}^r \times \widetilde{\Pi}^k(M)$ via the reduction $[\Gamma] : \widetilde{\Phi}^r \times \widetilde{\Pi}^k(M) \to \widetilde{\Phi}^{r+k}$.

Most important are the cases when either $r = 0$ or $k = 1$. We shall refer to $(0,k)$-connections in Φ as *k-th order connections in Φ* (if $k > 1$), to distinguish them from total connections in Φ: An *r-th order total connection in* Φ is a connection of type $(r-1,1)$ in Φ. The diagram (10) shows that an r-th order total connection in Φ gives rise to a number of s-th order total connections in Φ (where $s < r$) via the functors $\pi_{s-1}^{r-1\to i}$, $i = s-1,\dots,r-1$ as in (11), altogether $r-s+1$ of them. It will also give rise to the same number of s-th order total connections in $M \times M$ via the functor $\mathbf{J}^r(b)$ followed by $\pi_{s-1}^{r-1\to i}$, $i = s-1,\dots,r-1$ as in (12). Of course, an s-th order total connection in $M \times M$ is the same as an $(s-1)$-st order total connection in $\Pi(M)$ or on M; in particular, a second order total connection in $M \times M$ is the usual linear connection on M.

More important is the r-th order connection in Φ generated by an r-th order total connection via the map $\sigma_{r-1} : Q^1(\widetilde{\Phi}^{r-1}) \to \widetilde{Q}^r(\Phi)$, defined in [KoláŠ 74], which takes $Q^1(\bar{\Phi}^{r-1})$ into $\bar{Q}^r(\Phi)$. This indicates that a total connection of order r is "more" than a connection (in the sense of [Ehresmann 56]) of the same order. The same is suggested by

Proposition 1 (c.f. also [KoláŠ, Virsik]). *There exists a canonical $\mathcal{FM}(M)$-equivalence*

$$K : Q^1(\Phi^1) \to \bar{Q}^2(\Phi) \times Q^1(\Pi(M)) \,. \tag{17}$$

Proof. Define $K(Z) = (k_1(Z, \mathbf{J}(\pi_0^1)Z), \mathbf{J}^2(b)Z)$, where for $Z = j_x^1\zeta$ we put (c.f. [KoláŠ 74]) $k_1(Z,T) = j_x^1(t \mapsto (\zeta(t)\cdot T) \circ \mathbf{J}(b)\zeta(t)^{-1})$. Its inverse is $\kappa_2 : \bar{Q}^2(\Phi) \times Q^1(\Pi(M)) \to Q^1(\Phi^1)$ also defined in [KoláŠ 74] by $\kappa_2(X,Y) = j_x^1(t \mapsto v(t) \circ \lambda(t) \cdot v^{-1}(x))$, where $X = j_x^1 v \in \bar{Q}^2(\Phi)_x$ and $Y = j_x^1\lambda \in Q^1(\Pi(M))_x$.

Observe that $\mathbf{J}(\pi_0^1)\kappa_2(X,Y) = j_x^1(t \mapsto \pi_0^1(v(t)\circ\lambda(t)) \cdot \pi_0^1 v^{-1}(x)) = j_x^1(t \mapsto \pi_0^1(v(t)\circ\lambda(t))) \cdot j_x^1[\pi_0^1 v^{-1}(x)] = \mathbf{J}(\pi_0^1)(X) \cdot \pi_0^2 X^{-1} = \mathbf{J}(\pi_0^1)(X)$, ie.

$$\mathbf{J}(\pi_0^1)\kappa_2(X,Y) = \pi_1^2 X \tag{18}$$

since X is semi-holonomic. Also, $\mathbf{J}^2(b)\kappa_2(X,Y) = j_x^1(t \mapsto \mathbf{J}(b)v(t)\circ\lambda(t)) = j_x^1(t \mapsto \lambda(t))$. Since $v(t) \in Q^1(\Phi)$ we have $\mathbf{J}(b)v(t) = j_t^1$, and so we can conclude that

$$\mathbf{J}^2(b)\kappa_2(X,Y) = Y \,. \tag{19}$$

To show that K and κ_2 are indeed mutually inverse first use (18) and the formula (17) of [KoláŠ 74] to obtain

$$\begin{aligned} K(\kappa_2(X,Y)) &= (k_1(\kappa_2(X,Y), \mathbf{J}(\pi_0^1)\kappa_2(X,Y)), \mathbf{J}^2(b)\kappa_2(X,Y)) \\ &= (k_1(\kappa_2(X,Y), \pi_1^2 X), \mathbf{J}^2(b)\kappa_2(X,Y)) \\ &= (X, \mathbf{J}^2(b)\kappa_2(X,Y)) = (X,Y) \end{aligned}$$

by (19) . As for the converse, use formula (16) of [KolářŠ 74] to obtain

$$\kappa_2(K(Z)) = \kappa_2(k_1(Z, \mathbf{J}(\pi_0^1)Z), \mathbf{J}^2(b)Z) = Z\,. \qquad \square$$

Thus a second order total connection uniquely determines and is uniquely determined by a second order semi-holonomic connection together with a linear connection on M. More generally, an r-th order total connection in Φ uniquely determines and is uniquely determined by a second order semi-holonomic connection in $\widetilde{\Phi}^{r-2}$ (ie. a $(2S, r-2)$-connection in Φ) together with a linear connection on M.

An r-th order connection C in Φ defines a reduction $[C] : \Phi \times \widetilde{\Pi}^r(M) \to \widetilde{\Phi}^r$, a right inverse of the canonical projections $\widetilde{\Phi}^r \to \Phi \times \widetilde{\Pi}^r(M)$ defined by (10). On the other hand, an r-th order total connection Γ defines a reduction $[\Gamma] : \widetilde{\Phi}^{r-1} \times \Pi(M) \to \widetilde{\Phi}^r$. This Γ actually gives rise to a whole sequence of reductions $[\mathbf{J}(\pi^s_{s-1}) \circ \Gamma] : \widetilde{\Phi}^{s-1} \times \Pi(M) \to \widetilde{\Phi}^s$ for $s = 1, 2, \ldots, r$, hence to a reduction

$$\{\Gamma\} : \Phi \times \Pi(M) \times \cdots \times \Pi(M)_{[r \text{ times}]} \to \widetilde{\Phi}^r \tag{20}$$

which is a right inverse of the natural projection (14). We have then the *total reduction*

$$\begin{aligned} \{\Gamma\} : \Phi{\times}\Pi(M) &\to \widetilde{\Phi}^r\,, \\ (Z, X) &\mapsto \{\Gamma\}(Z, X, \ldots, X)\,. \end{aligned} \tag{21}$$

Note that we denote both (20) and (21) by the same symbol, ie. write simply $\{\Gamma\}(Z, X)$ instead of $\{\Gamma\}(Z, X, \ldots, X)$.

Two connections in Φ, C of order r and C_1 of order s, can be composed to obtain their product $C * C_1$ which is again a connection in Φ of order $r + s$. This composition is associative (c.f. [KolářŠ 74]). Of special interest is the case $C * h$, where h is a first order connection in Φ. In this case we can write explicitly

$$(C * h)(x) = j^1_x(u \mapsto C(u) \cdot j^r_u[h_x(u)]) \quad \text{with} \quad h(x) = j^1_x h_x, h_x(u) \in \Phi\,, \tag{22}$$

where the dot $\cdot$ denotes the jet prolongation of the groupoid multiplication in Φ (c.f. [Virsik 71]). If $h = \pi^r_1 \circ C$ then $C' = C * h$ is called the prolongation of C (c.f. [Ehresmann 56]). An r-th order connection C is said to be decomposable if $C = h_1 * \cdots * h_r$, where $h_1, \ldots, h_r$ are all first order connections in Φ. If this happens then necessarily

$$h_i = \pi_1^{r \to i} \circ C = \mathbf{J}(\Pi_0^{i-1}) \circ \pi^r_i \circ C : M \to Q^1(\Phi)\,. \tag{23}$$

A decomposable connection C is called simple if $h_1 = \cdots = h_r$. If C is any r-th order connection which is semi-holonomic then all the connections (21) coincide; in particular a decomposable connection is semi-holonomic iff it is simple. Also, if $C * h$ is semi-holonomic then necessarily $C * h = C'$ and C (as well C') is simple, ie. of the form $h*\cdots*h$ and it is holonomic iff h is curvature-free (c.f. [Ehresmann 56] and [Virsik 71]).

Let now Γ be an r-th order total connection in Φ. Recall that this is a first order connection in $\widetilde{\Phi}^{r-1}$ hence $\Gamma : M \to Q^1(\widetilde{\Phi}^{r-1})$ ie. it is a smooth map $\Gamma : M \to \mathbf{J}^r(M,\Phi) = \mathbf{J}^{r-1}(J^1(M,\Phi))$ satisfying

$$\text{(24)} \qquad \text{(i) } \pi^r_{r-1}\circ\Gamma(x) = j^{r-1}_x(\sim); \quad \text{(ii) } \mathbf{J}^r(a)\circ\Gamma(x) = j^1_x[j^{r-1}_x]; \quad \text{(iii) } \mathbf{J}(b\circ\pi^{r-1}_0)\circ\Gamma(x) = j^1_x\,.$$

Γ gives rise to the reduction $[\Gamma] : \widetilde{\Phi}^{r-1}\times\Pi(M)\to\widetilde{\Phi}^r$ as well as to the total reduction (21).Each one of these can be used to transport connections: If Γ' is another r-th order total connection in Φ and ξ a connection in $\Pi(M)$, ie. a linear connection on M, we get an $(r+1)$-st order total connection $[\Gamma]\langle\Gamma'\times\xi\rangle = Q^1([\Gamma])\circ\Gamma'\times\xi$ in Φ. If $\Gamma' = \Gamma$ we shall write $\Gamma\bullet\xi$, ie.

$$\text{(25)} \qquad \Gamma\bullet\xi = [\Gamma]\langle\Gamma\times\xi\rangle\,.$$

The explicit formula for $\Gamma\bullet\xi : M\to Q^1(\widetilde{\Phi}^r)$ is given by

$$\text{(26)} \qquad \begin{aligned} &(\Gamma\bullet\xi)(x) = \mathbf{J}([\Gamma])(\Gamma(x),\xi(x)) \quad \text{or, more precisely,}\\ &(\Gamma\bullet\xi)(x) = Q^1([\Gamma])(\Gamma(x),\xi(x))\,. \end{aligned}$$

The functor $[\Gamma] : \widetilde{\Phi}^{r-1}\times\Pi(M)\to\widetilde{\Phi}^r$ is given in (16) with $k=1$ and $\widetilde{\Phi}^{r-1}$ replacing $\widetilde{\Phi}^r$, ie. by

$$\text{(27)} \qquad \begin{aligned} &[\Gamma] : \widetilde{\Phi}^{r-1}\times\Pi(M)\to\widetilde{\Phi}^r\\ &(Z,X)\mapsto\Gamma(y)\circ X\cdot j^1_x[Z]\cdot\Gamma(x)^{-1} \end{aligned}$$

where $\cdot$ denotes the prolongation of composition in $\widetilde{\Phi}^{r-1}$, $x = \alpha X = \alpha^{r-1}Z$ and $y = \pi^1_0 X = b\circ\pi^{r-1}_0 Z$. Also, $[\Gamma]$ is a right inverse of the canonical projections $\widetilde{\Phi}^r\to\widetilde{\Phi}^{r-1}\times\Pi(M)$, explicitly

$$\text{(28)} \qquad (\pi^r_{r-1}\circ[\Gamma])(Z,X) = Z \quad \text{and} \quad (\mathbf{J}(b\circ\pi^{r-1}_0)\circ[\Gamma])(Z,X) = X\,.$$

Lemma 1. $(\mathbf{J}(\pi^{r-1}_{p-1})\circ[\Gamma])(Z,X) = [\mathbf{J}(\pi^{r-1}_{p-1})\circ\Gamma](\pi^{r-1}_{p-1}Z,X)$ *whenever* $1\le p<r$.

Proof. We shall prove it first for $r=2$ and $p=1$. Since the composition in Φ^1 satisfies $\pi^1_0(u\cdot v) = \pi^1_0(u)\cdot\pi^1_0(v)$, we can write

$$\begin{aligned} &(\mathbf{J}(\pi^1_0\circ[\Gamma])(Z,X) = \mathbf{J}(\pi^1_0)(\Gamma(y)\circ X\cdot j^1_x[Z]\cdot\Gamma(x)^{-1})\\ &= \mathbf{J}(\pi^1_0)(\Gamma(y)\circ X)\cdot j^1_x[\pi^1_0 Z]\cdot\mathbf{J}(\pi^1_0)\Gamma(x)^{-1} = [\mathbf{J}(\pi^1_0)\circ\Gamma])(\pi^1_0 Z,X) \end{aligned}$$

as required. Applying this to $\widetilde{\Phi}^{r-2}$ instead of Φ we get the required result for arbitrary $r > 1$ and $p = r-1$. Assuming this for some $p < r$ we derive it for $p-1$ since $(\mathbf{J}(\pi^{r-1}_{p-2}) \circ [\Gamma])(Z, X) =$

$$\begin{aligned}&(\mathbf{J}(\pi^{p-1}_{p-2}) \circ \mathbf{J}(\pi^{r-1}_{p-1}) \circ [\Gamma])(Z, X) = \mathbf{J}(\pi^{p-1}_{p-2}) \circ [\mathbf{J}(\pi^{r-1}_{p-1}) \circ \Gamma](\pi^{r-1}_{p-1} Z, X)\\ &= [\mathbf{J}(\pi^{p-1}_{p-2}) \circ \mathbf{J}(\pi^{r-1}_{p-1}) \circ \Gamma](\pi^{p-1}_{p-2} \circ \pi^{r-1}_{p-1} Z, X) = [\mathbf{J}(\pi^{r-1}_{p-2}) \circ \Gamma](\pi^{r-1}_{p-2} Z, X)\end{aligned}$$

and this completes the proof. □

Note that if we write $\Gamma_{(p)} = \mathbf{J}(\pi^{r-1}_{p-1}) \circ \Gamma : M \to Q^1(\widetilde{\Phi}^{p-1})$ for the underlying total connection of order p, the formula just proved can be written as

$$(\mathbf{J}(\pi^{r-1}_{p-1}) \circ [\Gamma])(Z, X) = [\Gamma_{(p)}](\pi^{r-1}_{p-1} Z, X) \quad \text{whenever} \quad 1 \le p < r\,. \tag{29}$$

The relation between the reductions $[\Gamma_{(s)}]$ and the total reductions $\{\Gamma_{(s)}\}$ is given by the recurrence formula

$$\{\Gamma_{(s)}\}(Z, X) = [\Gamma_{(s)}](\{\Gamma_{(s-1)}\}(Z, X), X) \tag{30}$$

for $s = 2, \dots r$, where $\Gamma_{(r)} = \Gamma$. By (28) we get from (30) also $(\pi^s_{s-1} \circ \{\Gamma_{(s)}\})(Z, X) = (\pi^s_{s-1} \circ [\Gamma_{(s)}])(\{\Gamma_{(s-1)}\}(Z, X), X) = \{\Gamma_{(s-1)}\}(Z, X)$, ie.

$$\pi^s_{s-1} \circ \{\Gamma_{(s)}\} = \{\Gamma_{(s-1)}\} \quad \text{for} \quad s = 2, \dots r\,. \tag{31}$$

If h is a first order connection in Φ and $\xi_1, \dots, \xi_{r-1}$ first order connections in $\Pi(M)$, we can define $h \bullet \xi_1 \bullet \dots \bullet \xi_{r-1}$ recurrently via $(h \bullet \xi_1 \bullet \dots \bullet \xi_{r-2}) \bullet \xi_{r-1}$. If $\Gamma = h \bullet \xi_1 \bullet \dots \bullet \xi_{r-1}$ then $h = \pi^{r-1}_0 \langle \Gamma \rangle$ and $\xi_i = (\mathbf{J}(b) \circ \pi^{r-1 \to i}_r)\langle \Gamma \rangle$, $i = 1, \dots, r-1$. Such Γ will be called a *decomposable r-th order total connection*. It is called *simple* if $\xi_1 = \dots = \xi_{r-1}$. A simple total connection of order r is thus $\Gamma = h \bullet \xi \bullet \dots \bullet \xi$ or briefly $\Gamma = h \bullet (\bullet \xi)^{r-1}$. Note that in the notation of [KoláŠ 74] $\Gamma \bullet \xi$ would be written as $p(\Gamma, \xi)$, the simple r-th order total connection $h \bullet (\bullet \xi)^{r-1}$ as $p^{r-1}(h, \xi)$, and Proposition 6 of [KoláŠ 74] says that a simple total connection is always semi-holonomic. Recall that if an r-th order total connection Γ is semi-holonomic (ie. of type $(r-1S, 1)$) then $[\Gamma] : \bar{\Phi}^{r-1} \times \Pi(M) \to (\bar{\Phi}^{r-1})^1$ whereas Γ has a semi-holonomic total reduction if $\{\Gamma\} : \Phi \times \Pi(M) \to \bar{\Phi}^r \subset \widetilde{\Phi}^r$.

Proposition 2. *Any r-th order total connection Γ in Φ has a semi-holonomic total reduction.*

Proof. We need to show that $\{\Gamma\}$ satisfies

$$\mathbf{J}(\pi^{k-1}_{s-1}) \circ \pi^r_k \circ \{\Gamma\} = \pi^r_s \circ \{\Gamma\} : \Phi \times \Pi(M) \to \widetilde{\Phi}^s \tag{32}$$

whenever $1 \le s < k \le r$ (see the semi-holonomity condition (9)). If $1 \le s < k \le r$ we can write, using (28) and (31),

$$\begin{aligned}(\mathbf{J}(\pi^{k-1}_{s-1}) \circ \pi^r_k \circ \{\Gamma\})(Z, X) &= (\mathbf{J}(\pi^{k-1}_{s-1}) \circ \{\Gamma_{(k)}\})(Z, X)\\ &= (\mathbf{J}(\pi^{k-1}_{s-1}) \circ [\Gamma_{(k)}])(\{\Gamma_{(k-1)}\}(Z, X), X)\end{aligned}$$

which, by (29), (31) and (30), gives

$$\begin{aligned}&[\Gamma_{(s)}](\pi^{k-1}_{s-1} \circ \{\Gamma_{k-1}\}(Z,X),X)\\ &= [\Gamma_s](\{\Gamma_{(s-1)}\}(Z,X),X) = \{\Gamma_{(s)}\}(Z,X) = (\pi^r_s \circ \{\Gamma\})(Z,X)\,,\end{aligned}$$

as required. □

Note that in Proposition 2 it was essential that $\{\Gamma\}$ was the functor (21) and not (20).

With each total connection $\Gamma : M \to Q^1(\widetilde{\Phi}^{r-1})$ we can associate total connections

$$\mathbf{J}(\mathbf{J}(\pi^{s-1}_{p-2}) \circ \pi^{r-1}_s) \circ \Gamma : M \to Q^1(\widetilde{\Phi}^{p-1}) \quad \text{for} \quad 1 \le p-1 \le s \le r-1 \tag{33}$$

as well as the first order total connection $\Gamma_{(1)} = \mathbf{J}(\pi^{r-1}_0) \circ \Gamma : M \to Q^1(\Phi)$. Note that we have denoted by $\Gamma_{(p)}$ the connection (33) corresponding to $s = p-1$. There are in (33) $r-p+1$ p-th order total connections, and Γ is semi-holonomic iff the connections (33) depend only on p and not on s (see again the semi-holonomity condition (9)). Thus Γ is semi-holonomic iff

$$\Gamma_{(p)} = \mathbf{J}(\mathbf{J}(\pi^{s-1}_{p-2}) \circ \pi^{r-1}_s) \circ \Gamma : M \to Q^1(\widetilde{\Phi}^{p-1}) \quad \text{for} \quad 1 < p \le s \le r-1\,, \tag{34}$$

where $\Gamma_{(p)} = \mathbf{J}(\pi^{r-1}_{p-1}) \circ \Gamma$.

In particular, with an r-th order total connection $\Gamma : M \to Q^1(\widetilde{\Phi}^{r-1})$ we can associate $r-1$ second order total connections

$$\mathbf{J}(\eta^{r-1}_s) \circ \Gamma : M \to Q^1(\Phi^1), \quad 1 \le s \le r-1\,, \tag{35}$$

where $\mathbf{J}(\eta^{r-1}_s) = \mathbf{J}(\pi^{s-1}_0) \circ \pi^{r-1}_s : \widetilde{\Phi}^{r-1} \to \Phi^1$. Hence we get also $r-1$ first order linear connections on M, namely

$$\Gamma_{Ms} = \mathbf{J}^2(b) \circ \mathbf{J}(\eta^{r-1}_s) \circ \Gamma : M \to Q^1(\Pi(M)), \quad 1 \le s \le r-1\,. \tag{36}$$

Lemma 2.

(i) $(\Gamma \bullet \xi)_{(p)} = \Gamma_{(p)}$ *for* $p = 1, \dots r$;
(ii) $(\Gamma \bullet \xi)_{Ms} = \Gamma_{Ms}$ *for* $s = 1, \dots r-1$ *and* $(\Gamma \bullet \xi)_{Mr} = \xi$.

Proof.

(i) From $(\Gamma \bullet \xi)(x) = \mathbf{J}([\Gamma])(\Gamma(x), \xi(x))$ and (28) we conclude that $(\Gamma \bullet \xi)_{(r)}(x) = \mathbf{J}(\pi^r_{r-1})(\Gamma \bullet \xi)(x) = \mathbf{J}(\pi^r_{r-1})(\Gamma \bullet \xi)(x) = (\mathbf{J}(\pi^r_{r-1}) \circ [\Gamma])(\Gamma(x), \xi(x))) = \mathbf{J}(\mathrm{pr}_1)(\Gamma(x), \xi(x))) = \Gamma(x)$, ie. $(\Gamma \bullet \xi)_{(r)} = \Gamma$. If $p \le r-1$ then $(\Gamma \bullet \xi)_{(p)} = \mathbf{J}(\pi^r_{p-1}) \circ (\Gamma \bullet \xi) = \mathbf{J}(\pi^{r-1}_{p-1}) \circ \mathbf{J}(\pi^r_{r-1}) \circ (\Gamma \bullet \xi) = \mathbf{J}(\pi^{r-1}_{p-1}) \circ (\Gamma \bullet \xi)_{(r)} = \mathbf{J}(\pi^{r-1}_{p-1}) \circ \Gamma = \Gamma_{(p)}$.

(ii) For $s = 2, \dots r-1$ we have $(\Gamma \bullet \xi)_{Ms} = \mathbf{J}^2(b) \circ \mathbf{J}(\eta^r_s) \circ (\Gamma \bullet \xi) = \mathbf{J}^2(b) \circ \mathbf{J}(\mathbf{J}(\pi^{s-1}_0) \circ \pi^r_s) \circ (\Gamma \bullet \xi) = \mathbf{J}^2(b) \circ \mathbf{J}(\mathbf{J}(\pi^{s-1}_0) \circ \mathbf{J}(\pi^{r-1}_s) \circ \mathbf{J}(\pi^r_{r-1})(\Gamma \bullet \xi) = \mathbf{J}^2(b) \circ$

$\mathbf{J}(\eta_s^{r-1}) \circ \Gamma = \Gamma_{M^s}$.
If $s = r$ we get $(\Gamma \bullet \xi)_{M^r}(x) = \mathbf{J}^2(b) \circ \mathbf{J}(\eta_r^r) \circ (\Gamma \bullet \xi)(x) = \mathbf{J}^2(b) \circ \mathbf{J}(\eta_r^r)\mathbf{J}([\Gamma])(\Gamma(x), \xi(x)) =$ $\mathbf{J}(\mathbf{J}(b) \circ \eta_r^r \circ [\Gamma])(\Gamma(x), \xi(x)) = \mathbf{J}(\mathbf{J}(b) \circ \mathbf{J}(\pi_0^{r-1}) \circ [\Gamma])(\Gamma(x), \xi(x)) = \mathbf{J}(\mathrm{pr}_2)(\Gamma(x), \xi(x)) =$ $\xi(x)$,
because

$$\begin{aligned}\mathbf{J}(b \circ \pi_0^{r-1}) \circ [\Gamma](Z, X) &= \mathbf{J}(b)(\mathbf{J}(\pi_0^{r-1}) \circ \Gamma(y) \circ X \cdot j_x^1[\pi_0^{r-1} Z] \cdot \mathbf{J}(\pi_0^{r-1}) \circ \Gamma(x)^{-1}) \\ &= \mathbf{J}(b)(\mathbf{J}(\pi_0^{r-1}) \circ \Gamma(y) \circ X = X, \quad \text{ie.} \quad \mathbf{J}(b \circ \pi_0^{r-1}) \circ [\Gamma] = \mathrm{pr}_2 . \qquad \square\end{aligned}$$

The following result states that simple total connections are practically the only ones that are semi-holonomic. Compare this with Theorem 5 and 6 of [Virsik, 71]

Proposition 3. *If Γ is an r-th order total connection then $\Gamma \bullet \xi$ is semi-holonomic iff $\xi = \Gamma_{M^s}$ for $s = 2, \dots r-1$ and $\Gamma = h \bullet (\bullet \xi)^{r-1}$, where $h = \mathbf{J}(\pi_0^{r-1}) \circ \Gamma$.*

Proof. By (34) $\Gamma \bullet \xi$ is semi-holonomic iff

$$(37) \quad (\Gamma \bullet \xi)_{(p)} = \mathbf{J}(\mathbf{J}(\pi_{p-2}^{s-1}) \circ \pi_s^r \circ (\Gamma \bullet \xi) : M \to Q^1(\widetilde{\Phi}^{p-1}) \quad \text{for} \quad 1 < p \le s \le r$$

By Lemma 2 the left hand side is $\Gamma_{(p)}$ and the right hand side gives

$$\begin{aligned}\mathbf{J}(\mathbf{J}(\pi_{p-2}^{s-1}) \circ \pi_s^r) \circ (\Gamma \bullet \xi) &= \mathbf{J}(\mathbf{J}(\pi_{p-2}^{s-1}) \circ \pi_s^{r-1}) \circ \mathbf{J}(\pi_{r-1}^r) \circ (\Gamma \bullet \xi) \\ &= \mathbf{J}(\mathbf{J}(\pi_{p-2}^{s-1}) \circ \pi_s^{r-1}) \circ \Gamma\end{aligned}$$

for $1 < p \le s \le r-1$, and

$$\mathbf{J}^2(\pi_{p-2}^{r-1}) \circ (\Gamma \bullet \xi) = \mathbf{J}(\mathbf{J}(\pi_{p-2}^{s-1}) \circ \pi_s^{r-1}) \circ \mathbf{J}(\pi_{r-1}^r) \circ (\Gamma \bullet \xi) = \mathbf{J}(\mathbf{J}(\pi_{p-2}^{s-1}) \circ \pi_s^{r-1}) \circ \Gamma$$

for $1 < p \le r$. Thus $\Gamma \bullet \xi$ is semi-holonomic iff $\Gamma_{(p)} = \mathbf{J}(\mathbf{J}(\pi_{p-2}^{s-1}) \circ \pi_s^{r-1}) \circ \Gamma$ for $1 < p \le s \le r-1$ and $\Gamma_{(p)} = \mathbf{J}^2(\pi_{p-2}^{r-1}) \circ (\Gamma \bullet \xi) : M \to Q^1(\widetilde{\Phi}^{p-1})$. This means that $\Gamma \bullet \xi$ is semi-holonomic iff Γ is semi-holonomic and

$$(38) \qquad \Gamma_{(p)} = \mathbf{J}^2(\pi_{p-2}^{r-1}) \circ (\Gamma \bullet \xi) : M \to Q^1(\widetilde{\Phi}^{p-1}) \quad \text{for} \quad 1 < p \le r .$$

On the other hand, (26) and Lemma 1 allow us to give $\mathbf{J}^2(\pi_{p-2}^{r-1}) \circ (\Gamma \bullet \xi)(x)$ the form

$$\begin{aligned}&\mathbf{J}(\mathbf{J}((\pi_{p-2}^{r-1}) \circ [\Gamma])(\Gamma(x), \xi(x)) = \mathbf{J}([\mathbf{J}(\pi_{p-2}^{r-1}) \circ \Gamma] \circ (\pi_{p-2}^{r-1} \times \mathrm{id}_{\Pi(M)}))(\Gamma(x), \xi(x)) = \\ &\mathbf{J}([\Gamma_{(p-1)}] \circ (\mathbf{J}(\pi_{r-2}^{r-1}) \times \mathrm{id}_{\Pi(M)})(\Gamma(x), \xi(x)) = \mathbf{J}([\Gamma_{(p-1)}])(\mathbf{J}(\pi_{p-2}^{r-1}) \circ \Gamma)(x), \xi(x)) = \\ &\mathbf{J}([\Gamma_{(p-1)}])(\Gamma_{(p-1)})(x), \xi(x)) = (\Gamma_{(p-1)} \bullet \xi)(x) .\end{aligned}$$

Thus (38) is equivalent to $\Gamma_{(p)} = \Gamma_{(p-1)} \bullet \xi$ for $1 < p \le r$.

To summarize: $\Gamma \bullet \xi$ is semi-holonomic iff Γ is semi-holonomic and $\Gamma = \Gamma_{(p-1)} \bullet \xi$ for all $p = 2, \dots r$. We conclude by induction that $\Gamma \bullet \xi$ is semi-holonomic iff $\Gamma = h \bullet \xi \bullet \dots \bullet \xi$ where $h = \Gamma_{(1)}$ as required. $\square$

Recall that if an r-th order total connection Γ is holonomic then the reduction $[\Gamma]$ satisfies $[\Gamma] : \Phi^{r-1} \times \Pi(M) \to (\Phi^{r-1})^1$. We have just seen that any total connection has a semi-holonomic total reduction. Similarly, we shall say that Γ has a *holonomic total reduction* if $\{\Gamma\} : \Phi \times \Pi \to \Phi^r \subset \bar{\Phi}^r$. KoláŠ proved

Proposition 4. *If h is a connection in Φ and ξ a linear connection on M then $h \bullet \xi$ has a holonomic total reduction, explicitly*

$$\{h \bullet \xi\} : \Phi \times \Pi(M) \to \Phi^1 \times \Pi(M) \to \Phi^2 \subset \widetilde{\Phi}^2$$

$$(Z, X) \mapsto ([h](Z, X), X) \mapsto [h \bullet \xi]([h](Z, X), X) \in \Phi^2 \,, \tag{39}$$

if and only if h is curvature-free and ξ is torsion-free.

See Proposition 5 and 6 in [KoláŠ 75], where $h \bullet \xi$ is denoted by $p(h, \xi)$, and is called torsion-free if (39) holds.

Lemma 3. *$\Gamma \bullet \xi : M \to Q^1(\widetilde{\Phi}^r)$ is curvature-free if and only if both $\Gamma : M \to Q^1(\widetilde{\Phi}^{r-1})$ and $\xi : M \to Q^1\Pi(M)$ are curvature-free.*

Proof. We shall use the obvious fact that if C is a first order connection in Φ and $\varphi : \Phi \to \Psi$ is a smooth functor then also the transported connection $\varphi\langle C\rangle$ is curvature free. Assuming $\Gamma \bullet \xi : M \to Q^1(\widetilde{\Phi}^r)$ is curvature-free we get then by Lemma 2 that both Γ and ξ are curvature free. Conversely, assuming both Γ and ξ curvature free we conclude easily that also $\Gamma \times \xi$ is curvature-free, so it suffices to apply (25). □

The same will hold if $Q^1(\widetilde{\Phi}^r)$ and $Q^1(\widetilde{\Phi}^{r-1})$ are replaced by their semi-holonomic and holonomic counterparts.

The following result is easily established from coordinate expressions of the prolongations in question.

Lemma 4. *If $Y \to M$ is a fibred manifold and $r > 2$ is an integer, then*

$$J_r(Y) = \bar{J}_r(Y) \cap J_1(J_{r-1}(Y)) \cap J_2(J_{r-2}(Y)) \,.$$

Proposition 5. *If $r > 2$ then the simple total connection $h \bullet \xi \bullet \cdots \bullet \xi = h \bullet (\bullet\xi)^{r-1}$ has a holonomic total reduction if and only if h is curvature free and ξ is both curvature and torsion free.*

Note that in the case of $r = 2$ we do not need ξ to be curvature free only torsion-free: this is the quoted result from [KoláŠ 75].

Proof of Proposition 5: Let $\Gamma = h \bullet (\bullet\xi)^{r-1}$ have a holonomic total reduction. By (31) the same is true about $\{\Gamma_{(3)}\} = \{(h \bullet \xi) \bullet \xi\}$ and so by Proposition 4 applied to Φ^1 the connection $h \bullet \xi$ is curvature-free and ξ is torsion-free. Apply Lemma 3 to $h \bullet \xi$ to conclude that h as well as ξ are also curvature-free.

Conversely, let h be curvature-free and ξ both curvature- and torsion-free, and let first $r = 3$. We get from (26) and (30) — written as $\{\Gamma_{(s)}\} = [\Gamma_{(s)}] \circ (\{\Gamma_{(s-1)}\}, \mathrm{pr}_2) : \Phi \times \Pi(M) \to \bar{\Phi}^s$ and applied to $\Gamma_{(s)} = h \bullet \xi$ —

$$\begin{aligned}
&(h \bullet \xi \bullet \xi)(x) = Q^1([h \bullet \xi]) \circ ((h \bullet \xi) \times \xi)(x) = \\
&Q^1([h \bullet \xi]) \circ (Q^1([h]) \times \mathrm{pr}_2)(h(x), \xi(x)) = \\
&Q^1(\{h \bullet \xi\}(h(x), \xi(x)) \in Q^1(\Phi^2)
\end{aligned}$$

since by Proposition 4 $\{h \bullet \xi\}(Z,X) \in \Phi^2$. Hence the reduction $[h \bullet \xi \bullet \xi]$ maps $\Phi^2 \times \Pi(M)$ into $(\Phi^2)^1$ and so $\{h \bullet \xi \bullet \xi\}(Z,X) = [h \bullet \xi \bullet \xi](\{h \bullet \xi\}(Z,X),X) \in (\Phi^2)^1$ for any $Z \in \Phi$ and $X \in \Pi(M)$. On the other hand, applying Proposition 4 to the groupoid Φ^1 we get $[(h \bullet \xi) \bullet \xi] : \Phi^1 \times \Pi \to (\Phi^1)^2$ and so this time $\{h \bullet \xi \bullet \xi\}(Z,X) = [(h \bullet \xi) \bullet \xi]([h](Z,X),X) \in (\Phi^1)^2$ for any $Z \in \Phi$ and $X \in \Pi(M)$. Since $\{h \bullet \xi \bullet \xi\}(Z,X) \in \bar{\Phi}^3$ by Proposition 2, it suffices to apply Lemma 4 to conclude that $\{h \bullet \xi \bullet \xi\} : \Phi \times \Pi(M) \to \Phi^3$.

Assume now that $\{h \bullet (\bullet\xi)^{r-2}\} : \Phi \times \Pi(M) \to \Phi^{r-1}$, where $r \geq 4$. Writing $h_\xi = h \bullet (\bullet\xi)^{r-3}$ we get similarly as before

$$(h_\xi \bullet \xi \bullet \xi)(x) = Q^1([h_\xi \bullet \xi]) \circ ((h_\xi \bullet \xi) \times \xi)(x) =$$
$$Q^1([h_\xi \bullet \xi]) \circ (Q^1([h_\xi]) \times \mathrm{pr}_2)(h_\xi(x), \xi(x)) = Q^1(\{h_\xi \bullet \xi\}(h_\xi(x), \xi(x)) \in Q^1(\Phi^{r-1})$$

since by the induction assumption $\{h_\xi \bullet \xi\}(Z,X) \in \Phi^{r-1}$. Hence the reduction $[h_\xi \bullet \xi \bullet \xi]$ maps $\Phi^{r-1} \times \Pi(M)$ into $(\Phi^{r-1})^1$ and so $\{h \bullet \xi \bullet \xi\}(Z,X) = [h \bullet \xi \bullet \xi](\{h \bullet \xi\}(Z,X),X) \in (\Phi^{r-1})^1$ for any $Z \in \Phi$ and $X \in \Pi(M)$. On the other hand, applying Proposition 4 to the groupoid Φ^{r-2} we get $[(h_\xi \bullet \xi) \bullet \xi] : \Phi^{r-2} \times \Pi \to (\Phi^{r-2})^2$ for any $Z \in \Phi$ and $X \in \Pi(M)$. Since $\{h_\xi \bullet \xi \bullet \xi\}(Z,X) \in \bar{\Phi}^r$ by Proposition 2, it suffices to apply Lemma 4 to conclude that $\{h_\xi \bullet \xi \bullet \xi\} = \{h \bullet (\bullet\xi)^{r-1}\} : \Phi \times \Pi(M) \to \Phi^r$. □

Using $\{\Gamma\}$ of (20) or (21) to transport connections, we get immediately

Proposition 6. *If h is a first order connection in Φ, and $\xi_1, \dots \xi_r$ linear connections on M then any r-th order total connection Γ in Φ will give rise to their lift, ie. an $(r+1)$-st order total connection $\{\Gamma\}\langle h \times \xi_1 \times \cdots \times \xi_r\rangle$ in Φ. In particular, if $\xi_1 = \cdots = \xi_r = \xi$, we get the total connection $\{\Gamma\}\langle h \times \xi\rangle$, using (21), rather than (20), for the lifting.*

Since $\{\Gamma\}$ of (20) is a right inverse of the canonical projection $\mathbf{\Phi\Pi}^r : \widetilde{\Phi}^r \to \Phi \times \Pi(M) \times \cdots \times \Pi(M)$ we conclude that — c.f. (14) —

$$(40) \quad (\pi_0^r \circ \{\Gamma\})\langle h \times \xi\rangle = h \text{ and } (\mathbf{J}(b) \circ \pi_0^{r \to i} \circ \{\Gamma\})\langle h \times \xi\rangle = \xi \text{ for } i = 1, \dots r\,.$$

Proposition 2 yields immediately

Proposition 7. *The lifted $(r+1)$-st order total connection $\{\Gamma\}\langle h\rangle$ is always semi-holonomic. It is even holonomic if Γ has a holonomic total reduction.*

From Proposition 3 we conclude that if $\Gamma \bullet \xi = \{\Gamma\}\langle h \times \xi\rangle$ then Γ is necessarily simple, ie. $\Gamma = h \bullet (\bullet\xi)^{r-1}$. Conversely, if $\Gamma = h \bullet (\bullet\xi)^{r-1}$ then $\Gamma \bullet \xi = h \bullet (\bullet\xi)^r$ and the recurrence relations

$$h \bullet (\bullet\xi)^s = [h \bullet (\bullet\xi)^{s-1}]\langle h \bullet (\bullet\xi)^{s-1} \times \xi\rangle, \quad s = 1, \dots r\,,$$

together with (30) show easily that also $\{\Gamma\}\langle h \times \xi\rangle$ equals $h \bullet (\bullet\xi)^r$. Thus we have

Proposition 8. *If Γ is an r-th order total connection in Φ, ξ a linear connection on M then the two $(r+1)$-st order total connections $\Gamma \bullet \xi$ and $\{\Gamma\}\langle h \times \xi\rangle$ coincide iff $\Gamma = h \bullet (\bullet\xi)^{r-1}$, where $h = \mathbf{J}(\pi_0^{r-1}) \circ \Gamma$.*

Proposition 9. *The simple connection $\Gamma = h \bullet (\bullet\xi)^2$ is holonomic provided h is curvature-free and ξ is torsion-free. For $r > 3$, the simple connection $\Gamma = h\bullet(\bullet\xi)^{r-1}$ is holonomic provided h is curvature-free and ξ is curvature-free as well as torsion-free.*

Proof. This follows immediately from Propositions 4 and 5. □

* * *

Let us now consider the special case of total connections on M, ie. when $\Phi = \Pi(M)$ or Φ is the trivial groupoid $\Pi^0(M) = M \times M$. The r-th prolongation of $\Pi(M)$ will be denoted by $\widetilde{(\Pi)}^r$ and similarly in the semi-holonomic and holonomic cases. Note that $\widetilde{(\Pi)}^r = \widetilde{\Pi}^{r+1}(M)$, whereas $\overline{\Pi}^{r+1}(M)$ and $\Pi^{r+1}(M)$ are in general proper subgroupoids of $\overline{(\Pi)}^r$ and $(\Pi)^r$ respectively. On the other hand, the r-th (semi-holonomic or holonomic) prolongation of $\Pi^0(M)$ is $\widetilde{\Pi}^r(M)$ ($\overline{\Pi}^r(M)$ or $\Pi^r(M)$). Thus an $(r+1)$-st order total connection in $\Pi^0(M)$ is the same as an r-th order total connection in $\Pi(M)$. In particular, it will give rise to a total reduction $\Pi(M) \times \Pi(M) \to \widetilde{(\Pi)}^r$ or to a total reduction $\Pi(M) \to \widetilde{\Pi}^{r+1}(M)$ depending on whether one takes for Φ the groupoid $\Pi(M)$ or $\Pi^0(M)$. On the other hand, an r-th order semi-holonomic or holonomic total connection in $\Pi(M)$ is not necessarily (reducible to) an $(r+1)$-st order semi-holonomic or holonomic total connection in $\Pi^0(M)$. We shall say that an r-th order total connection on M, ie. in $\Pi(M)$, is *strongly semi-holonomic* or *strongly holonomic* if it is a semi-holonomic or holonomic respectively $(r+1)$-st order total connection in $\Pi^0(M)$.

An r-th order total connection Γ on M gives rise to a reduction $[\Gamma] : \widetilde{\Pi}^r(M) \times \Pi(M) \to \widetilde{\Pi}^{r+1}(M)$ (c.f. (16)). If Γ is semi-holonomic then $[\Gamma]$ maps $\overline{(\Pi)}^{r-1} \times \Pi(M)$ into $(\overline{(\Pi)}^{r-1})^1$ and if it is strongly semi-holonomic then it maps $\overline{\Pi}^r(M) \times \Pi(M)$ into $\overline{\Pi}^r(M)^1 \subset (\overline{(\Pi)}^{r-1})^1$. By Proposition 2 the total reduction $\{\Gamma\}$ always maps $\Pi(M) \times \Pi(M)$ into $\overline{(\Pi)}^r$ and by the same result the total reduction of Γ, seen as an $(r+1)$-st order total connection in $\Pi^0(M)$, is a functor $\Pi(M) \to \overline{\Pi}^{r+1}(M)$. This generalises Proposition 9 of [KoláŠ 74], which assumes that Γ is semi-holonomic. In fact, it is not hard to see that our $\{\Gamma\} : \Pi(M) \to \overline{\Pi}^{r+1}(M)$ corresponds to $f(\Gamma)$ of [KoláŠ 74].

The concept of a simple r-th order total connection depends again on whether we take for Φ the groupoid $\Pi(M)$ or $\Pi^0(M)$: a decomposable r-th order total connection in $\Pi(M)$ is of the form $\xi_0 \bullet \xi_1 \bullet \cdots \bullet \xi_{r-1}$, where $\xi_0, \xi_1, \ldots \xi_{r-1}$ are linear connections on M, ξ_0 corresponding to the connection h in Φ which is now $\Pi(M)$. It is simple if $\xi_1 = \xi_2 = \cdots = \xi_{r-1}$. The same connection $\xi_0 \bullet \xi_1 \bullet \cdots \bullet \xi_{r-1}$ can also be seen as a decomposable $(r+1)$-st order total connection in $\Pi^0(M)$, where the role of the connection h in $\Pi^0(M)$ is played by the trivial connection. If it is simple, ie. $\xi_0 = \xi_1 = \cdots = \xi_{r-1}$, we shall say that $\xi_0 \bullet \xi_1 \bullet \cdots \bullet \xi_{r-1}$ is *strongly simple*. Proposition 3 applied to these two cases says then that if Γ is an r-th order total connection in $\Pi(M)$ then $\Gamma \bullet \xi$ is semi-holonomic iff Γ is simple and $\Gamma \bullet \xi$ is strongly semi-holonomic iff Γ is strongly simple. Compare this with Proposition 7 of [Koláš 74] which says that $\xi \bullet \xi \cdots \bullet \xi = \xi \bullet (\bullet\xi)^{r-1}$ is not only a semi-holonomic r-th order total connection in $\Pi(M)$ but also a semi-holonomic $(r+1)$-st order total connection in $\Pi^0(M)$, ie. that $\xi \bullet \xi \bullet \cdots \bullet \xi$ is strongly semi-holonomic for any linear connection ξ on M.

Applying Lemma 3, Propositions 4, 5 and 9 to $\Phi = \Pi(M)$ we obtain immediately

Proposition 10. *The simple connection $\xi_0 \bullet \xi$ has a holonomic total reduction $\{\xi_0 \bullet \xi\} : \Pi(M) \times \Pi(M) \to (\Pi)^2$ iff ξ_0 is curvature-free and ξ is torsion-free; for $r > 2$ the simple connection $\xi_0 \bullet (\bullet\xi)^{r-1}$ has a holonomic total reduction $\{\xi_0 \bullet (\bullet\xi)^{r-1}\} : \Pi(M) \times \Pi(M) \to (\Pi)^r$ iff ξ_0 is curvature-free and ξ is torsion-free as well as curvature-free. For any $r > 1$ the r-th order simple connection $\xi_0 \bullet (\bullet\xi)^{r-1}$ is curvature-free if and only if both ξ_0 and ξ are curvature-free. If ξ_0 is curvature-free and ξ is torsion-free then $\xi_0 \bullet \xi \bullet \xi$ is a holonomic total connection and if ξ is also curvature-free then $\xi_0 \bullet (\bullet\xi)^{r-1}$ is a holonomic r-th order total connection for any $r > 2$.*

Applying the same results to $\Phi = \Pi^0(M)$ we obtain similarly:

Proposition 11. *The connection ξ on M has a holonomic total reduction $\{\xi\} : \Pi(M) \to \Pi^2(M)$ iff ξ is torsion-free, and for $r > 1$ the r-th order strongly simple total connection $(\bullet\xi)^r$ in $\Pi(M)$ has a holonomic total reduction $\Pi(M) \to \Pi^r(M)$ iff ξ is torsion-free as well as curvature-free. If ξ is torsion-free then $\xi \bullet \xi$ is a strongly holonomic 2-nd order total connection in $\Pi(M)$ and if ξ is also curvature-free then $(\bullet\xi)^r$ is a strongly holonomic r-th order total connection in $\Pi(M)$ for any $r > 1$.*

Remark. Yuen introduced the concept of torsion for connections in higher order semi-holonomic frame bundles, and Koláš generalised this to connections in the first prolongation of any principal bundle (c.f. [Yuen 71] and [Koláš 75]). In the language of Lie groupoids one can speak of the torsion of a semi-holonomic total connection in $\Pi(M)$ of arbitrary order $r > 1$, and of the torsion of a second order total connection in Φ where Φ is any Lie groupoid. For a total connection Γ being torsion free in this sense seems to be closely related to the holonomity of its total reduction $\{\Gamma\}$. More exactly, Koláš has recently proved what in our terminology amounts to the result that an r-th order holonomic total connection in $\Pi(M)$ is torsion-free in the sense of [Yuen 71] iff it has a holonomic total reduction $\Pi(M) \to \Pi^r(M) \subset \overline{\Pi}^r(M)$, ie. when Γ is seen as an $(r+1)$-st order total connection in $\Pi^0(M)$ (c.f. [Koláš , to appear]). On the other hand, Proposition 5 of [Koláš 75] can be interpreted as saying that a second order total connection in Φ is torsion-free iff its total reduction $\Phi \times \Pi(M) \to \overline{\Phi}^2$ is holonomic.

There is a one-to-one correspondence between r-th order connections (in the sense of [Ehresmann 56]) in $\Pi(M)$ and r-th order total connections in $\Pi(M)$, ie.

an $\mathcal{FM}(M)$-equivalence

$$\widetilde{\varrho}_r : \widetilde{Q}^r(\Pi(M)) \to Q^1(\widetilde{\Pi}^r(M)) \tag{41}$$

(c.f [KoláŠ 74]). Let us summarize here some properties of $\widetilde{\varrho}_r$ derived in [KoláŠ 74]. For any Lie groupoid Φ one constructs $\mathcal{FM}(M)$-morphisms

$$\kappa_{r+1} \equiv \kappa^{\Phi}_{r-1} : \widetilde{Q}^{r+1}(\Phi) \times Q^1(\widetilde{\Pi}^r(M)) \to Q^1(\widetilde{\Phi}^r) \quad \text{and} \tag{42}$$

$$k_r \equiv k^{\Phi}_r : Q^1(\widetilde{\Phi}^r) \times \widetilde{Q}^r(\Phi) \to \widetilde{Q}^{r+1}(\Phi) \tag{43}$$

(for $r = 1$, see the proof of Proposition 1). If $\Phi = \Pi(M)$ one defines $\widetilde{\varrho}_r$ of (41) recurrently by $\widetilde{\varrho}_1 = \mathrm{id}$ and $\widetilde{\varrho}_r(X) = \kappa_r(X, \widetilde{\varrho}_{r-1}(\pi^r_{r-1}X))$, where $\kappa_r = \kappa^{\Pi(M)}_r$. It is shown that this indeed defines an $\mathcal{FM}(M)$-equivalence whose inverse is given again recurrently by $\widetilde{\varrho}_1^{-1} = \mathrm{id}$ and $\widetilde{\varrho}_r^{-1}(Z) = k_{r-1}(Z, \widetilde{\varrho}_{r-1}^{-1})(\mathbf{J}(\pi^r_{r-1}), Z)$, where $k_{r-1} = k^{\Phi}_{r-1}$. Actually, $\widetilde{\varrho}_r^{-1}$ is nothing but the map $\sigma_{r-1} : Q^1(\widetilde{\Phi}^{r-1}) \to \widetilde{Q}^r(\Phi)$ referred to earlier, applied to $\Phi = \Pi(M)$. Moreover, $\widetilde{\varrho}_r$ maps $\overline{Q}^r(\Pi(M))$ onto $Q^1(\overline{\Pi}^r(M))$. Proposition 4 of [KoláŠ 74] can be stated as:

Proposition 12. *If C is an r-th order connection in Φ, C_0 a first order connection in Φ and L a first order connection in $\widetilde{\Pi}^r(M)$ then*

$$\kappa_{r+1}(C * C_0, L) = [C]\langle C_0 \times L\rangle \tag{44}$$

where $\kappa_{r+1} \equiv \kappa^{\Phi}_{r+1}$.

Applying this to $\widetilde{\Phi}^{r-1}$ instead of Φ and restricting ourselves to first order connections, the connection $[C]\langle C_0 \times L\rangle$ with $C = C_0 = \Gamma$ and $L = \xi$ can be written as $\Gamma \bullet \xi$ and we get immediately

Proposition 13. *If Γ is an r-th order total connection in Φ, and ξ a linear connection on M then*

$$\kappa_2(\Gamma * \Gamma, \xi) = \Gamma \bullet \xi \tag{45}$$

where κ_2 is taken with respect to the groupoid $\widetilde{\Phi}^{r-1}$, ie. $\kappa_2 : \widetilde{Q}^2(\widetilde{\Phi}^{r-1}) \times Q^1(\Pi(M)) \to Q^1(\widetilde{\Phi}^r)$.

Proposition 8 of [KoláŠ 74] can be stated as follows.

Proposition 14. *If h is a first order connection in Φ, and ξ a linear connection on M then*

$$\kappa_{r+1}\big(\underbrace{h * \cdots * h}_{(r+1)-\text{times}}, \varrho_r\big(\underbrace{\xi * \cdots * \xi}_{r-\text{times}}\big)\big) = h \bullet \underbrace{\xi \bullet \cdots \bullet \xi}_{r-\text{times}}. \tag{46}$$

In particular, for $\Phi = \Pi(M)$ and $h = \xi$,

$$\widetilde{\varrho}_r(\underbrace{\xi * \cdots * \xi}_{(r+1)-\text{times}}) = \underbrace{\xi \bullet \cdots \bullet \xi}_{(r+1)-\text{times}}. \tag{47}$$

Thus the equivalence (41) maps " $*$-powers of ξ" onto "$\bullet$-powers of ξ" for any linear connection ξ on M.

Remark. It seems that formula (47) cannot be extended to different linear connections on M. In fact, even for two linear connections ξ_0 and ξ_1 on M we get $\widetilde{\varrho}_2(\xi_0 * \xi_1) = \kappa_2(\xi_0 * \xi_1, \xi_0)$ since $\pi_1^2(\xi_0 * \xi_1) = \xi_0$ and $\varrho_1 = \mathrm{id}$. By Proposition 12 this yields $\widetilde{\varrho}_2(\xi_0 * \xi_1) = [\xi_0]\langle\xi_1 \times \xi_0\rangle \neq [\xi_0]\langle\xi_0 \times \xi_1\rangle = \xi_0 \bullet \xi_1$.

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George Virsik
Department of Mathematics
Monash University
Clayton, Victoria 3168
AUSTRALIA