ARCHIVUM MATHEMATICUM (BRNO)
Tomus 31 (1995), 217 – 238

CONJUGACY CRITERIA AND PRINCIPAL SOLUTIONS OF SELF–ADJOINT DIFFERENTIAL EQUATIONS

Ondřej Došlý, Jan Komenda

Abstract. Oscillation properties of the self-adjoint, two term, differential equation

$$(-1)^n(p(x)y^{(n)})^{(n)} + q(x)y = 0 \tag{*}$$

are investigated. Using the variational method and the concept of the principal system of solutions it is proved that (*) is conjugate on $\mathbf{R} = (-\infty, \infty)$ if there exist an integer $m \in \{0, 1, \dots, n-1\}$ and $c_0, \dots, c_m \in \mathbf{R}$ such that

$$\int_{\infty}^{0} x^{2(n-m-1)} p^{-1}(x)\, dx = \infty = \int_0^{\infty} x^{2(n-m-1)} p^{-1}(x)\, dx$$

and

$$\limsup_{x_1 \downarrow -\infty, x_2 \uparrow \infty} \int_{x_1}^{x_2} q(x)(c_0 + c_1 x + \cdots + c_m x^m)^2\, dx \le 0, \quad q(x) \not\equiv 0.$$

Some extensions of this criterion are suggested.

I. INTRODUCTION

In this paper we study oscillation properties of solutions of the self-adjoint differential equation

$$(-1)^n(p(x)y^{(n)})^{(n)} + q(x)y = 0, \tag{1.1}$$

where $x \in I = (a, b)$, $-\infty \le a < b \le \infty$, $p(x) > 0$ on I and $p^{-1}, q \in L_{\text{loc}}(I)$. Particularly, we give conditions on the functions p, q which guarantee that (1.1) is conjugate on I.

Recall that two points $x_1, x_2 \in I$ are said to be *conjugate* relative to (1.1) if there exists a nontrivial solution y of (1.1) such that $y^{(i)}(x_1) = 0 = y^{(i)}(x_2)$,

1991 *Mathematics Subject Classification*: 34C10.
Key words and phrases: conjugate points, principal system of solutions, variational method, conjugacy criteria.
Received February 17, 1995.

$i = 0, 1, \dots, n-1$. Equation (1.1) is said to be *conjugate* on I whenever there exists at least one pair of points of I which are conjugate relative to (1.1); in the opposite case equation (1.1) is said to be *disconjugate* on I. If there exists $c \in I$ such that (1.1) is disconjugate on (c, b) then this equation is said to be *nonoscillatory at* b. In the opposite case (1.1) is said to be *oscillatory at* b. Oscillation and nonoscillation of (1.1) at the left end point a are defined in the same way.

The problem of conjugacy on a given interval has been recently studied for various differential equations. Hawking and Penrose [8] proved that the second order equation

$$-y'' + q(x)y = 0 \tag{1.2}$$

is conjugate on $\mathbf{R}$ provided $q(x) \le 0$ and $q \not\equiv 0$ on $\mathbf{R}$. This result was extended by Tipler [13] who showed that (1.2) is conjugate on $\mathbf{R}$ if

$$\limsup_{x_1 \downarrow a,\, x_2 \uparrow b} \int_{x_1}^{x_2} q(x)\, dx < 0. \tag{1.3}$$

Concerning the more general second order equation

$$-(p(x)y')' + q(x)y = 0,$$

Müller-Pfeiffer [9] proved that this equation is conjugate on an arbitrary interval I if

$$\int^b p^{-1}(x)\, dx = \infty = \int_a p^{-1}(x)\, dx. \tag{1.4}$$

and (1.3) hold. A similar criterion was proved in [9] for the fourth order equation and it was conjectured that the result may be extended to (1.1). This conjecture was shown to be correct in [4] where the following statement was proved.

Theorem A. *Suppose that there exist an integer* $m \in \{0, 1, \dots, n-1\}$ *and* $c_0, \dots, c_m \in \mathbf{R}$ *such that*

$$\int_{\infty}^{0} x^{2(n-m-1)} p^{-1}(x)\, dx = \infty = \int_0^{\infty} x^{2(n-m-1)} p^{-1}(x)\, dx. \tag{1.5}$$

If

$$\limsup_{x_1 \downarrow -\infty, x_2 \uparrow \infty} \int_{x_1}^{x_2} q(x)(c_0 + c_1 x + \dots + c_m x^m)^2\, dx < 0 \tag{1.6}$$

then (1.1) is conjugate on $\mathbf{R}$.

In [11] it was proved that (1.3) may be replaced by a weaker condition

$$\limsup_{x_1 \downarrow a,\, x_2 \uparrow b} \int_{x_1}^{x_2} q(x)\, dx \le 0, \quad q(x) \not\equiv 0. \tag{1.7}$$

In this paper we combine Theorem A with (1.7); we show that (1.1) is conjugate on $\mathbf{R}$ if (1.5) holds and (1.6) is replaced by the weaker condition

$$\limsup_{x_1\downarrow-\infty, x_2\uparrow\infty} \int_{x_1}^{x_2} q(x)(c_0+c_1x+\cdots+c_mx^m)^2\,dx \le 0, \quad q(x)\not\equiv 0.$$

In contrast to [4] where a rather tedious proof of Theorem A is presented, here we used the method based on the concept of the principal system of solutions introduced in [5].

The paper is organized as follows. In the next section we recall basic properties of solutions of self-adjoint, even order, differential equations and their relation to linear Hamiltonian systems. Section 3 is devoted to the investigation of the principal system of solutions of the equation

$$(p(x)y^{(n)})^{(n)} = 0$$

and in Section 4 these results are used to prove the main statement of the paper – a conjugacy criterion for (1.1). The last section contains some remarks and comments concerning possible extension of the results of the paper.

2. PRELIMINARY RESULTS

In this section we recall some results concerning oscillation properties of linear differential equations and their relation to linear Hamiltonian systems (LHS).

Let y be a solution of the self-adjoint equation

$$\sum_{k=0}^{n}(-1)^k\left(p_k(x)y^{(k)}\right)^{(k)} = 0, \tag{2.1}$$

where $p_n>0$ on I and $p_0, p_1, \dots, p_{n-1}, p_n^{-1} \in L_{\mathrm{loc}}(I)$. Let $u_1=y,\dots,u_n=y^{(n-1)}$, $v_n=p_ny^{(n)}$, $v_{n-k}=-v'_{n-k+1}+p_{n-k}y^{(n-k)}$, $k=1,\dots,n-1$, $u=(u_1,\dots,u_n)^T$, $v=(v_1,\dots,v_n)^T$ (the superscript "T" denotes the transpose of a vector or a matrix). Then (u,v) is a solution of the LHS

$$u' = Au + B(x)v, \quad v' = C(x)u - A^Tv, \tag{2.2}$$

where A, B, C are $n\times n$ matrices with entries

$$\begin{aligned} A_{ij} &= \begin{cases} 1, & \text{for } j=i+1, i=1,\dots,n-1,\\ 0, & \text{elsewhere},\end{cases}\\ B(x) &= \operatorname{diag}\{0,\dots,0,p_n^{-1}(x)\},\\ C(x) &= \operatorname{diag}\{p_0(x),\dots,p_{n-1}(x)\}, \end{aligned} \tag{2.3}$$

see e. g. [3, Chap. II]. We say that the solution (u, v) of (2.3) is *generated* by the solution y of (1.2). Conjugate points relative to (2.1) are defined in the same way as for (1.1).

Two points x_1, x_2 are said to be *conjugate relative* (2.2) if there exists a solution (u, v) of this system such that $u(x_1) = 0 = u(x_2)$ and $u(x)$ is not identically zero between x_1 and x_2. Consequently, x_1,x_2 are conjugate relative (1.1) if and only if they are conjugate relative to (2.3) with A, B, C given by (2.4). These definitions of conjugate points relative to (2.1) and (2.2) are motivated by the calculus of variations, where the nonexistence of conjugate points implies positive definiteness of the functional of the second variation, see e. g. [2]. Conjugacy, disconjugacy, oscillation and nonoscillation of (2.1) and (2.2) are defined via conjugate points in the same way as for (1.1).

Now let us recall some facts concerning linear Hamiltonian systems. Simultaneously with (2.2) consider the matrix system

$$U' = AU + B(x)V, \quad V' = C(x)U - A^T V \tag{2.4}$$

Here U, V are $n \times n$ matrices of real-valued functions. A solution (U, V) of (2.4) is said to be *self-conjugate* whenever $U^T(x)V(x) = V^T(x)U(x)$. Two solutions (U_1, V_1), (U_2, V_2) of (2.4) are said to be *linearly independent* if every solution U, V of (2.4) can be expressed in the form $(U, V) = (U_1C_1 + U_2C_2, V_1C_1 + V_2C_2)$, where C_1, C_2 are constant $n \times n$ matrices. If $(U_1, V_1), (U_2, V_2)$ are self-conjugate solutions, it can be shown that they are linearly independent if and only if the constant matrix $U_1^T(x)V_2(x) - V_1^T(x)U_2(x)$ is nonsingular. A self-conjugate solution (U_b, V_b) is said to be *principal* at b if $U_b(x)$ is nonsingular near b and there exists a solution (U_1, V_1), linearly independent of (U_b, V_b), such that $U_1(x)$ is nonsingular near b and $\lim_{x\to b-} U_1^{-1}(x)U_b(x) = 0$. The solution (U_1, V_1) linearly independent of (U_b, V_b) is said to be *nonprincipal at b*. It can be proved that (U_b, V_b) is the principal solution at b if and only if

$$\lim_{x\to b-} \left[\int^x U_b^{-1}(s)B(s)U_b^{T-1}(s)\, ds \right]^{-1} = 0$$

and that the principal solution is determined uniquely up to a right multiple by a constant nonsingular $n \times n$ matrix.

Let $y_1, \dots, y_n$ be solutions of (2.1) and $(u_1, v_1), \dots, (u_n, v_n)$ be the solutions of (2.2) generated by $y_1, \dots, y_n$. If (U, V) is the solution of (2.4) such that $u_1, \dots, u_n$ and $v_1, \dots, v_n$ form the columns of U and V respectively, we say that the solution (U, V) of (2.4) is *generated* by the solutions $y_1, \dots, y_n$ of (2.1). If $y_1, \dots, y_n$ generate the principal solution at b of (2.4) we call $y_1, \dots, y_n$ *the principal system of solutions* of (2.1) at b. If $y_1, \dots, y_n$ generate the nonprincipal solution at b of (2.4) we call $y_1, \dots, y_n$ *the nonprincipal system of solutions* of (2.1) at b.

Now we give some auxiliary results, which we use in the proof of Theorem 4.1. The conjugacy criterion given in this theorem is based on the following variational lemma.

Lemma 1. *([7]). Equation (2.1) is conjugate on $I = (a, b)$ if and only if there exists a function $y \in W^{2,n}(I)$, supp $y \subset I$, such that*

$$I(y; a, b) = \int_a^b \left[\sum_{k=0}^{n} p_k(x)(y^{(k)})^2 \right] dx < 0.$$

The following properties of equation (2.1) and LHS (2.2) are well known and their proofs can be found in [3, Chap. II].

Lemma 2. *Let (2.1) be disconjugate on I and let $x_1, x_2 \in I$, c_i, $d_i \in \mathbf{R}^n$, $i = 0, \ldots, n-1$, be arbitrary. There exists unique solution y of (2.1) satisfying the boundary conditions $y^{(i)}(x_1) = c_i$, $y^{(i)}(x_2) = d_i$, $i = 0, \ldots, n-1$.*

Lemma 3. *Let (U, V) be a self-conjugate solution of (2.4) such that the matrix $U(x)$ is nonsingular for $x \in I_0 \subseteq I$ and $c \in I_0$. Then*

$$\begin{aligned} U_1(x) =& U(x) \int_c^x U^{-1}(s) B(s) U^{T-1}(s) ds, \\ V_1(x) =& U^{T-1}(x) + V(x) \int_c^x U^{-1}(s) B(s) U^{T-1}(s) ds \end{aligned}$$

is also a self-conjugate solution of (2.4) and $V_1^T(x)U(x) - U_1^T(x)V(x) = I$ (the identity matrix).

We need also a result concerning transformations of the equation (2.1), which can be found in [1].

Lemma 4. *Let $h(x) \in C^n(I)$ be a positive real-valued function such that its quasiderivatives $h^{[n]} = p_n h^{(n)}$, $h^{[n+1]} = -\left(h^{[n]}\right)' + p_{n-1} h^{(n-1)}, \ldots, h^{[2n-1]} = -\left(h^{[2n-2]}\right)' + p_1 h'$ are absolutely continuous on I. The transformation $y = hz$ transforms equation (2.1) into the equation*

$$\sum_{k=0}^{n} (-1)^k (P_k(x) z^{(k)})^{(k)} = 0,$$

where $P_n = p_n h^2$, $P_0 = h \sum_{k=0}^{n} (-1)^k (p_k h^{(k)})^{(k)}$.

Throughout the paper we shall also use another definition of disconjugacy and nonoscillation for linear differential equations, introduced by Nehari. To distinguish these concepts from the above given ones, we shall speak about *N-disconjugacy* and *N-nonoscillation.*

Consider a linear differential equation

$$L(y) = y^{(n)} + q_{n-1}(x) y^{(n-1)} + \cdots + q_0(x) y = 0, \tag{2.5}$$

where $q_k(x) \in C(I)$. This equation is said to be *N-disconjugate* on an interval $I_0 \subseteq I$ whenever every nontrivial solution of this equation has at most $(n-1)$

zeros on I_0, every zero counted according to its multiplicity, (2.5) is said to be *N-nonoscillatory* at b if there exists $c \in I$ such that this equation is N-disconjugate on (c, b).

Recall briefly oscillation properties of solutions of (2.5). A system of solutions $y_1, \dots, y_n$ of (2.5) is said to form the *Markov system of solutions on an interval* $I_0 \subseteq I$ if the Wronskians

$$W(y_1, \dots, y_k) = \begin{vmatrix} y_1 & \dots & y_k \\ \vdots & & \vdots \\ y_1^{(k-1)} & \dots & y_k^{(k-1)} \end{vmatrix}, \quad k = 1, \dots, n$$

are positive throughout I_0. The proofs of the following lemmata may be found in [3, Chap. III]

Lemma 5. *Equation (2.5) is N-disconjugate on an interval $(d, b) \subseteq I$ if and only if there exists a Markov system of solutions of (2.5) on (d, b). This system can be found in such a way that it satisfies the additional conditions*

$$\begin{aligned} &y_i > 0 \text{ on } (d, b), i = 1, \dots, n, \\ &y_{k-1} = o(y_k) \text{ for } x \to b, k = 2, \dots, n. \end{aligned} \tag{2.6}$$

Lemma 6. *Let $y_1, \dots, y_{2n}$ be the linearly independent solutions of (2.1) such that*

$$\lim_{x \to b} \frac{y_k(x)}{y_{k+1}(x)} = 0, \quad k = 1, \dots, 2n - 1.$$

Then $y_1, \dots, y_n$ form the principal system of solutions at b and $y_{n+1}, \dots, y_{2n}$ the nonprincipal system of solutions at b of (2.1), particularly, for any two n-tuples, $1 \le i_1 < i_2 \dots < i_n \le 2n$, $1 \le j_1 < j_2 \dots < j_n \le 2n$, such that $i_k \le j_k$, $k = 1, \dots, n$, and at least one inequality is strict, we have

$$\lim_{x \to b} \frac{W(y_{i_1}, \dots, y_{i_n})}{W(y_{j_1}, \dots, y_{j_n})} = 0.$$

III. PRINCIPAL SYSTEM OF SOLUTIONS

In this section we describe explicitely the so-called ordered system of solutions at $\pm\infty$ of the equation

$$(p(x) y^{(n)}(x))^{(n)} = 0. \tag{3.1}$$

This description of ordered system plays crucial role in the proof of the conjugacy criterion for (1.1) given in the next section.

In order to simplify the formulation of our results, we introduce the following terminology.

Definition. A solution y of (3.1) is said to be *polynomial* if $y^{(n)} \equiv 0$. In the opposite case, i. e., $y^{(n)} \not\equiv 0$, it is said to be *nonpolynomial.*

Equation (3.1) has n linearly independent polynomial solutions $1, \ldots, x^{n-1}$ and n linearly independent nonpolynomial solutions

$$\bar{y}_1 = \int_0^x (x-t)^{n-1} p^{-1}(t)dt, \ldots, \bar{y}_n = \int_0^x (x-t)^{n-1} t^{n-1} p^{-1}(t)dt.$$

We also use the following notation. If $f, g : [T, \infty) \to \mathbf{R}$ for some $T \in \mathbf{R}$ we write

$$f \prec g \text{ as } t \to \infty \quad \iff \quad \lim_{t \to \infty} \frac{f(t)}{g(t)} = 0.$$

Recall that by Lemma 5 equation (3.1) always possesses a system of solutions $y_1, \ldots, y_n, \tilde{y}_1, \ldots, y_n$ such that

$$y_1 \prec y_2 \prec \cdots \prec y_n \prec \tilde{y}_1 \prec \tilde{y}_2 \prec \cdots \prec \tilde{y}_n. \tag{3.2}$$

In this case, by Lemma 6, $y_1, \ldots, y_n$ is the principal system of solutions at ∞ and $\tilde{y}_1, \ldots, \tilde{y}_n$ is the nonprincipal system. The system of solutions satisfying (3.2) is said to be *ordered at* ∞.

To describe explicitely ordered system of solution of (3.1) at ∞, we distinguish four (mutually disjoint) cases, each covered by one of the next four theorems.

Theorem 3.1. *Suppose that* $m \in \{0, 1, \ldots, n-1\}$, $n \leq 2(m+1)$, $k := n - m - 1$ *and*

$$\int^\infty x^{2k} p^{-1}(x)dx = \infty, \int^\infty x^{2k-1} p^{-1}(x)dx < \infty, \tag{3.2}$$

If

$$
\begin{aligned}
y_1(x) &= \int_0^x (x-t)^{n-1} t^k p^{-1}(t)dt - \sum_{j=0}^{k-1} \binom{n-1}{j} (-1)^j x^{n-1-j} \int_0^\infty t^{k+j} p^{-1}(t)dt, \\
y_2(x) &= \int_0^x (x-t)^{n-1} t^{k+1} p^{-1}(t)dt - \sum_{j=0}^{k-2} \binom{n-1}{j} (-1)^j x^{n-1-j} \\
&\quad \times \int_0^\infty t^{k+j+1} p^{-1}(t)dt, \\
&\ \vdots \\
y_k(x) &= \int_0^x (x-t)^{n-1} t^{2k-1} p^{-1}(t)dt - x^{n-1} \int_0^\infty t^{2k-1} p^{-1}(t)dt, \\
y_{k+1}(x) &= \int_0^x (x-t)^{n-1} t^{2k} p^{-1}(t), dt \\
&\ \vdots \\
y_{m+1}(x) &= \int_0^x (x-t)^{n-1} t^{k+m} p^{-1}(t)dt = \int_0^x (x-t)^{n-1} t^{n-1} p^{-1}(t)dt, \\
z_1(x) &= \int_0^x (x-t)^{n-1} p^{-1}(t)dt - \sum_{j=0}^{2k-1} \binom{n-1}{j} (-1)^j x^{n-1-j} \int_0^\infty t^j p^{-1}(t)dt, \\
z_2(x) &= \int_0^x (x-t)^{n-1} t p^{-1}(t)dt - \sum_{j=0}^{2k-2} \binom{n-1}{j} (-1)^j x^{n-1-j} \int_0^\infty t^{j+1} p^{-1}(t)dt, \\
&\ \vdots \\
z_k(x) &= \int_0^x (x-t)^{n-1} t^{k-1} p^{-1}(t)dt - \sum_{j=0}^{k} \binom{n-1}{j} (-1)^j x^{n-1-j} \int_0^\infty t^{j+k-1} p^{-1}(t)dt
\end{aligned}
$$

then the ordered system of solutions of (3.1) at ∞ is

$$
\begin{gathered}
1 \prec \cdots \prec z_1 \prec x^{2m-n+2} \prec z_2 \prec \cdots \prec x^{m-1} \prec z_k \prec x^m \prec \\
\prec y_1 \prec x^{m+1} \prec \cdots \prec y_{n-m-1} \prec x^{n-1} \prec y_{n-m} \prec \cdots \prec y_{m+1}.
\end{gathered}
$$

Proof. First we prove the ordering of nonprincipal system of solutions whose smallest solution we prove to be y_1. We have

$$
y_1^{(m+1)} = \frac{(n-1)!}{(n-m-2)!} \int_0^x (x-t)^{n-m-2} t^k p^{-1}(t)dt -
$$

$$-\sum_{j=0}^{k-1}(-1)^j \frac{(n-1)!(n-j-1)!}{j!(n-1-j)!(n-m-2-j)!}x^{n-m-j-2}\int_0^\infty t^{k+j}p^{-1}(t)dt =$$
$$= \frac{(n-1)!}{(k-1)!}\left[\int_0^x (x-t)^{k-1}t^k p^{-1}(t)dt - \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j x^{k-1-j}\int_0^\infty t^{k+j}p^{-1}(t)dt\right] =$$
$$= \frac{(n-1)!}{(k-1)!}\left[\int_0^x (x-t)^{k-1}t^k p^{-1}(t)dt - \int_0^\infty (x-t)^{k-1}t^k p^{-1}(t)dt\right] =$$
$$= \frac{(n-1)!}{(k-1)!}\int_x^\infty -(x-t)^{k-1}t^k p^{-1}(t)dt = (-1)^k \frac{(n-1)!}{(k-1)!}\int_x^\infty (t-x)^{k-1}t^k p^{-1}(t)dt.$$

If $x>0$, we have $\int_x^\infty (t-x)^{k-1}t^k p^{-1}(t)dt \le \int_x^\infty t^{k-1}t^k p^{-1}(t)dt = \int_x^\infty t^{2k-1}p^{-1}(t)dt$, hence $\lim_{x\to\infty} y_1^{(m+1)}(x) = 0$ and thus $y_1 \prec x^{m+1}$. Further,

$$\lim_{x\to\infty}\frac{y_1(x)}{x^m} = \frac{1}{m!}\lim_{x\to\infty} y_1^{(m)}(x) = \frac{1}{m!}\lim_{x\to\infty}\int_0^x y_1^{(m+1)}(t)dt$$

and

$$|\int_0^x y_1^{(m+1)}(t)dt| = \int_0^x \left[\int_t^\infty (s-t)^{k-1}s^k p^{-1}(s)ds\right]dt =$$
$$= \int_0^x \left[\int_0^s (s-t)^{k-1}s^k p^{-1}(s)dt\right]ds +$$
$$+\int_x^\infty \left[\int_0^x (s-t)^{k-1}s^k p^{-1}(s)dt\right]ds = \int_0^x s^k p^{-1}(s)\left[-\frac{(s-t)^k}{k}\right]_0^s ds +$$
$$+\int_x^\infty s^k p^{-1}(s)\left[-\frac{(s-t)^k}{k}\right]_0^x ds = \frac{1}{k}\int_0^x s^k s^k p^{-1}(s)ds +$$
$$+\frac{1}{k}\int_x^\infty s^k p^{-1}(s)\left[-(s-x)^k + s^k\right]ds.$$

Since by our assumption $\lim_{x\to\infty}\int_0^x t^{2k}p^{-1}(t)dt = \infty$, the first integral in the last expression tends to ∞ as $x\to\infty$. The second integral is positive (for $s>x>0$ we have $s^k > (s-x)^k$), hence $\lim_{x\to\infty}|y_1^{(m)}(x)| = \infty$ which means $x^m \prec y_1$.

By a similar computation one may verify that

$$\lim_{x\to\infty}\frac{y_j(x)}{x^{m+j}} = 0, \quad \lim_{x\to\infty}\frac{y_j(x)}{x^{m+j-1}} = \infty,$$

$j=1,\dots,k=n-m-1$, hence $x^{m+j-1} \prec y_j \prec x^{m+j}$.

To prove the remaining inequalities, we proceed in the same way. As an example, we prove that $x^{2m-n+1} = x^{n-2k-1} \prec z_1 \prec x^{2m-n+2} = x^{2n-k}$. We have

$$
\begin{aligned}
z_1^{(n-2k)}(x) &= \int_0^x (x-t)^{2k-1} \frac{(n-1)!}{(2k-1)!} p^{-1}(t)dt - \\
&- \sum_{j=0}^{2k-1} \frac{(n-1)!}{j!(2k-j-1)!} (-1)^j x^{2k-j-1} \int_0^\infty t^j p^{-1}(t)\,dt = \\
= \frac{(n-1)!}{(2k-1)} &\left[\int_0^x (x-t)^{2k-1} p^{-1}(t)dt - \sum_{j=0}^{2k-1} \binom{2k-1}{j} (-1)^j x^{2k-1-j} \int_0^\infty t^j p^{-1}(t)dt \right] = \\
&= \frac{(n-1)!}{(2k-1)!} \left[\int_0^x (x-t)^{2k-1} p^{-1}(t)dt - \int_0^\infty (x-t)^{2k-1} p^{-1}(t)dt \right] = \\
&= -\frac{(n-1)!}{(2k-1)!} \int_x^\infty (x-t)^{2k-1} p^{-1}(t)dt.
\end{aligned}
$$

Further, $|\int_x^\infty (x-t)^{2k-1} p^{-1}(t)dt| \le \int_x^\infty t^{2k-1} p^{-1}(t)dt \to 0$ as $x \to \infty$ and hence $\lim_{x\to\infty} \frac{z_1(x)}{x^{n-2k}} = 0$, i. e., $z_1 \prec x^{n-2k}$. Next we show that $x^{n-2k-1} \prec z_1$. It holds

$$
\begin{aligned}
\lim_{x\to\infty} \frac{z_1(x)}{x^{n-2k-1}} &= \frac{1}{(n-2k-1)!} \lim_{x\to\infty} z_1^{(n-2k-1)}(x) = \\
&= \lim_{x\to\infty} \frac{1}{(n-2k-1)!} \lim_{x\to\infty} \int_0^x z_1^{(n-2k)}(t)dt = \\
&= \frac{1}{(n-2k-1)!} \lim_{x\to\infty} \int_0^x \left(\int_t^\infty (s-t)^{2k-1} p^{-1}(s)ds \right) dt
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{x\to\infty} \int_0^x \left(\int_t^\infty (s-t)^{2k-1} p^{-1}(s)ds \right) dt = \\
&= \lim_{x\to\infty} \left[\int_0^x \left(\int_0^s (s-t)^{2k-1} p^{-1}(s)dt \right) ds + \int_x^\infty \left(\int_0^x (s-t)^{2k-1} p^{-1}(s)dt \right) ds \right] = \\
&= -\lim_{x\to\infty} \left[\int_0^x p^{-1}(s) \left[\frac{(s-t)^{2k}}{2k} \right]_0^s ds + \int_x^\infty \left[\frac{(s-t)^{2k}}{2k} \right]_0^x p^{-1}(s)ds \right] = \\
&= \lim_{x\to\infty} \left[\int_0^x p^{-1}(s) \frac{s^{2k}}{2k} ds + \int_x^\infty p^{-1}(s) \left(\frac{s^{2k}}{2k} - \frac{(s-x)^{2k}}{2k} \right) ds \right] = \infty,
\end{aligned}
$$

since the integrand in the last integral is positive for $x > 0$. Consequently, we have $x^{2m-n+1} = x^{n-2k-1} \prec z_1 \prec x^{2m-n+2} = x^{2n-k}$. □

The remaining three statements of this section may be proved in the same way as Theorem 3.1.

Theorem 3.2. *Suppose that (3.2) holds and $n \geq 2(m+1)$. Let*

$$\begin{aligned}
y_1 &= \int_0^x (x-t)^{n-1} t^k p^{-1}(t)dt - \sum_{j=0}^{k-1} \binom{n-1}{j} (-1)^j x^{n-1-j} \int_0^\infty t^{k+j} p^{-1}(t)dt \\
y_2 &= \int_0^x (x-t)^{n-1} t^{k+1} p^{-1}(t)dt - \sum_{j=0}^{k-2} \binom{n-1}{j} (-1)^j x^{n-1-j} \\
&\quad \times \int_0^\infty t^{k+j+1} p^{-1}(t)dt \\
&\quad \vdots \\
y_{m+1} &= \int_0^x (x-t)^{n-1} t^{k+m} p^{-1}(t)dt - \\
&\quad - \sum_{j=0}^{k-m-1} \binom{n-1}{j} (-1)^j x^{n-1-j} \int_0^\infty t^{k+m+j} p^{-1}(t)dt, \\
z_1(x) &= \int_x^\infty (x-t)^{n-1} p^{-1}(t)dt \\
&\quad \vdots \\
z_{k-m} &= \int_x^\infty (x-t)^{n-1} t^{-n+2k} p^{-1}(t)dt) \\
z_{k-m+1} &= \int_0^x (x-t)^{n-1} t^{2k-n+1} p^{-1}(t)dt - \\
&\quad - \sum_{j=0}^{n-2} \binom{n-1}{j} (-1)^j x^{n-1-j} \int_0^\infty t^{2k-n+j+1} p^{-1}(t)dt \\
&\quad \vdots \\
z_k &= \int_0^x (x-t)^{n-1} t^{k-1} p^{-1}(t)dt - \\
&\quad - \sum_{j=0}^{k} \binom{n-1}{j} (-1)^j x^{n-1-j} \int_0^\infty t^{j+k-1} p^{-1}(t)dt.
\end{aligned}$$

then the ordered system of solutions of (3.1) at ∞ is

$$\begin{aligned}
&z_1 \prec \cdots \prec z_{k-m} \prec 1 \prec z_{k-m+1} \prec x \prec \cdots \prec z_k \prec x^m \prec \\
&\prec y_1 \prec x^{m+1} \prec \cdots \prec x^{2m} \prec y_{m+1} \prec x^{2m+1} \prec \cdots \prec x^{n-1}.
\end{aligned}$$

Theorem 3.3. *Let m and k be the same as in Theorem 3.1, $n < 2m+3$, i. e., $k < m+2$ and*

$$\int^{\infty} x^{2k-2}p^{-1}(x)dx < \infty, \quad \int^{\infty} x^{2k-1}p^{-1}(x)dx = \infty. \tag{3.4}$$

If

$$y_1(x) = \int_0^x (x-t)^{n-1}t^k p^{-1}(t)dt - \sum_{j=0}^{k-2}\binom{n-1}{j}(-1)^j x^{n-1-j}\int_0^{\infty} t^{k+j}p^{-1}(t)\,dt,$$

$$y_2(x) = \int_0^x (x-t)^{n-1}t^{k+1}p^{-1}(t)dt - \sum_{j=0}^{k-3}\binom{n-1}{j}^j x^{n-1-j}\int_0^{\infty} t^{k+j+1}p^{-1}(t)\,dt,$$

$$\vdots$$

$$y_{k-1}(x) = \int_0^x (x-t)^{n-1}t^{2(k-1)}p^{-1}(t)dt - x^{n-1}\int_0^{\infty} t^{2(k-1)}p^{-1}(t)\,dt$$

$$y_k(x) = \int_0^x (x-t)^{n-1}t^{2k-1}p^{-1}(t)\,dt$$

$$\vdots$$

$$y_{m+1}(x) = \int_0^x (x-t)^{n-1}t^{n-1}p^{-1}(t)\,dt.$$

$$z_1(x) = \int_0^x (x-t)^{n-1}p^{-1}(t)dt - \sum_{j=0}^{2k-2}\binom{n-1}{j}(-1)^j x^{n-1-j}\int_0^{\infty} t^j p^{-1}(t)dt$$

$$z_2(x) = \int_0^x (x-t)^{n-1}tp^{-1}(t)\,dt - \sum_{j=0}^{2k-3}\binom{n-1}{j}(-1)^j x^{n-1-j}\int_0^{\infty} t^{j+1}p^{-1}(t)\,dt$$

$$\vdots$$

$$z_k(x) = \int_0^x (x-t)^{n-1}t^{k-1}p^{-1}(t)\,dt - \sum_{j=0}^{k-1}\binom{n-1}{j}(-1)^j x^{n-1-j}\int_0^{\infty} t^{j+k-1}p^{-1}(t)\,dt.$$

then the ordered system of solutions of (3.1) at ∞ is

$$1 \prec \cdots \prec x^{2m-n+2} \prec z_1 \prec x^{2m-n+3} \prec \cdots \prec z_{k-1}x^m \prec z_k \prec$$
$$\prec x^{m+1} \prec y_1 \prec x^{m+2} \prec \cdots \prec y_{k-1} \prec x^{n-1} \prec y_k \prec \cdots \prec y_{m+1}.$$

Theorem 3.4. *Let (3.4) holds and $n \geq 2m+3$, i. e., $k \geq m+2$. If*

$$y_1(x) = \int_0^x (x-t)^{n-1}t^k p^{-1}(t)dt - \sum_{j=0}^{k-2}\binom{n-1}{j}(-1)^j x^{n-1-j}\int_0^{\infty} t^{k+j}p^{-1}(t)dt$$

$$
\begin{aligned}
y_2(x) &= \int_0^x (x-t)^{n-1} t^{k+1} p^{-1}(t)dt - \sum_{j=0}^{k-3} \binom{n-1}{j} (-1)^j x^{n-1-j} \\
&\quad \times \int_0^\infty t^{k+j+1} p^{-1}(t)dt \\
&\vdots \\
y_{m+1} &= \int_0^x (x-t)^{n-1} t^{k+m} p^{-1}(t)dt - \sum_{j=0}^{k-m-2} \binom{n-1}{j} (-1)^j x^{n-1-j} \\
&\quad \times \int_0^\infty t^{k+m+1} p^{-1}(t)\, dt, \\
z_1 &= \int_x^\infty (x-t)^{n-1} p^{-1}(t)dt \\
&\vdots \\
z_{-n+2k} &= \int_x^\infty (x-t)^{n-1} t^{-n+2k-1} p^{-1}(t)dt \\
z_{2k-n+1} &= \int_0^x (x-t)^{n-1} t^{2k-n} p^{-1}(t)dt - \sum_{j=0}^{n-2} \binom{n-1}{j} (-1)^j x^{n-1-j} \\
&\quad \times \int_0^\infty t^{j+2k-n} p^{-1}(t)dt \\
&\vdots \\
z_k &= \int_0^x (x-t)^{n-1} t^{k-1} p^{-1}(t)dt - \sum_{j=0}^{k-1} \binom{n-1}{j} (-1)^j x^{n-1-j} \\
&\quad \times \int_0^\infty t^{k+j-1} p^{-1}(t)dt.
\end{aligned}
$$

then the ordered system of solutions of (3.1) at ∞ is

$$
\begin{gathered}
z_1 \prec \cdots \prec z_{k-m-1} \prec 1 \prec z_{k-m} \prec x \prec \cdots \prec z_{k-1} \prec x^m \prec z_k \prec \\
\prec x^{m+1} \prec y_1 \prec \cdots \prec y_{m+1} \prec x^{2m} \prec \cdots \prec x^{n-1}.
\end{gathered}
$$

Remark 3.1. To determine an ordered system of solutions at $-\infty$ one may use the same method. It suffices in the above theorems to replace $\int_0^\infty$ by $\int_{-\infty}^0$. As a corollary of previous theorems we have the following statement which plays crucial role in the proof of the conjugacy criterion for (1.1) given in the next section.

Corollary 3.1. *Let $\mathcal{V}^+$, $\mathcal{V}^-$ denote the subspaces of the solution space of (3.1) generated by the principal system of solution at ∞ and $-\infty$, respectively. If for* *some* $m \in \{0, \dots, n-1\}$

$$\int_{-\infty}^{0} x^{2(n-m-1)} p^{-1}(x)\, dx = \infty = \int_{0}^{\infty} x^{2(n-m-1)} p^{-1}(x)\, dx$$

then

$$\mathcal{V}^+ \cap \mathcal{V}^- \supseteq \mathit{Lin}\{1, x, \dots, x^m\}.$$

Remark 3.2. The statement of Corollary 3.1 is implicitely given without proof in [4]. In that paper it is claimed that if the first integral in (3.3) is divergent and $\int_0^\infty x^{2k-2} p^{-1}(x)\, dx < \infty$ then the principal solution at ∞ of the LHS corresponding to (3.1) is of the form

$$U_0(x) = H(x) \left(C_1 + \int_0^x H^{-1}(t) B(t) H^{T^{-1}}(t)\, dt \right), \quad V_0(x) = H^{T^{-1}}(x) C_2,$$

where H is the fundamental matrix of the system $H' = AH$ satisfying $H(0) = I$,

$$C_1 = \begin{pmatrix} I_{m+1} & 0 \\ 0 & \int_0^\infty \bar{B}_3(t) dt \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-m-1} \end{pmatrix},$$

I_{m+1}, I_{n-m-1} being identity matrices of the dimension given by their subscript and $\bar{B}_3(t)$ is the $(n-m-1) \times (n-m-1)$ matrix which we get if we write the matrix $H^{-1}(t) B(t) H^{T^{-1}}(t)$ in the form

$$H^{-1} B H^{T^{-1}} = \begin{pmatrix} \bar{B}_1 & \bar{B}_2 \\ \bar{B}_2^T & \bar{B}_3 \end{pmatrix}$$

with $(m+1) \times (m+1)$ and $(m+1) \times (n-m-1)$ matrices $\bar{B}_1,\ \bar{B}_2$. If

$$\tilde{C}_1 = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-m-1} \end{pmatrix}, \tilde{C}_2 = \begin{pmatrix} I_{m+1} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\tilde{U}(x) = H(x) \left(\tilde{C}_1 + \int_0^x H^{-1}(t) B(t) H^{T^{-1}}(t)\, dt\, \tilde{C}_2 \right), \quad \tilde{V}(x) = H^{T^{-1}}(x) \tilde{C}_2$$

then the matrix $U_0^T \tilde{V} - V_0^T \tilde{U}$ is nonsingular, i. e., (U_0, V_0) and $(\tilde{U}, \tilde{V})$ form the base of the solution space of (2.2) and by a direct computation one may verify that

$$\tilde{U}^{-1}(x) U_0(x) = \begin{pmatrix} \left(\int_0^x \bar{B}_1\right)^{-1} & \left(\int_0^x \bar{B}_1\right)^{-1} \left(\int_0^x \bar{B}_2\right) \\ -\left(\int_0^x \bar{B}_2^T\right) \left(\int_0^x \bar{B}_1\right)^{-1} & \left(\int_x^\infty \bar{B}_3\right) \end{pmatrix}.$$

To prove that (U_0, V_0) is the principal solution at ∞, one needs to show that $\tilde{U}^{-1}(x) U(x) \to 0$ as $x \to \infty$. After some computations it is possible to see that this is a difficult problem. Moreover, this approach does not give any information about the ordered system of solutions.

IV. CONJUGACY CRITERION

In the following theorem we give the conjugacy criterion which extends the result of [11] to equation (1.1) and generalizes also the main result of [4] given in Theorem A.

Theorem 4.1. *Suppose there exist an integer $m \in \{0, \dots, n-1\}$ and real constants $c_0, \dots, c_m$ such that*

$$\int_{-\infty}^{0} x^{2(n-m-1)} p^{-1}(x)dx = \infty, \quad \int_{0}^{\infty} x^{2(n-m-1)} p^{-1}(x)dx = \infty$$

and the function $h(x) = c_0 + \cdots + c_m x^m$ satisfies

$$\limsup_{t_1 \downarrow -\infty, t_2 \uparrow \infty} \int_{t_1}^{t_2} q(x)h^2(x)dx \le 0. \tag{4.1}$$

If $q \not\equiv 0$ in $\mathbf{R}$ then (1.1) is conjugate on $\mathbf{R}$.

Proof. The proof is based on the variational principle given in Lemma 1. It suffices to find a function $y \in W^{2,n}(\mathbf{R})$ with $\operatorname{supp} y \subset \mathbf{R}$ such that the quadratic functional $I(y; -\infty, \infty)$ is negative.

The equation (3.1) can be rewritten in the form of LHS (2.3) with $C = 0$, i. e.,

$$u' = Au + B(x)v, \quad v' = -A^T v. \tag{4.2}$$

and its matrix analogy is

$$U' = AU + B(x)V, \quad V' = -A^T V, \tag{4.3}$$

where $A, B(x)$ are $n \times n$ matrices given by (2.3). Let $\bar{x} \in \mathbf{R}$, $d > 0$, $\rho > 0$ be such that $q(x) \le -d$ for $|x - \bar{x}| \le \rho$ (the existence of such quantities follows from (4.1) and the fact that $q \not\equiv 0$) and let $x_1, x_2 \in \mathbf{R}$ be such that $x_1 < \bar{x} - \rho, \bar{x} + \rho < x_2$. Further, let Δ be any C^n function which is positive in $(\bar{x} - \rho, \bar{x} + \rho)$ and is equal to zero outside this interval. Finally, choose $x_0, x_3 \in \mathbf{R}$ such that $x_0 < x_1$ and $x_2 > x_3$.

Now, denote by f, g the solution of the boundary value problem

$$(p(x)y^{(n)}(x))^{(n)} = 0$$
$$f^{(i)}(x_0) = 0,\ f^{(i)}(x_1) = h^{(i)}(x_1),\ g^{(i)}(x_3) = 0,\ g^{(i)}(x_2) = h^{(i)}(x_2),$$
$$i = 0, \dots, n-1,$$

respectively. As the equation (3.1) is disconjugate, by Lemma 2 this problem has unique solution for every x_i, $i = 0, \dots, 3$. Define the test function y as follows

$$y(x) = \begin{cases} 0, & x < x_0, \\ f(x), & x_0 \le x \le x_1, \\ h(x)(1 + \delta\Delta(x)), & x_1 < x < x_2, \\ g(x), & x_2 \le x \le x_3, \\ 0, & x > x_3, \end{cases}$$

where $\delta > 0$. Then $y \in W^{2,n}(\mathbf{R})$ and $\operatorname{supp} y \subset \mathbf{R}$. Denote by (u_1, v_1) and (u_2, v_2) solutions of (4.2) corresponding to the solutions f, g of (3.1). If we denote $c = (c_0, \dots, c_m, 0, \dots, 0)^T \in \mathbf{R}^n$ then by Lemma 3

$$
\begin{aligned}
u_1(x) &= U_L(x)\left(\int_{x_0}^{x} U_L^{-1} B U_L^{T-1} ds\right)\left(\int_{x_0}^{x_1} U_L^{-1} B U_L^{T-1} ds\right)^{-1} c \\
v_1(x) &= \left(V_L(x)\int_{x_0}^{x} U_L^{-1} B U_L^{T-1} ds + U_L^{T-1}(x)\right)\left(\int_{x_0}^{x_1} U_L^{-1} B U_L^{T-1} ds\right)^{-1} c \\
u_2(x) &= U_R(x)\left(\int_{x}^{x_3} U_R^{-1} B U_R^{T-1} ds\right)\left(\int_{x_2}^{x_3} U_R^{-1} B U_R^{T-1} ds\right)^{-1} c \\
v_2(x) &= \left(V_R(x)\int_{x}^{x_3} U_R^{-1} B U_R^{T-1} ds - U_R^{T-1}(x)\right)\left(\int_{x_2}^{x_3} U_R^{-1} B U_R^{T-1} ds\right)^{-1} c,
\end{aligned}
$$

where $(U_R, V_R), (U_L, V_L)$ denote the principal solutions of (2.4) at ∞ and $-\infty$, such that U_L and U_R contain at first $m+1$ columns functions $1, \dots, x^m$ and their derivatives up to order $n-1$. Hence, u_1 and u_2 satisfy the following boundary conditions:

$$
\begin{aligned}
u_1(x_0) &= 0 \qquad u_1(x_1) = \tilde{h}(x_1) \\
u_2(x_3) &= 0 \qquad u_2(x_2) = \tilde{h}(x_2),
\end{aligned}
$$

where

$$\tilde{h}(x) = (h(x), h'(x), \dots, h^{(n-1)}(x))^T = U_L(x)c = U_R(x)c.$$

Let us compute

$$
\begin{aligned}
\int_{x_0}^{x_1} p(f^{(n)})^2 dx &= \int_{x_0}^{x_1} v_1^T B v_1 dx = \int_{x_0}^{x_1} v_1^T (u_1' - A u_1) dx = \\
&= \left[v_1^T u_1\right]_{x_0}^{x_1} - \int_{x_0}^{x_1} (v_1^{T\prime} u_1 + v_1^T A u_1) dx = \left[v_1^T u_1\right]_{x_0}^{x_1} - \\
&- \int_{x_0}^{x_1} v_1^T(-A + A) u_1 dx = v_1^T(x_1) u_1(x_1).
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
\int_{x_2}^{x_3} p(g^{(n)})^2 dx &= \int_{x_2}^{x_3} v_2^T B v_2 dx = \int_{x_2}^{x_3} v_2^T (u_2' - A u_2) dx = \\
&= \left[v_2^T u_2\right]_{x_2}^{x_3} - \int_{x_2}^{x_3} (v_2^{T\prime} u_2 + v_2^T A u_2) dx = \left[v_2^T u_2\right]_{x_2}^{x_3} - \\
&- \int_{x_2}^{x_3} v_2^T(-A + A) u_2 dx = -v_2^T(x_2) u_2(x_2)
\end{aligned}
$$

Using these results, we have

$$\int_{x_0}^{x_1} p(f^{(n)})^2 dx = v_1^T(x_1)u_1(x_1) = c^T \left(\int_{x_0}^{x_1} U_L^{-1} B U_L^{T-1} ds\right)^{-1} \times$$
$$\times \left[\int_{x_0}^{x_1} U_L^{-1} B U_L^{T-1} ds\, V_L^T(x_1) + U_L^{-1}(x_1)\right] U_L(x_1) \left(\int_{x_0}^{x_1} U_L^{-1} B U_L^{T-1} ds\right) \times$$
$$\times \left(\int_{x_0}^{x_1} U_L^{-1} B U_L^{T-1} ds\right)^{-1} c = c^T \left(\int_{x_0}^{x_1} U_L^{-1} B U_L^{T-1} ds\right)^{-1} c.$$

Here the fact that $V_L(x)c \equiv 0$ has been used. In a similar way we obtain

$$\int_{x_2}^{x_3} p(g^{(n)})^2 dx = c^T \left(\int_{x_2}^{x_3} U_R^{-1} B U_R^{T-1} ds\right)^{-1} c.$$

As $U_L(x)$ and $U_R(x)$ are the first components of the principal solutions of (4.3) at $-\infty$ and ∞, we have

$$\left(\int_{x_0}^{x_1} U_L^{-1} B U_L^{T-1} ds\right)^{-1} \to 0 \quad \text{for } x_0 \to -\infty \tag{4.4}$$

and

$$\left(\int_{x_2}^{x_3} U_R^{-1} B U_R^{T-1} ds\right)^{-1} \to 0 \quad \text{for } x_3 \to \infty \tag{4.5}$$

Now we estimate $\int_{x_0}^{x_1} qf^2 dx$ and $\int_{x_2}^{x_3} qg^2 dx$. To do it we prove the monotonicity of the functions f/h and g/h on (x_0, x_1) and (x_2, x_3). Suppose, that $z_1, \dots, z_{2n}$ is an ordered system of solutions at ∞ of (3.1), i. e., $z_i = o(z_{i+1})$ for $x \to \infty$, $i = 1, \dots, 2n-1$. By Lemma 5 these solutions form a Markov system of solutions in some neighbourhood of ∞, i. e., all Wronskians $W(z_1, \dots, z_k)$ are positive, $k = 1, \dots, 2n$, if x is sufficiently large. By Lemma 6 $z_1, \dots, z_n$ form principal system of solutions of (3.1) at ∞. Because $h(x) = c_0 + \dots + c_m x^m$ is contained in the principal system of solutions of (3.1) at ∞, there exist constants d_i, $i = 1, \dots, n$, such that $h(x) = d_1 z_1 + \dots + d_n z_n$. Now denote $k = \max\{i : i \in \{1, \dots, n\}; d_i \neq 0\}$ and replace z_k by h. This new system of solutions $z_1, \dots, z_{k-1}, h, z_{k+1}, \dots, z_{2n}$ has the same properties as $z_1, \dots, z_{2n}$. By Lemma 4 the transformation $y = h\bar{y}$ transforms (3.1) into the equation

$$(-1)^n (ph^2 \bar{y}^{(n)})^{(n)} + \dots + (P_1 \bar{y}')' + h(ph^{(n)})^{(n)} \bar{y} = 0,$$

i. e., $w = \bar{y}'$ is the solution of

$$(-1)^n (ph^2 w^{(n-1)})^{(n)} + \dots + (P_1 w)' = 0. \tag{4.6}$$

We shall show that this equation is N-disconjugate on $[x_2, x_3]$, if x_2 is sufficiently large. Put

$$w_1 = -(z_1/h)', \dots, w_{k-1} = -(z_{k-1}/h)', w_k = (z_{k+1}/h)', \dots, w_{2n-1} = (z_{2n}/h)'.$$

As $z_1, \dots, z_{k-1}, h, z_{k+1}, \dots, z_{2n}$ form the Markov system of solutions of (3.1), we have

$$w_1 = -(z_1/h)' = h^{-2} W(z_1, h) > 0,\ W(w_1, w_2) = h^{-3} W(z_1, z_2, h) > 0$$

$$W(w_1, \dots, w_i) = (-1)^i W((z_1/h)', \dots, (z_i/h)') = (-1)^i h^{-(i+1)} W(h, z_1, \dots, z_i) = (-1)^{2i} h^{-i-1} W(z_1, \dots, z_i, h) > 0 \text{ for } i < k$$

$$\begin{aligned} W(w_1, \dots, w_i) = {} & (-1)^k W((z_1/h)', \dots, (z_{k-1}/h)', (z_{k+1}/h)', \dots, (z_i/h)') = \\ & (-1)^k h^{-(i+1)} W(h, z_1, \dots, z_i) = \\ & (-1)^{2k} h^{-i-1} W(z_1, \dots, z_{k-1}, h, z_{k+1}, \dots, z_i) > 0 \\ & \text{for } k \le i < 2n. \end{aligned}$$

Hence, $w_1, \dots, w_{2n-1}$ form the Markov system of solutions of (4.6) on $[x_2, x_3]$ for x_2 sufficiently large. By Lemma 5 the equation (4.6) is N-disconjugate on $[x_2, x_3]$, i. e., every solution of (4.6) has at most $2n-2$ zeros on this interval. The function $(g/h)'$ is a solution of (4.6) and has the zeros of multiplicity $n-1$ both at x_2 and x_3. It follows that $(g/h)'$ cannot have zero on (x_2, x_3), i. e., g/h is monotonic on this interval. In a similar way the existence of a Markov system and disconjugacy of (4.6) on $[x_0, x_1]$ for x_1 sufficiently close to $-\infty$ can be proved. Hence f/h is monotonic on (x_0, x_1). Now, by the second mean value theorem of integral calculus there exists $\xi_1 \in (x_0, x_1)$ such that

$$\int_{x_0}^{x_1} q f^2 dx = \int_{x_0}^{x_1} q h^2 (f/h)^2 dx = \int_{\xi_1}^{x_1} q h^2 dx.$$

Similarly

$$\int_{x_2}^{x_3} q g^2\, dx = \int_{x_2}^{x_3} q h^2 (g/h)^2\, dx = \int_{x_2}^{\xi_2} q h^2\, dx,$$

for some $\xi_2 \in (x_2, x_3)$.

Further let us compute the integral

$$\begin{aligned} \int_{x_1}^{x_2} \left\{ p(y^{(n)})^2 + q y^2 \right\} dx = {} & \int_{x_1}^{x_2} \left\{ p[(h(1+\delta\Delta))^{(n)}]^2 + q h^2 (1+\delta\Delta)^2 \right\} dx = \\ = {} & \int_{x_1}^{x_2} \left\{ p[(h\delta\Delta)^{(n)}]^2 + q h^2 (1 + 2\delta\Delta + \delta^2\Delta^2) \right\} dx \le \\ \le {} & \delta^2 \int_{\bar{x}-\rho}^{\bar{x}+\rho} p[(h\Delta)^{(n)}]^2 dx + \int_{x_1}^{x_2} q h^2 dx + \\ & + 2\delta(-d) \int_{\bar{x}-\rho}^{\bar{x}+\rho} \Delta h^2\, dx + \delta^2 \int_{\bar{x}-\rho}^{\bar{x}+\rho} q h^2 \Delta^2\, dx < \\ < {} & \delta^2 K_1 + \int_{x_1}^{x_2} q h^2\, dx - 2\delta d K_2, \end{aligned}$$

where $K_1 = \int_{\bar{x}-\rho}^{\bar{x}+\rho} p[(h\Delta(x))^{(n)}]^2 dx$ and $K_2 = \int_{\bar{x}-\rho}^{\bar{x}+\rho} \Delta h^2 dx > 0$. Consequently, using the previous estimates,

$$
\begin{aligned}
I(y,-\infty,\infty) &= I(y,x_0,x_3) = I(y,x_0,x_1) + I(y,x_1,x_2) + I(y,x_2,x_3) \leq \\
&\leq \delta^2 K_1 - 2\delta d K_2 + \int_{x_1}^{x_2} qh^2 dx + c^T(\int_{x_0}^{x_1} U_L^{-1} B U_L^{T-1} ds)^{-1} c + \\
&+ c^T(\int_{x_2}^{x_3} U_R^{-1} B U_R^{T-1} ds)^{-1} c + \int_{\xi_1}^{x_1} qh^2 dx + \int_{x_2}^{\xi_2} qh^2 dx \\
&= \delta^2 K_1 + \int_{\xi_1}^{\xi_2} qh^2 dx - 2\delta d K_2 + \\
&+ c^T(\int_{x_0}^{x_1} U_L^{-1} B U_L^{T-1} ds)^{-1} c + c^T(\int_{x_2}^{x_3} U_R^{-1} B U_R^{T-1} ds)^{-1} c
\end{aligned}
$$

In view of (4.4) and (4.5) we can choose x_1, x_2 and x_0, x_3 in such a way that $c^T(\int_{x_0}^{x_1} U_L^{-1} B U_L^{T-1} ds)^{-1} c < \delta^2/3$, $c^T(\int_{x_2}^{x_3} U_R^{-1} B U_R^{T-1} ds)^{-1} c < \delta^2/3$ and (4.1) implies that $\int_{t_1}^{t_2} qh^2 dx < \delta^2/3$, whenever $t_1 < x_1$ and $t_2 > x_2$. Consequently,

$$
I(y,x_0,x_3) \leq \delta^2 K_1 - 2\delta d K_2 + \delta^2 = \delta(\delta(K_1+1) - 2dK_2),
$$

where $K_2 > 0$, i. e., $I(y,a,b) < 0$ if $\delta > 0$ is sufficiently small. This completes the proof. □

V. REMARKS ON RELATED PROBLEMS

By a slight modification of the proof of Theorem 4.1 we may establish the following criterion for conjugacy of the equation

$$
\sum_{j=0}^{n} (p_j(x) y^{(j)})^{(j)} + q(t) y = 0, \tag{5.1}
$$

on an arbitrary interval $I = (a,b)$, $a, b \in \mathbf{R}$. In this criterion equation (5.1) is viewed as a perturbation of (2.1).

Theorem 5.1. *Let (2.1) be disconjugate on I, $m \in \{1,\dots,n\}$ and suppose that there exist solutions $y_1,\dots,y_m$ of (3.1) which are involved at principal system of solutions both at a and b. If (2.1) is N-nonoscillatory at a and b, there exist $c_1,\dots,c_m \in \mathbf{R}$ such that the function $h(x) = c_1 y_1(x) + \cdots + c_m y_m(x)$ satisfies*

$$
\limsup_{t_1 \downarrow a, t_2 \uparrow b} \int_{t_1}^{t_2} q(x) h^2(x) dx \leq 0
$$

and $q \not\equiv 0$ on $I = (a, b)$, then (1.1) is conjugate on I.

As an example of application of the previous theorem consider the equation

$$(-1)^n y^{(2n)} + p(t) y = 0 \tag{5.1}$$

in the interval $I = (0, \infty)$. Equation (5.1) can be viewed as a perturbation of the Euler equation

$$(-1)^n y^{(2n)} - \frac{\mu_{2n}}{t^{2n}} y = 0, \tag{5.2}$$

where $\mu_{2n} = P_{2n}(\frac{2n-1}{2})$ and

$$P_{2n}(\lambda) = (-1)^n \lambda(\lambda - 1) \dots (\lambda - 2n + 1).$$

This approach has been used in [6] in order to investigate oscillation properties of (5.1).

First of all, recall basic properties of the Euler equation (5.2). The real function $y = P_{2n}(x)$ has n local minima and $(n-1)$ local maxima if n is even (n maxima and $(n-1)$ minima for n odd) and its graph is symmetric with respect to the line $x = \frac{2n-1}{2}$. The equation

$$P_{2n}(\lambda) = P_{2n}\left(\frac{2n-1}{2}\right)$$

has exactly $2n-2$ simple roots, denote them $\lambda_1 < \lambda_2 < \dots < \lambda_{n-1} < 2n-1-\lambda_{n-1} < 2n-1-\lambda_{n-2} < \dots < 2n-1-\lambda_1$, and one double root $\lambda_n = \frac{2n-1}{2}$. The solutions of (5.2) are of the form $y_i = t^{\lambda_i}$, $\quad i = 1, \dots, n-1, \quad y_n = t^{\frac{2n-1}{2}}, \quad y_{n+1} = t^{\frac{2n-1}{2}} \lg t, \quad y_{n+i+1} = t^{2n-1-\lambda_i}, \quad i = 1, \dots, n-1$. It is not difficult to verify that these solutions form the ordered system at ∞ as well as at 0 and that $y_n = t^{\frac{2n-1}{2}}$ is the only solution of (5.2) (up to a multiply by a nonzero real constant) which is contained both in principal system of solutions at $t = 0$ and $t = \infty$. Hence, as a corollary of Theorem 5.1, we have the following statement.

Theorem 5.2. *Suppose that*

$$\limsup_{t_1 \downarrow 0, t_2 \uparrow \infty} \int_{t_1}^{t_2} t^{\frac{2n-1}{2}} \left[p(t) + \frac{\mu_{2n}}{t^{2n}} \right] dt \le 0$$

and

$$p(t) + \frac{\mu_{2n}}{t^{2n}} \not\equiv 0 \quad \text{on } (0, \infty).$$

Then (5.2) is conjugate on $I = (0, \infty)$.

Remark 5.1. i) Observe that Theorem 5.1 is not applicable if one wants to investigate conjugacy of the equation (5.1) on the interval $I = (0, \infty)$, considered as a perturbation of the equation $y^{(2n)} = 0$ instead of (5.2). The reason is that in this case the dimension of the solution space of the equation $y^{(2n)} = 0$ generated by solutions which are contained both in principal systems at 0 and ∞ is zero.

ii) The test function used in the proof of Theorem 4.1 is the best possible in the sense that for the functions f, g the positive contributions of $\int_{x_0}^{x_1} p(f^{(n)})^2 dx$, and $\int_{x_2}^{x_3} p(g^{(n)})^2 dx$ are the smallest as possible, since f, g are the solutions of the equation (3.1) which is the Euler-Lagrange equation corresponding to the quadratic functional

$$I_1(y) = \int_a^b p(x)(y^{(n)}(x))^2 dx.$$

iii) The open problem is whether Theorem 5.1 holds without the assumption that (2.1) is N-nonoscillatory at a and b. This assumption was needed to estimate the integrals $\int_{x_0}^{x_1} qf^2$, $\int_{x_2}^{x^3} qg^2$ by the second mean value theorem of integral calculus (we needed the monotonicity of f/h and g/h). Observe also that this assumption may be dropped if $q(x) \le 0$ near a and b.

References

[1] Ahlbrandt, C.D., Hinton, D.B., Lewis, R.T., *The effect of variable change on oscillation and disconjugacy criteria with application to spectral theory and asymptotic theory*, J. Math. Anal. Appl. **81** (1981), 234-277.

[2] Bliss, G. A., Lectures on the Calculus of Variations, Chicago Press, Chicago 1946.

[3] Coppel, W. A., Disconjugacy, Lecture Notes in Math. No 220, Springer Verlag, Berlin-Heidelberg 1971.

[4] Došlý, O., *The existence of conjugate points for self-adjoint linear differential equations*, Proc. Roy. Soc. Edinburgh **113A** (1989), 73-85.

[5] Došlý, O., *Oscillation criteria and discreteness of the spectrum of self-adjoint, even order, differential operators*, Proc. Roy. Soc. Edinburgh **119A** (1991), 219-231.

[6] Došlý, O., Osička, J., *Kneser-type oscillation criteria for self-adjoint two-term differential equations*, Georgian Math. J. **2** (1995), 241-259.

[7] Glazman, I. M., Direct Methods of Qualitative Analysis of Singular Differential Operators, Jerusalem 1965.

[8] Hawking, S. W., Penrose, R., *The singularities of gravity collapse and cosmology*, Proc. Roy. Soc. London **A314** (1970), 529-548.

[9] Müller-Pfeiffer, E., *Existence of conjugate points for second and fourth order differential equations*, Proc. Roy. Soc. Edinburgh **89A** (1981), 281-91.

[10] Müller-Pfeiffer, E., *An oscillation theorem for self- adjoint differential equations*, Math. Nachr. **108** (1982), 79-92.

[11] Müller-Pfeiffer, E., *Nodal Domains for One- and Two- Dimensional Elliptic Differential Equations*, Z. Anal. Anwendungen **7** (1988), 135-139.

[12] Reid, W. T., Sturmian Theory for Ordinary Differential Equations, Acad. Press, New-York 1980.

[13] Tipler, F. J., *General relativity and conjugate ordinary differential equations*, J. Diff. Equations **30** (1978), 165-174.

Ondřej Došlý, Jan Komenda
Department of Mathematical Analysis
Faculty of Science
Masaryk University
Janákovo nám. 2a
662 95 Brno, CZECH REPUBLIC