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SOME NATURAL OPERATORS ON VECTOR FIELDS

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Abstract. We determine all natural operators transforming vector fields on a manifold M to vector fields on $T^*T^2_1M$, $\dim M \geq 2$, and all natural operators transforming vector fields on M to functions on $T^*TT^2_1M$, $\dim M \geq 3$. We describe some relations between these two kinds of natural operators.

0. Preliminaries

We present a contribution to the theory of natural operators and we follow the basic terminology used in [6]. Our starting point was a paper by Kobak, [2], in which all natural operators $T \to TT^*T$ were determined. In Section 1 we find all natural operators $T \to TT^*T^2_1$, where T^2_1 denotes the bundle of $(1,2)$-velocities. In Section 2 we solve a related problem of finding of all natural operators transforming vector fields into functions on $T^*TT^2_1$. Our approach is heavily based on the technique of Weil bundles, [6].

All natural bundles and operators are considered on $\mathcal{M}f_m$, the category of smooth m-dimensional manifolds and their local diffeomorphisms. Let $\mathcal{M}f$ be the category of smooth manifolds and smooth maps and $\mathcal{FM}$ be the category of fibered manifolds.

Let $A = E(k)/I$ be a Weil algebra, where $E(k)$ is the algebra of germs of smooth functions $\mathbb{R}^k \to \mathbb{R}$ at zero and I is an ideal of finite codimension. We remind the covariant definition of the Weil bundle functor $T^A : \mathcal{M}f \to \mathcal{FM}$, [6],[3]. Two maps $f, g : \mathbb{R}^k \to M$ satisfying $f(0) = g(0) = x$ are said to be I-equivalent, if for every germ $h : M \to \mathbb{R}$ at x it holds $h \circ f - h \circ g \in I$. Classes of such an equivalence relation are called A-velocities and are denoted by $j^A f$. They are the elements of $T^A M$. For a smooth map $f : M \to N$ we define $T^A f : T^A M \to T^B M$ by $T^A f(j^A g) = j^A(f \circ g)$ for all $j^A g \in T^A M$.

Given two Weil algebras A, B, we denote by $\mathrm{Hom}(A, B)$ the set of all algebra homomorphisms. A classical result reads there is a bijection between the elements

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of $\mathrm{Hom}(A,B)$ and natural transformations $T^A \to T^B$. We shall need the following form of the result. Let $A = E(k)/I$, $B = E(p)/J$ and $f : (\mathbb{R}^k, 0) \to (\mathbb{R}^p, 0)$ be a smooth map. Then $j^A f \in T_0^A \mathbb{R}^p$ is said to be a B-admissible A-velocity iff $j^A(g \circ f) = 0_A$ for all $g \in J$. It can be easily seen, that if $j^A f \in T_0^A \mathbb{R}^p$ is a B-admissible A-velocity, then $j^A(g \circ f)$ depends only on $j^B g$ for every $g : \mathbb{R}^p \to M$. The main result of [3] is that every B-admissible A-velocity $X = j^A f$ defines a natural transformation $i^X : T^B M \to T^A M$ by $j^B g \mapsto j^A(g \circ f)$. Moreover, every natural transformation $T^B \to T^A$ is of this type. It is proved in [6], that all those results remain valid if we restrict ourselves to the category $\mathcal{M}f_m$.

The group $\mathrm{Aut}(A)$ of all algebra automorphisms is a closed subgroup in $\mathrm{GL}(A)$, so it is a Lie subgroup. Every element D of its Lie algebra $\mathcal{A}ut(A)$ is tangent to a one-parameter subgroup $d(t)$ and determines a vector field $D(M)$ tangent to $(d(t))_M$ in $t = 0$ on $T^A M$. Thus we have an absolute natural operator $T \to TT^A$ such that $X \mapsto D(M)$ for every vector field X. This operator is denoted by $\mathrm{op}(D)$, [6] ,[5] .

Furthermore, for every natural bundle F we have the flow operator $\mathcal{F}$, defined by $\mathcal{F}(X) = \frac{\partial}{\partial t}|_0 F(Fl_t^X)$.

According to [6],[5], we have the following action of A on tangent vectors of $T^A M$. If $m : \mathbb{R} \times TM \to TM$ is the multiplication of the tangent vectors on M by reals, applying the functor T^A we obtain $T^A m : T^A\mathbb{R} \times T^A TM \to T^A TM$. Since $T^A TM = T^{A \otimes \mathbb{D}} M$ and $T^A \mathbb{R} = A$, where $\mathbb{D}$ is the algebra of dual numbers, we have constructed a map $A \times TT^A M \to TT^A M$. The coordinate expression of the action of $c \in A$ is $c(a_1, \dots, a_m, b_1, \dots, b_m) = (a_1, \dots, a_m, cb_1, \dots, cb_m)$ for all $a_1, \dots, a_m, b_1, \dots, b_m \in A$. This is a natural affinor [5] and we denote it by $af_M(c) : TT^A M \to TT^A M$.

Proposition 1 ([6]). *All natural operators $T \to TT^A$ are of the form $af(c) \circ \mathcal{T}^A + \mathrm{op}(D)$ for all $c \in A$, $D \in \mathcal{A}ut(A)$.*

In the special case $A = \mathbb{R}[x]/\langle x^{r+1}\rangle = \mathbb{D}_1^r$ we have $T^A M = T_1^r M = J_0^r(\mathbb{R}, M)$. Using the standard coordinates $(x^i, y_1^i, \dots, y_r^i, X^i, Y_1^i, \dots, Y_r^i)$ on $TT_1^r M$, we find $af_M(x+\langle x^{r+1}\rangle)(X^i, Y_1^i, \dots, Y_r^i) = (0, X^i, Y_1^i, \dots, Y_{r-1}^i)$. Let Q_M denote $af_M(x+\langle x^{r+1}\rangle)$.

Proposition 2 ([6]). *All natural operators $T \to TT_1^r$ are linearly generated by $\mathcal{T}_1^r, Q \circ \mathcal{T}_1^r, \dots, Q^r \circ \mathcal{T}_1^r, L, Q \circ L, \dots, Q^{r-1} \circ L$, where L is the generalized Liouville vector field having the coordinate form $X^i = 0, Y_s^i = s y_s^i$.*

1. Natural Operators Transforming Vector Fields to $T^*T_1^2$

According to Proposition 1 we have five generating natural operators $T \to TT_1^2$ and according to [2] we have two generating natural operators $T \to TT^*$, the flow operator $\mathcal{T}^*(x^i, p_i) = X^i \frac{\partial}{\partial x_i} - X_i^j p_j \frac{\partial}{\partial p_i}$ and the Liouville field $\mathcal{L}(x^i, p_i) = p_i \frac{\partial}{\partial p_i}$, where (x^i, p_i) are the standard coordinates on T^*M.

Composing these two sets of generators we obtain the following natural operators $T \to TT^*T_1^2$: $A_1 = \mathcal{T}^* \circ \mathcal{T}_1^2$, $A_2 = \mathcal{T}^* \circ (Q \circ \mathcal{T}_1^2)$, $A_3 = \mathcal{T}^* \circ (Q^2 \circ \mathcal{T}_1^2)$ and absolute operators $A_4 = \mathcal{T}^* \circ L$, $A_5 = \mathcal{T}^* \circ (Q \circ L)$, $A_6 = \mathcal{L}$.

Let the canonical coordinates x^i on $\mathbb{R}^m$ induce the coordinates $y^i = \frac{\partial x^i}{\partial \tau}$, $z^i = \frac{\partial^2 x^i}{\partial \tau^2}$ on $T_1^2\mathbb{R}^m$, while the additional coordinates on $T^*T_1^2\mathbb{R}^m$ are defined by $p_i dx^i + q_i dy^i + r_i dz^i$. Further, let x^i induce the additional coordinates ω_i on $T^*\mathbb{R}^m$ and $u^i = \frac{\partial x^i}{\partial \tau}$, $\gamma_i = \frac{\partial \omega_i}{\partial \tau}$, $w^i = \frac{\partial^2 x^i}{\partial \tau^2}$, $\delta_i = \frac{\partial^2 \omega_i}{\partial \tau^2}$ on $T_1^2T^*\mathbb{R}^m$.

We have the natural equivalence $s : T_1^2T^* \to T^*T_1^2$ of Cantrijn *et al* [1]

$$(1) \qquad \begin{gathered} (x^i, \omega_i, v^i, \gamma_i, w^i, \delta_i) \mapsto (x^i, y^i, z^i, p_i, q_i, r_i) \\ y^i = v^i, z^i = w^i, p_i = \delta_i, q_i = 2\gamma_i, r_i = \omega_i \end{gathered}$$

Thus we have two other natural operators: $A_7 = Ts((Q \circ \mathcal{T}_1^2) \circ \mathcal{L} \circ s^{-1})$ and $A_8 = Ts((Q^2 \circ \mathcal{T}_1^2) \circ \mathcal{L} \circ s^{-1})$.

Then the coordinate expressions of our operators are

$$A_1(X) = X^i \frac{\partial}{\partial x^i} + X^i_j y^j \frac{\partial}{\partial y^i} + (X^i_j z^j + X^i_{jk} y^j y^k) \frac{\partial}{\partial z^i} - (X^j_i p_j + X^j_{ik} y^k q_j + \\ + X^j_{ik} z^k r_j + X^j_{ikl} y^k y^l r_j) \frac{\partial}{\partial p_i} - (X^j_i q_j + 2X^j_{ik} y^k r_j) \frac{\partial}{\partial q_i} - X^j_i r_j \frac{\partial}{\partial r_i}$$

$$A_2(X) = X^i \frac{\partial}{\partial y^i} + 2X^i_j y^j \frac{\partial}{\partial z^i} - (X^j_i q_j + 2X^j_{ik} y^k r_j) \frac{\partial}{\partial p_i} - 2X^j_i r_j \frac{\partial}{\partial q_i}$$

$$A_3(X) = 2X^i \frac{\partial}{\partial z^i} - 2X^j_i r_j \frac{\partial}{\partial p_i} \qquad A_4 = y^i \frac{\partial}{\partial y^i} + 2z^i \frac{\partial}{\partial z^i} - q_i \frac{\partial}{\partial q_i} - 2r_i \frac{\partial}{\partial r_i}$$

$$A_5 = 2y^i \frac{\partial}{\partial z^i} - 2r_i \frac{\partial}{\partial q_i} \qquad A_6 = p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i} + r_i \frac{\partial}{\partial r_i}$$

$$A_7 = 2r_i \frac{\partial}{\partial q_i} + q_i \frac{\partial}{\partial p_i} \qquad A_8 = 2r_i \frac{\partial}{\partial p_i}$$

Let $p_M : FM \to M$ be a natural bundle of order r. According to the general theory, [6] ,there is a bijective correspondence between natural operators $A_M : T \to TFM$ and natural transformations $\mathcal{A}_M : J^rTM \times_M FM \to TFM$ over the identity of FM, which is given by $\mathcal{A}_M(j^r_x X, y) = A_M X(y)$, $x = p_M(y)$. Furthermore, there is a bijection between these natural transformations and equivariant maps of the standard fibers in question. Since $T^*T_1^2$ is a natural bundle of order three, we are searching for equivariant maps $(J^3T)_0\mathbb{R}^m \times (T^*T_1^2)_0\mathbb{R}^m \to (TT^*T_1^2)_0\mathbb{R}^m$. Let the additional coordinates on $TT^*T_1^2$ be

$$(2) \qquad W^i = dx^i,\ Y^i = dy^i,\ Z^i = dz^i,\ P_i = dp_i,\ Q_i = dq_i,\ R_i = dr_i$$

We evaluate the necessary transformation laws of the action of G^4_m on the standard fibers. Denote by $(a^i_{j_1}, \dots, a^i_{j_1\dots j_r})$ the canonical coordinates on G^r_m and indicate by tilde the coordinates of the inverse element. The action of G^4_m on $(T^*T^2_1)_0\mathbb{R}^m$ looks as follows

$$\bar{y}^i = a^i_j y^j \quad \bar{z}^i = a^i_j z^j + a^i_{jk} y^j y^k \quad \bar{r}_i = \tilde{a}^j_i r_j \quad \bar{q}_i = \tilde{a}^j_i q_j + 2\tilde{a}^j_{ik} a^k_l y^l r_j$$

$$(3) \qquad \bar{p}_i = \tilde{a}^j_i p_j + \tilde{a}^j_{ik} a^k_l y^l q_j + \tilde{a}^j_{ik} a^k_l z^l r_j + \tilde{a}^j_{ik} a^k_{lm} y^l y^m r_j + \tilde{a}^j_{ikl} a^k_m a^l_n y^m y^n r_j$$

Let $B^{r+1}_m = \{j^{r+1}_0 f;\ j^r_0 f = j^r_0 id_{\mathbb{R}^m}\}$. Then

$$\bar{q}_i = q_i - 2a^j_{ik} y^k r_j \qquad \text{for the action of } B^2_m$$

$$(4) \qquad \bar{p}_i = p_i - a^j_{ikl} y^k y^l r_j \qquad \text{for the action of } B^3_m$$

and

$$(5) \qquad \bar{X}^i_{j_1\dots j_r} = X^i + a^i_{j_1\dots j_r k} X^k \qquad \text{for the action of } B^{r+1}_m$$

$$\bar{X}^i_{j_1 j_2} = X^i_{j_1 j_2} - a^i_{kl} a^l_{j_1 j_2} X^k - a^k_{j_1 j_2} X^i_k + a^i_{j_1 k} X^k_{j_2} + a^i_{j_2 k} X^k_{j_1} \qquad \text{for the action of } B^2_m,$$

where $X^i_{j_1\dots j_r}$ indicates the r-jets of a vector field X. Furthermore

$$(6) \qquad \bar{W}^i = a^i_j W^j \qquad \bar{R}_i = \tilde{a}^j_i R_j \qquad \text{and it holds}$$

$$\bar{Y}^i = a^i_j Y^j \qquad \bar{Q}_i = \tilde{a}^j_i Q_j \qquad \bar{Z}^i = a^i_j Z^j \qquad \bar{P}_i = \tilde{a}^j_i R_j$$

whenever all the previous coordinates are zeros. Moreover, only P_i are changed by B^4_m and it holds

$$(7) \qquad \bar{P}_i = P_i - a^j_{iklm} y^k y^l W^m r_j$$

Finally we need the following lemma. Let

$$V_{p,q} = \underbrace{V \times \dots \times V}_{p-\text{times}} \times \overbrace{V^* \times \dots \times V^*}^{q-times},$$

where V denotes the vector space $\mathbb{R}^m$ with the standard action of G^1_m.

Lemma 3 ([6]). *(a) All smooth G^1_m-equivariant maps $V_{p,q} \to V$ are of the form*

$$\sum_{j=1}^{p} g_j(\langle x_k, y_l\rangle) x_j,$$

where $g_j : \mathbb{R}^{pq} \to \mathbb{R}$ are any smooth functions, $j, k = 1, \dots, p$, $l = 1, \dots, q$.

(b) All smooth G^1_m-equivariant maps $V_{p,q} \to V^*$ are of the form

$$\sum_{l=1}^{q} h_l(\langle x_k, y_h\rangle) y_l,$$

where $h_l : \mathbb{R}^{pq} \to \mathbb{R}$ are any smooth functions, $k = 1, \dots, p$, $h, l = 1, \dots, q$.

The proof of the main result essentially uses the following two lemmas.

Lemma 4. *Let $h : (J^3T)_0\mathbb{R}^m \times (T^*T_1^2)_0\mathbb{R}^m \to \mathbb{R}^m$ be an equivariant smooth mapping, $m \geq 2$. Then it holds*

$$W^i = g_1(I_1, \dots, I_5)X^i + g_2(I_1, \dots, I_5)y^i \tag{8}$$

where g_1, g_2 are any smooth functions $\mathbb{R}^5 \to \mathbb{R}$ and $I_1, \dots, I_5$ are invariants of the form

$$I_1 = X^i p_i + X^i_j y^j q_i + (X^i_j z^j + X^i_{jk} y^j y^k) r_i \quad I_2 = X^i q_i + 2X^i_j y^j r_i \tag{9}$$

$$I_3 = X^i r_i \qquad I_4 = y^i q_i + 2z^i r_i \qquad I_5 = y^i r_i$$

Proof. The first formula from (5) implies, that $W^i = h^i(j_0^3X, y^i, z^i, p_i, q_i, r_i)$ does not depend on $X^i_{j_1 j_2 j_3}$. Therefore we are searching equivariant maps $(J^2T)_0\mathbb{R}^m \times (T^*T_1^2)_0\mathbb{R}^m \to \mathbb{R}^m$.

Let S_0 be $C_0 \times (T^*T_1^2)_0\mathbb{R}^m$, where C_0 is the set of all 2-jets of constant vector fields on $\mathbb{R}^m$ at zero. Since S_0 is G_m^1-invariant and $W^i = a^i_j W^j$, the equivariance and Lemma 1 yield $W^i = \alpha_1 X^i + \alpha_2 y^i + \alpha_3 z^i$ on S_0, where α_1, α_2, α_3 are some functions of $X^i p_i, X^i q_i, X^i r_i, y^i p_i, y^i q_i, y^i r_i, z^i p_i, z^i q_i, z^i r_i$. Since $X^i p_i, X^i q_i, X^i r_i$ coincide with I_1, I_2, I_3 on S_0, $\alpha_1, \alpha_2, \alpha_3$ can be considered as functions of arguments I_1, I_2, I_3, I_4, I_5 and $y^i p_i, z^i p_i, z^i q_i, z^i r_i$.

Let $S_1 \subseteq S_0$ be the subset of all elements of S_0 satisfying the following conditions: X^i and y^i as well as X^i and z^i as well as y^i and z^i are linearly independent vectors and r_i is a non-zero vector. Obviously, S_1 is a dense subset of S_0. Let $i : G_m^1 \to G_m^3$ be the canonical injection. Fixing $X^i, y^i, z^i, p_i, q_i, r_i$ we can find some $j_0^3 f \in i(G_m^1)$ transforming X^i to δ^i_1, z^i to δ^i_2, while the other values are transformed to the bared ones. This is possible on S_1 due to the conditions from its definition.

Let ℓ denote, in general, the left action of the r-th order differential group on the standard fiber of an r-th order natural bundle. We have $h^i(j_0^2X, y^i, z^i, p_i, q_i, r_i) = \ell(j_0^3 f^{-1}, \ell(j_0^3 f, h^i(j_0^2X, y^i, z^i, p_i, q_i, r_i))) = \ell(j_0^3 f^{-1}, \alpha_1\delta^i_1 + \alpha_2\bar{y}^i + \alpha_3\delta^i_2)$, where the arguments of $\alpha_1, \alpha_2, \alpha_3$ are $I_1, \dots I_5$ and $\bar{y}^i\bar{p}_i, \bar{z}^i\bar{p}_i, \bar{z}^i\bar{q}_i, \bar{z}^i\bar{r}_i$ satisfying $\bar{z}^i = \delta^i_2$. It follows from the equivariance of h and the fact, that the last four arguments of $\alpha_1, \alpha_2, \alpha_3$ are G_m^1-invariants.

The definition of S_1 implies, that there is $j_0 \geq 2$ such that $y^{j_0} \neq 0$. Let $i_1 : B_m^2 \to G_m^3$ denote the canonical inclusion. Taking $j_0^3 f_1 \in i_1(B_m^2)$ with all $a^i_{jk} = 0$ except $a^2_{j_0 j_0}$ we can annihilate all expresions with z_i. It follows from (5) that $j_0^3 f_1$ stabilizes $j_0^2(\frac{\partial}{\partial x^1})$. But we changed the value of $\bar{y}^i\bar{p}_i$, which can be annihilated by taking a suitable $j_0^3 f_2 \in B_m^3$ with all $a^i_{jkl} = 0$ except $a^{k_0}_{j_0 j_0 j_0}$, where k_0 is an index such that $\bar{r}_{k_0} \neq 0$. It follows directly from (4) and (5), that $j_0^3 f_2$ stabilizes $j_0^2 \frac{\partial}{\partial x^1}$.

Thus we obtain, that $W^i = \ell(j_0^3 f^{-1}, \alpha_1\delta^i_1 + \alpha_2\bar{y}^i)$ on S_1, where the last four arguments of α_1, α_2 are zeros, while the invariants are not changed. So we have

$$\begin{aligned} W^i &= \ell(j_0^3 f^{-1}, \alpha_1(I_1, \dots, I_5, 0, 0, 0, 0)\delta^i_1 + \alpha_2(I_1, \dots, I_5, 0, 0, 0, 0)\bar{y}^i) = \\ &= \alpha_1(I_1, \dots, I_5, 0, 0, 0, 0)X^i + \alpha_2(I_1, \dots, I_5, 0, 0, 0, 0)y^i, \end{aligned}$$

which follows from the equivariance of the map h. Substituting $g_i(I_1,\dots,I_5)$ for $\alpha_i(I_1,\dots,I_5,0,0,0,0)$, $i=1,2$, we have

$$W^i = g_1(I_1,\dots,I_5)X^i + g_2(I_1,\dots,I_5)y^i \text{ on } S_1. \tag{10}$$

Since S_1 is dense in S_0, this holds on S_0 as well. Taking into account the equivariance of h, (10) can be extended to $(J^2T)_0\mathbb{R}^m \times (T^*T^2_1)_0\mathbb{R}^m$, which completes the proof. □

The following lemma is the dualization of Lemma 4 and since its proof is almost the same as that of Lemma 4, we omit it.

Lemma 5. *Let $h : (J^3T)_0\mathbb{R}^m \times (T^*T^2_1)_0\mathbb{R}^m \to \mathbb{R}^{m*}$ be an equivariant smooth mapping, $m \geq 2$. Then*

$$R_i = g_1(I_1,\dots,I_5)r_i \tag{11}$$

where $g : \mathbb{R}^5 \to \mathbb{R}$ is a smooth function.

Proposition 6. *For $m \geq 2$, every natural operator $A : T \to TT^*T^2_1$ is of the form $A = \sum_{j=1}^{8} h_j(I_1,\dots,I_5)A_j$, where $h_j : \mathbb{R}^5 \to \mathbb{R}$ are some smooth functions and*

$$\begin{array}{lll} A_1 = \mathcal{T}^* \circ \mathcal{T}^2_1 & A_2 = \mathcal{T}^* \circ (Q \circ \mathcal{T}^2_1) & A_3 = \mathcal{T}^* \circ (Q^2 \circ \mathcal{T}^2_1) \\ A_4 = \mathcal{T}^* \circ L & A_5 = \mathcal{T}^* \circ (Q \circ L) & A_6 = \mathcal{L} \\ A_7 = Ts((Q \circ \mathcal{T}^2_1) \circ \mathcal{L} \circ s^{-1}) & & A_8 = Ts((Q^2 \circ \mathcal{T}^2_1) \circ \mathcal{L} \circ s^{-1}) \end{array} \tag{12}$$

Proof. In the whole proof we use the coordinates (2). Let $A : T \to TT^*T^2_1$ be a natural operator and h be the corresponding equivariant map. Since $\bar{W}^i = a^i_j W^j$, applying Lemma 4 we get $W^i = g_1(I_1,\dots,I_5)X^i + g_2(I_1,\dots,I_5)y^i$. Taking the natural operator $B_1 = A - g_1(I_1,\dots,I_5)\mathcal{T}^* \circ \mathcal{T}^2_1$ we get its equivariant map in the form $W^i = g_2(I_1,\dots,I_5)y^i$.

First of all we prove, that g_2 is the zero function. Let $\alpha = (j^3_0(\frac{\partial}{\partial x^1}), \delta^i_2, z^i, p_i, q_i, r_i)$ be an element of $(J^3T)_0\mathbb{R}^m \times (T^*T^2_1)_0\mathbb{R}^m$ satisfying the existence of a non-zero r_i. Let j_0 be the least index, for which $r_{j_0} \neq 0$, and let $j^4_0 f \in B^4_m$ satisfy $a^i_{jklm} = 0$ except $a^{j_0}_{2222}$. Then the formula (7) implies, that we can change the value of P_2 stabilizing α, whenever $g_2(p_1, q_1, r_1, q_2 + 2z^i r_i, r_2) \neq 0$. Thus we obtain, that g_2 is the zero function on $\mathbb{R}^5$.

Now, put $h_1 = g_1$ and consider the natural operator B_1. Since its equivariant map satisfies $W^i = 0$, the formula (6) and Lemma 4 yield $Y^i = g_3(I_1,\dots,I_5)X^i + g_4(I_1,\dots I_5)y^i$. We can subtract $g_3(I_1,\dots,I_5)A_2 + g_4(I_1,\dots I_5)A_4$ and write $h_2 = g_3$ and $h_4 = g_4$. We can iterate these steps using the formula (6), Lemmas 4 and 5. This way we prove our claim. □

2. Natural Operators $T \to C^\infty(T^*TT_1^2, \mathbb{R})$

In this part we are searching all natural operators transforming vector fields to functions on $T^*TT_1^2$. We use essentially the following result by Kolář , [4]. Let F be a natural bundle, $Y : FM \to TFM$ be a vector field and $\widetilde{Y}$ denote the function $T^*FM \to \mathbb{R}$ defined by $\widetilde{Y}(w) = \langle Y(p(w)), w\rangle$, where p is the cotangent bundle projection. Let F have the following properties I,II,III.

I. The set N_F of all natural operators $T \to TFM$ is a finite dimensional vector space. (*This property is satisfied for every Weil bundle.*)

Let N_F^* be the dual vector space and $\mathsf{Nop}(T, T^*F \times \mathbb{R})$ denote the set of all natural operators $T \to C^\infty(T^*F, \mathbb{R})$. For every smooth function $h : N_F^* \to \mathbb{R}$ Kolář constructed the following natural operator Dh : $T \to C^\infty(T^*F, \mathbb{R})$. Fixing a basis $A_1, \dots, A_n$ of N_F, its dual vector space N_F^* can be identified with $\mathbb{R}^n$ and we can put $(\mathrm{Dh})_M X = h(\widetilde{A_{1,M}X}, \dots, \widetilde{A_{n,M}X}) : T^*FM \to \mathbb{R}$. Thus we obtain a mapping $C^\infty(N_F^*, \mathbb{R}) \to \mathsf{Nop}(T, T^*F \times \mathbb{R})$.

II. There exists a smooth function $j : N_F^* \to (T^*F)_0\mathbb{R}^m$ satisfying

$$\langle A, u\rangle = \widetilde{A(\frac{\partial}{\partial x^1})}(ju) \tag{13}$$

for every $A \in N_F$, $u \in N_F^*$.

Let $\mathit{Diff}_0^1\,\mathbb{R}^m$ denote the stability group of the origin and the vector field $\frac{\partial}{\partial x^1}$.

III. The orbit of $j(N_F^*)$ with respect to $\mathit{Diff}_0^1\,\mathbb{R}^m$ is dense in $(T^*F)_0\mathbb{R}^m$.

Proposition 7 ([4]). *If the assumptions I, II, III are satisfied, then all natural operators $T \to C^\infty(T^*F, \mathbb{R})$ are of the form* Dh *for all $h \in C^\infty(N_F^*, \mathbb{R})$.*

This result enables searching for natural operators $T \to C^\infty(T^*F, \mathbb{R})$, where $F = T^A$ is a Weil bundle. Let T^A be of order r. In order to find all the natural operators $T \to TT^*T^A$ we can use the following procedure consisting of four steps.

(a) We find a base $B_1, \dots B_k$ of all natural operators $T \to TT^A$.

(b) We take some immersion element $i \in T_0^A\mathbb{R}^m$. Over the element i we have a space P in $(T^*T^A)_0\mathbb{R}^m$, on which the stabilizing group H of i and $j_0^r(\frac{\partial}{\partial x^1})$ acts.

(c) We compute $I_i = \widetilde{B_i}(\frac{\partial}{\partial x^1})|P$. If possible, we choose coordinates $w_1, \dots, w_k$, $z_1, \dots, z_l$ on P such that $w_i = I_i$.

(d) We prove, that we can annihilate $z_1, \dots, z_l$ on a dense subset of P by the group H.

Then every natural operator $T \to C^\infty(T^*F, \mathbb{R})$ is smoothly generated by $\widetilde{B_1}, \dots, \widetilde{B_k}$. Indeed we can define

$$j : N_F^* \to (T^*F_0)\mathbb{R}^m, \qquad b_1B^1 + \cdots + b_kB^k \mapsto (b_1, \dots b_k, 0, \dots, 0) \tag{14}$$

which clearly satisfies (13). The denseness of the orbit $j(N_F^*)$ is guaranteed by (d).

Now we use this procedure for the bundle TT_1^2. First of all we find all natural operators $T \to TTT_1^2$. Since $TT_1^2 = T^{\mathbb{D}\otimes\mathbb{D}_1^2}$, where $\mathbb{D}\otimes\mathbb{D}_1^2 = \mathbb{R}[t,\tau]/\langle t^2,\tau^3\rangle$, every element from TT_1^2M is of the form $x^i + z_1^i\tau + \frac{1}{2}z_2^i\tau^2 + y^i t + w_1^i t\tau + \frac{1}{2}w_2^i t\tau^2$, where $(x^i, z_1^i, z_2^i, y^i, w_1^i, w_2^i)$ are the canonical coordinates on TT_1^2M.

Lemma 8. *All natural operators $T \to TTT_1^2$ are linearly generated by the following ones*

$$N_1 = \mathcal{T}\circ\mathcal{T}_1^2 \qquad N_2 = af(\tau + \langle t^2,\tau^3\rangle)(\mathcal{T}\circ\mathcal{T}_1^2)$$

$$N_3 = af(t + \langle t^2,\tau^3\rangle)(\mathcal{T}\circ\mathcal{T}_1^2) \qquad N_4 = af(\tau^2 + \langle t^2,\tau^3\rangle)(\mathcal{T}\circ\mathcal{T}_1^2)$$

$$N_5 = af(t\tau + \langle t^2,\tau^3\rangle)(\mathcal{T}\circ\mathcal{T}_1^2) \qquad N_6 = af(t\tau^2 + \langle t^2,\tau^3\rangle)(\mathcal{T}\circ\mathcal{T}_1^2)$$

$$N_7 = y^i\frac{\partial}{\partial y^i} + w_1^i\frac{\partial}{\partial w_1^i} + w_2^i\frac{\partial}{\partial w_2^i} \qquad N_8 = z_1^i\frac{\partial}{\partial z_1^i} + 2z_2^i\frac{\partial}{\partial z_2^i} + w_1^i\frac{\partial}{\partial w_1^i} + 2w_2^i\frac{\partial}{\partial w_2^i}$$

$$N_9 = y^i\frac{\partial}{\partial w_1^i} + 2w_1^i\frac{\partial}{\partial w_2^i} \qquad N_{10} = 2y^i\frac{\partial}{\partial w_2^i}$$

$$N_{11} = 2z_1^i\frac{\partial}{\partial z_2^i} + 2z_2^i\frac{\partial}{\partial w_2^i} \qquad N_{12} = z_1^i\frac{\partial}{\partial w_1^i} + 2z_2^i\frac{\partial}{\partial w_2^i}$$

$$N_{13} = 2z_1^i\frac{\partial}{\partial w_2^i}$$

Proof. By Proposition 1 we have to determine the absolute operators. In our case $A = \mathbb{D}\otimes\mathbb{D}_1^2$. Every A-velocity in question is of the form

$$\begin{aligned}(15)\qquad & at + b\tau + c\tau^2 + dt\tau + et\tau^2\\ & ft + g\tau + h\tau^2 + jt\tau + kt\tau^2\end{aligned}$$

Taking into account the conditions of admissibility we obtain $b = 0$, $ac = 0$ and $3fg^2 = 0$. Since every A-admissible A-velocity induces a homomorphism $A \to A$ and we are searching for curves in $\mathrm{Aut}(A)$ in a neighbourhood of the unit, we can restrict ourselves to the connected component of the unit in $\mathrm{Aut}(A)$. Then we have $c = 0$ and $f = 0$. Renaming the parameters in (15), all considered automorphisms $A \to A$ are given by

$$\begin{aligned}(16)\qquad & t \mapsto at + bt\tau + ct\tau^2\\ & \tau \mapsto d\tau + e\tau^2 + ft\tau + gt\tau^2\end{aligned}$$

By Proposition 1 we find the operators $N_7, \dots, N_{13}$ in the form of the curves in $\mathrm{Aut}(A)$ defined by reparametrization, e.g. N_7 by reparametrization $t \mapsto at, \tau \mapsto \tau$ or N_8 by reparametrization $\tau \mapsto b\tau, t \mapsto t$. □

Now we prove the main result of this Section.

Proposition 9. *All natural operators* $T\mathbb{R}^m \to C^\infty(T^*TT_1^2\mathbb{R}^m, \mathbb{R})$, $m \geq 3$, *are of the form*

$$h(\widetilde{N_1}, \widetilde{N_2}, \dots, \widetilde{N_{13}}), \tag{17}$$

where $h : \mathbb{R}^{13} \to \mathbb{R}$ *is an arbitrary smooth function and* $N_1, \dots, N_{13}$ *are the natural operators from Lemma 8.*

Proof. We apply the procedure explained before Lemma 8. According to the immersion theorem, we can consider i in the form

$$y^i = \delta_2^i, z_1^i = \delta_3^i, z_2^i = w_1^i = w_2^i = 0$$

for all $i = 1, \dots, m$. Let $q_i dx^i + r_i^1 dz_1^i + r_i^2 dz_2^i + p_i dy^i + s_i^1 dw_1^i + s_i^2 dw_2^i$ define the additional coordinates on $T^*TT_1^2M$. Taking the space P over the element i, we obtain the following values of $I_i = \widetilde{N_i}(\frac{\partial}{\partial x^1})|P$

$$I_1 = q_1,\ I_2 = r_1^1,\ I_3 = p_1,\ I_4 = r_1^2,\ I_5 = s_1^1,\ I_6 = s_1^2$$
$$I_7 = p_2,\ I_8 = r_3^1,\ I_9 = s_2^1,\ I_{10} = s_2^2,\ I_{11} = r_3^2,\ I_{12} = s_3^1,\ I_{13} = s_3^2.$$

The stabilizing group $H \subseteq G_m^4$ of the element i and $\frac{\partial}{\partial x^1}$ can be considered as a subgroup of $id_\mathbb{R} \times \mathit{Diff}_0\,\mathbb{R}^{m-1}$. The group H acts in the following way:

$$\begin{aligned} &\bar{z}_1^i = a_j^i z_1^j \quad \bar{z}_2^i = a_j^i z_2^j + a_{jk}^i z_1^j z_1^k \quad \bar{y}^i = a_j^i y^j \\ &\bar{w}_1^i = a_j^i w_1^j + a_{jk}^i z_1^j y^k \quad \bar{w}_2^i = a_j^i w_2^j + a_{jk}^i z_2^j y^k + 2a_{jk}^i z_1^j w_1^k + a_{jkl}^i z_1^j z_1^k y^l \end{aligned} \tag{18}$$

for $i, j \geq 2$. It is useful to annihilate the excessive coordinates extra for $m = 3$ and $m \geq 4$.

$m = 3$: We must annihilate p_3, r_2^1, r_2^2 and q_2, q_3. It follows from the action of H, that $a_j^i = \delta_j^i$, and for $i, j \geq 2$ it holds $a_{33}^i = a_{23}^i = a_{233}^i = 0$. Taking into account the action of $B_m^4 \cap H$ on $T^*TT_1^2$, we have $\bar{q}_2 = q_2 - a_{2233}^j s_j^2$, $\quad \bar{q}_3 = q_3 - a_{2333}^j s_j^2$, so we can annihilate q_2, q_3 by means of a_{2233}^2, a_{2333}^2 in the case $s_2^2 \neq 0$. Furthermore $B_m^3 \cap H$ turns p_3 to $\bar{p}_3 = p_3 - a_{333}^j s_j^2$ and r_2^1 to $\bar{r}_2^1 - 2a_{223}^j s_j^2$. Thus we can annihilate p_3 and r_2^1 by means of a_{333}^2 and a_{223}^2 if $s_2^2 \neq 0$. It remains to annihilate r_2^2. Since $B_m^2 \cap H$ turns r_2^2 to $\bar{r}_2^2 = r_2^2 - a_{22}^j s_j^2$, we can achieve $r_2^2 = 0$ by means of a_{22}^2 in the case of non-zero s_2^2. Since the condition $s_2^2 \neq 0$ determines a dense subset in P, our claim is proved for $m = 3$.

In the case $m \geq 4$ we put $a_j^i = \delta_j^i$. Analogously to the case $m = 3$ we obtain $a_{33}^i = a_{23}^i = a_{233}^i = 0$ from (18). We can annihilate q_i for $i \geq 2$ by means of a_{i233}^2 in the case $s_2^2 \neq 0$, p_i by a_{i33}^2 for $i \geq 3$ and r_i^1 by a_{i23}^2 for $i = 2$ or $i \geq 4$ in the case $s_2^2 \neq 0$. It remains to annihilate r_i^2 for $i = 2$ or $i \geq 4$, which can be done by means of a_{i2}^2 in the case $s_2^2 \neq 0$. Since the condition $s_2^2 \neq 0$ defines a dense subset of P, our claim is proved for the case $m \geq 4$ too. $\square$

Now we show, how the generating operators $T \to TT^*T_1^2$ can be found by means of the natural operators $T \to C^\infty(T^*TT_1^2, \mathbb{R})$. Let G be a natural bundle. A natural operator $T \to C^\infty(T^*G, \mathbb{R})$ is called a natural T-function. Every natural operator $D : T \to TG$ determines a natural T-function $\tilde{D}_M : T^*GM \to \mathbb{R}$, defined by $\tilde{D}_M(w) = \langle D_M(qw), w\rangle$, $w \in T^*GM$, $q : T^*G \to G$, which is linear on fibers. Conversely, let f_M be a natural T-function linear on fibers $T^*(GM)$. Then $f_M|T_z^*(GM)$, where $z \in GM$, is identified with an element $\tilde{f}_M(z)$ from the dual vector space $T_z(GM)$. Thus we obtain a natural operator $\tilde{f}_M : T \to TG$ and a canonical bijection between natural operators $T \to TG$ and natural T-functions, which are linear on fibers of $T^*(GM)$.

Let x^i be the standard coordinates on $\mathbb{R}^m$ and $p_i dx^i$ define the additional coordinates p_i on $T^*\mathbb{R}^m$. Let x^i, p_i induce the coordinates $X_1^i = dx^i, P_i = dp_i$ on $TT^*\mathbb{R}^m$. We can also define the additional coordinates ξ_i, η^i on $T^*T^*\mathbb{R}^m$ by $\xi_i dx^i + \eta^i dp_i$. Furthermore, let x^i induce the coordinates $Y^i = dx^i$ on $T\mathbb{R}^m$ and the additional coordinates α_i, β_i on $T^*T\mathbb{R}^m$ be defined by $\alpha_i dx^i + \beta_i dY^i$.

We have the natural equivalence $s : TT^* \to T^*T$ by Modugno, Stefani, [8], and the natural equivalence $t : TT^* \to T^*T^*$ by Kolář, Radziszewski, [7],

$$s(x^i, p_i, X_1^i, P_i) = (x^i, Y^i, \alpha_i, \beta_i), \text{ where } Y^i = X_1^i, \alpha_i = P_i, \beta_i = p_i \tag{19}$$

$$t(x^i, p_i, X_1^i, P_i) = (x^i, p_i, \xi_i, \eta^i), \quad \text{where } \xi_i = P_i, \eta^i = -X_1^i$$

.

Let the standard coordinates x^i on $\mathbb{R}^m$ induce the coordinates $z_1^i = \frac{\partial x^i}{\partial \tau}$, $z_2^i = \frac{\partial^2 x^i}{\partial \tau^2}$ on $T_1^2\mathbb{R}^m$ and the additional coordinates on $T^*T_1^2\mathbb{R}^m$ be defined by $p_i dx^i + s_i^1 dz_1^i + s_i^2 dz_2^i$. Further, define the additional coordinates on $T^*T^*T_1^2\mathbb{R}^m$ by $q_i dx^i + r_i^1 dz_1^i + r_i^2 dz_2^i - y^i dp_i - w_1^i ds_i^1 - w_2^i ds_i^2$.

Clearly, $N : T \to C^\infty(T^*TT_1^2, \mathbb{R})$ is a natural operator if and only if $A = N \circ s \circ t^{-1}$ is a natural operator $T \to C^\infty(T^*T^*T_1^2, \mathbb{R})$.

Transforming all the generating natural operators $T \to C^\infty(T^*TT_1^2, \mathbb{R})$ into the generating natural operators $T \to C^\infty(T^*T^*T_1^2, \mathbb{R})$ and among the transformed ones selecting those, which are linear on fibers over $T^*T_1^2$, we finally obtain the natural operators $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8$ from Section 1.

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