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PRE-SOLID VARIETIES OF SEMIGROUPS

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Abstract. Pre-hyperidentities generalize the concept of a hyperidentity. A variety V is said to be pre-solid if every identity in V is a pre-hyperidentity. Every solid variety is pre-solid. We consider pre-solid varieties of semigroups which are not solid, determine the smallest and the largest of them, and some elements in this interval.

1. Introduction

An identity $t \approx t'$ is called a *hyperidentity* in a variety V if whenever the operation symbols occuring in t and t' are replaced by any terms of the appropriate arity, the identity which results holds in V ([14]). Hyperidentities can be defined more precisely using the concept of a *hypersubstitution* ([2]).

We fix a type $\tau = (n_i)_{i \in I}, n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i \in I}$, where f_i is n_i-ary. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet X, and let $Alg(\tau)$ be the class of all algebras of type τ. Then a mapping

$$\sigma : \{f_i | i \in I\} \to W_\tau(X)$$

which assigns to every n_i-ary operation symbol f_i an n_i-ary term will be called a hypersubstitution of type τ (for short, a hypersubstitution). For a term $t \in W_\tau(X)$ by $\hat{\sigma}[t]$ we define the application of the hypersubstitution σ to the term t. The term $\hat{\sigma}[t]$ can be defined inductively by:

(i) $\hat{\sigma}[x] := x$ for any variable x in the alphabet X, and
(ii) $\hat{\sigma}[f_i(t_1, \cdots, t_{n_i})] := \sigma(f_i)^{\mathcal{W}_\tau(X)}(\hat{\sigma}[t_1], \cdots, \hat{\sigma}[t_{n_i}])$.

It is clear that $\sigma(f_i)^{\mathcal{W}_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra $\mathcal{W}_\tau(X)$.

An identity $t \approx t'$ where t, t' are terms of type τ is a hyperidentity of type τ (for short a hyperidentity) in an algebra $\mathcal{A} \in Alg(\tau)$ if $\hat{\sigma}[t] \approx \hat{\sigma}[t']$ is an identity in $\mathcal{A}$ for every hypersubstitution σ.

An important example of a hyperidentity is the type $\tau = (2)$ medial hyperidentity

$$F(F(x, y), F(z, t)) \approx F(F(x, z), F(y, t))$$

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which is a hyperidentity in the variety of all medial semigroups (defined by the identity $xyzt \approx xzyt$ ([15])).
On the basis of a hypersubstitution in [2] a closure operator Ξ defined on classes of algebras of type τ and of sets of equations was introduced. If $t \approx t'$ is an equation then by $\Xi[t \approx t']$ we denote the set $\{\hat{\sigma}[t] \approx \hat{\sigma}[t'] | \sigma$ is a hypersubstitution of type $\tau\}$ and by $\Xi[\Sigma]$ for a set Σ of equations the union of the sets $\Xi[t \approx t']$ for $t \approx t'$ in Σ. Let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ be an algebra in $Alg(\tau)$ and let σ be a hypersubstitution. Then we make the following definitions:

$$\begin{array}{lll} \sigma[\mathcal{A}] & := & (A; (\sigma(f_i)^A)_{i \in I}), \\ \Xi[\mathcal{A}] & := & \{\sigma[\mathcal{A}] \mid \sigma \text{ is a hypersubstitution of type } \tau\}, \\ \Xi[K] & := & \bigcup\limits_{\mathcal{A} \in K} \Xi[\mathcal{A}]. \end{array}$$

A variety V of type τ is *solid* if $\Xi[V] = V$. In [2] it was shown that a variety V is solid if and only if every identity in V is a hyperidentity. Equivalently, solid varieties can be characterized as classes of algebras satisfying a given set of equations as hyperidentities (*hyperequational classes*). Although the concept of a solid variety is very strong there are infinitely many solid varieties of semigroups ([3]). All solid varieties of a given type τ form a lattice which is a sublattice of the lattice of all varieties of type τ.
Note that the clone of a solid variety V (denoted by *clone* V see [14]) is a free heterogeneous algebra with respect to the variety of heterogeneous algebras generated by *clone* V ([6]). There is a lot of equations which cannot be hyperidentities in a nontrivial algebra (or variety). For instance, substituting one of the binary projections for F in the commutative law $F(x,y) \approx F(y,x)$ we get $x \approx y$ which is satisfied only in a trivial algebra. This observation motivates to weaken the concept of a hyperidentity. The simplest way for weakeness could be to substitute only terms different from variables. The set of all term functions of an algebra which are different from projections forms a so-called pre-iterative algebra ([11]). Therefore these weaker hyperidentities are called pre-hyperidentities in [7]. At first we will set up the preliminary results and notation we will need and then we will consider the set of all pre-solid varieties of semigroups.

2. Basic Concepts

According to the ideas explained in the introduction we define a *pre-hypersubstitution* of type τ as a mapping

$$\sigma_p : \{f_i | i \in I\} \to W_\tau(X) \setminus X$$

which assigns to every operation symbol f_i an n_i-ary term which is different from a variable. (Note that we consider the first n_i variables $x_0, \cdots, x_{n_i-1}$ of the standard alphabet $X = \{x_0, \cdots, x_{n_i-1}, \cdots\}$ as n_i-ary terms. A composed term is called n_i-ary if it is built up from operation symbols of the correponding arities and variables from this alphabet X.)

The extension $\hat{\sigma}_p[t]$ of a pre-hypersubstitution to a term t is defined inductively by

(i) $\hat{\sigma}_p[x] := x$ for any variable x in the alphabet X, and
(ii) $\hat{\sigma}_p[f_i(t_1, \cdots, t_{n_i})] := \sigma_p(f_i)^{\mathcal{W}_\tau(X)}(\hat{\sigma}_p[t_1], \cdots, \hat{\sigma}_p[t_{n_i}])$.

The expression $\sigma_p(f_i)^{\mathcal{W}_\tau(X)}$ on the right hand side of (ii) is the operation induced by $\sigma(f_i)$ on the term algebra $\mathcal{W}_\tau(X)$.
If $t \approx t'$ is an equation, then we denote by $\Xi_p[t \approx t']$ the set

$$\{\hat{\sigma}_p[t] \approx \hat{\sigma}_p[t'] \mid \sigma_p \text{ is a pre-hypersubstitution }\}$$

If Σ is a set of equations, we use $\Xi_p[\Sigma]$ for the union of the sets $\Xi_p[t \approx t']$, for $t \approx t'$ in Σ.

Let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ be an algebra in $Alg(\tau)$, and let K be a class of algebras of type τ. Then we define:

$$\begin{array}{lcl} \sigma_p[\mathcal{A}] & := & (A; (\sigma_p(f_i)^A)_{i \in I}), \\ \Xi_p[\mathcal{A}] & := & \{\sigma_p[\mathcal{A}] \mid \sigma_p \text{ is a pre-hypersubstitution of type } \tau\}, \\ \Xi_p[K] & := & \bigcup\limits_{\mathcal{A} \in K} \Xi_p[\mathcal{A}]. \end{array}$$

In [7] we proved the following proposition:

Result 2.1. *Ξ_p is a closure operator on sets of equations Σ and on classes of algebras K of type τ, i.e.*

(i) $\Sigma \subseteq \Xi_p[\Sigma]$,
(ii) $\Sigma' \subseteq \Sigma \Rightarrow \Xi_p[\Sigma'] \subseteq \Xi_p[\Sigma]$,
(iii) $\Xi_p[\Xi_p[\Sigma]] = \Xi_p[\Sigma]$,
(i') $K \subseteq \Xi_p[K]$,
(ii') $K' \subseteq K \Rightarrow \Xi_p[K'] \subseteq \Xi_p[K]$,
(iii') $\Xi_p[\Xi_p[K]] = \Xi_p[K]$. □

Since every pre-hypersubstitution is a hypersubstitution we have

Result 2.2. *Let K be a class of algebras of type τ and let Σ be a set of equations of type τ. Then*

(i) $\Xi_p[\Sigma] \subseteq \Xi[\Sigma]$ *and*
(ii) $\Xi_p[K] \subseteq \Xi[K]$. □

Using the concept of a pre-hypersubstitution we define pre-hyperidentities in the following way:

Definition 2.3. Let $\mathcal{A} \in Alg(\tau)$ be an algebra of type τ. Then the identity $t \approx t'$, where t, t' are terms of type τ is a pre-hyperidentity of type τ in $\mathcal{A}$ ($\mathcal{A}$ pre-hypersatisfies $t \approx t'$) if $\hat{\sigma}_p[t] \approx \hat{\sigma}_p[t']$ is an identity of $\mathcal{A}$ for every pre-hypersubstitution σ_p.

Clearly, every hyperidentity of type τ is a pre-hyperidentity of this type. In general, the converse is false.

For a class K of algebras of type τ and for a set Σ of equations of this type we fix the following notations:
IdK - the class of all identities of K,
$HIdK$- the class of all hyperidentities of K,
H_pIdK- the class of all pre-hyperidentities of K,
$Mod\Sigma = \{\mathcal{A} \in Alg(\tau) | \mathcal{A}$ satisfies $\Sigma\}$ - the variety defined by Σ
$HMod\Sigma = \{\mathcal{A} \in Alg(\tau) | \mathcal{A}$ hypersatisfies $\Sigma\}$ - the hyperequational class defined by Σ,
$H_pMod\Sigma = \{\mathcal{A} \in Alg(\tau) | \mathcal{A}$ pre-hypersatisfies $\Sigma\}$ - the pre-hyperequational class defined by Σ,
$VarK = ModIdK$ - the variety generated by K,
$HVarK = HModHIdK = \{\mathcal{A} \in Alg(\tau) | \mathcal{A}$ hypersatisfies $HIdK\}$ - the hypervariety of type τ generated by K.
For these sets we get the following inclusions:

$$HIdK \subseteq H_pIdK, \;\; HMod\Sigma \subseteq H_pMod\Sigma.$$

By definition every hyperidentity or every pre-hyperidentity is an identity. Very natural there arises the problem to find algebras or varieties for which every identity is a hyperidentity or such that every identity is a pre-hyperidentity.

Definition 2.4. Let V be a variety of type τ. Then V is called pre-solid if $\Xi_p[V] = V$.

In [7] pre-solid varieties were characterized in the following manner:

Result 2.5. ([7]) *Let $K \subseteq Alg(\tau)$ be a variety. Then the following conditions are equivalent:*

(i) *K is a pre-hyperequational class,*
(ii) *K is pre-solid,*
(iii) *$IdK \subseteq H_pIdK$, i.e. every identity of K is a pre-hyperidentity,*
(iv) *$\Xi_p[IdK] = IdK$, i.e. IdK is closed under pre-hypersubstitutions.* □

For a given type τ by $\mathcal{L}(\tau)$ we denote the lattice of all varieties of this type and by $S(\tau)$ the set of all solid varieties of this type. $S_p(\tau)$ is the set of all pre-solid varieties of type τ. Then we have the following results:

Result 2.6. ([10], [13])

(i) *The set $S(\tau)$ forms a sublattice of $\mathcal{L}(\tau)$,*
(ii] *The set $S_p(\tau)$ forms a sublattice of $\mathcal{L}(\tau)$ containing $S(\tau)$ as a sublattice.*
(iii) *If τ is a finite type then the lattice $\mathcal{S}(\tau)$ is atomic. The unique atom is the variety RA_τ of all rectangular algebras of type τ. (RA_τ is the variety generated by all algebras of type τ whose fundamental operations are projections).* □

3. Solid and Pre-solid Varieties of Semigroups

By $\mathcal{L}(S)$ we denote the lattice of all semigroup varieties. To describe a bit more of the structure of all pre-solid varieties of semigroups we note that a variety of semigroups to be solid it must satisfy the associative law as a hyperidentity. That means, the greatest solid variety of semigroups is the hypermodel class of the associative law: $HMod\{F(F(x,y),z) \approx F(x,F(y,z))\}$. In [1] we obtained an equational basis for this variety:
$HMod\{F(F(x,y),z) \approx F(x,F(y,z))\} = V_{HS}$ with $V_{HS} = Mod\{I_1 \cup I_2 \cup \{x^2 \approx x^4\}\}$ where I_1 and I_2 are the following sets of identities:

$$I_1 := \{(x^{k_1}y^{k_2}\cdots x^{k_{n-1}}y^{k_n})^{k_1}z^{k_2}\cdots(x^{k_1}y^{k_2}\cdots x^{k_{n-1}}y^{k_n})^{k_{n-1}}z^{k_n}$$
$$\approx x^{k_1}(y^{k_1}z^{k_2}\cdots y^{k_{n-1}}z^{k_n})^{k_2}\cdots x^{k_{n-1}}(y^{k_1}z^{k_2}\cdots y^{k_{n-1}}z^{k_n})^{k_n} | n \in \{2,4,6\}$$
$$for\ 1 \leq k_1, \cdots, k_n \leq 3\}.$$

$$I_2 := \{(x^{k_1}(y^{k_1}z^{k_2}y^{k_3}\cdots z^{k_{n-1}}y^{k_n})^{k_2}\cdots(y^{k_1}z^{k_2}y^{k_3}\cdots z^{k_{n-1}}y^{k_n})^{k_{n-1}}x^{k_n}$$
$$\approx (x^{k_1}y^{k_2}x^{k_3}\cdots y^{k_{n-1}}x^{k_n})^{k_1}z^{k_2}\cdots(x^{k_1}y^{k_2}x^{k_3}\cdots y^{k_{n-1}}x^{k_n})^{k_n} | n \in \{3,5\}$$
$$for\ 1 \leq k_1, \cdots, k_n \leq 3\}.$$

Note that L. Polák recently proved that $V_{HS} = Mod\{x(yz) \approx (xy)z, xyxzxyx \approx xyzyx, x^2 \approx x^4, xy^2z^2 \approx xyz^2yz^2, x^2y^2z \approx x^2yx^2yz\}$.

By $S(V_{HS})$ we denote the lattice of all solid semigroup varieties (indeed, the set of all solid semigroup varieties forms a sublattice of $\mathcal{L}(S)$ since $S(V_{HS})$ is the intersection of the subvariety lattice of the variety V_{HS} and the lattice of all solid varieties of type $\tau = (2)$.)
According to Result 2.6 every non trivial solid semigroup variety contains the variety RB of all rectangular bands which is defined by the identities $x(yz) \approx (xy)z,\ x^2 \approx x,\ xyz \approx xz$. Then we have:

Result 3.1. ([5]) *The variety RB is the least nontrivial element of $S(V_{HS})$.* □

We call an equation $t \approx t'$ to be left-most (right-most) if the left-most (right-most) variables in t and t' are the same. An equation $t \approx t'$ will be called outer-most if it is left-most and right-most. These notions were used by E. Graczyńska in [9]. Let $Out(2)$ be the set of all outer-most equations of type $\tau = (2)$. Clearly, $IdRB = Out(2)$. As a consequence, a hyperidentity must be an outer-most equation. The following useful fact is obvious:

Proposition 3.2. *Let V be a nontrivial variety of semigroups and let $t \approx t'$ be an outer-most equation of type $\tau = (2)$. Then $t \approx t'$ is a hyperidentity in V if and only if $t \approx t'$ is a pre-hyperidentity in V.*

Proof. Clearly, if $t \approx t'$ is a hyperidentity satisfied in V then it is a pre-hyperidentity in V. Let $t \approx t'$ be a pre-hyperidentity satisfied in V. Then for every pre-hypersubstitution σ_p we have $\sigma_p[t] \approx \sigma_p[t'] \in IdV$. A hypersubstitution which is no pre-hypersubstitution assigns to the binary fundamental operation symbol a binary projection. Since $t \approx t'$ is outer-most the equation $\sigma[t] \approx \sigma[t']$ is equal to $x \approx x$ or to $y \approx y$ which is an identity in V. Altogether, for every hypersubstitution σ we have $\sigma[t] \approx \sigma[t'] \in IdV$ and $t \approx t'$ is a hyperidentity in V. □

Clearly, the associative law is an outermost equation and applying Proposition 3.2 we have:

Corollary 3.3. *The variety V_{HS} is pre-solid and for any pre-solid variety V of semigroups, $V \subseteq V_{HS}$.*

Proof. As a solid variety V_{HS} is pre-solid. Since V_{HS} is the hyperequational class generated by the associative law it is also the pre-hyperequational class generated by the associative law and thus the greatest pre-solid variety of semigroups. □

Let $S_p(V_{HS})$ be the set of all pre-solid semigroup varieties. We want to discuss the following question:
Are there pre-solid semigroup varieties in the interval between RB and V_{HS} which are not solid?
The answer is given in [7], namely

Lemma 3.4. ([7]) *Let V be a variety of type $\tau = (2)$ such that $RB \subseteq V$. Then V is solid iff V is pre-solid.*

Proof. If V is solid then V is also pre-solid. Let V be pre-solid. The inclusion $RB \subseteq V$ means that every identity in V is outer-most. Therefore, by Proposition 3.2 every pre-hyperidentity is a hyperidentity and since every identity is a pre-hyperidentity V must be solid. □

Lemma 3.4 shows that a pre-solid variety of semigroups which is not solid must be outside of the interval between RB and V_{HS}. Now we ask for the greatest pre-solid semigroup variety which is not solid.

Proposition 3.5. *The variety $V_{PS} := Mod\{(xy)z \approx x(yz), xyxzxyx \approx xyzyx, x^2 \approx y^2, x^3 \approx y^3\}$ is pre-solid, but not solid.*

Proof. Since the identity $x^2 \approx y^2$ is no outer-most equation and since V_{PS} is nontrivial the variety V_{PS} is not solid. We are going to show that V_{PS} is the pre-hyperequational class defined by the associative law, by $F(x,x) \approx F(y,y)$, and by $F(x, F(x,x)) \approx F(y, F(y,y))$. Now, $V_{PS} \subseteq Mod\{I_1 \cup I_2 \cup \{x^2 \approx x^4\}\}$ since $I_1 \cup I_2 \cup \{x^2 \approx x^4\} \subseteq \{(xy)z \approx x(yz), xyxzxyx \approx xyzyx\} \cup \{uv^2w \approx u'v'^2w' | u, v, w, u', v', w'$ are binary terms $\} \subseteq IdMod\{(xy)z \approx x(yz), xyxzxyx \approx xyzyx, x^2 \approx y^2, x^3 \approx y^3\}$ because of $uv^2w \approx u^3w \approx w^4 \approx u'^3w' \approx u'v'^2w'$. Since $Mod\{I_1 \cup I_2 \cup \{x^2 \approx x^4\}\} = H_pMod\{F(x, F(y,z)) \approx F(F(x,y),z)\}$ the associative law is a pre-hyperidentity in V_{PS}. We have to check that $F(x,x) \approx F(y,y)$ and $F(F(x,x),x) \approx F(F(y,y),y)$ are pre-hyperidentities in V_{PS}. For every binary term different from a variable there are natural numbers $r, s > 1$ with $t(x,x) = x^r$, $t(y,y) = y^r$, respectively with $t(t(x,x),x) = x^s$ and $t(t(y,y),y) = y^s$. Because of

$x^2 \approx y^2, x^3 \approx y^3 \in IdV_{PS}$ we get $t(x,x) \approx t(y,y), t(t(x,x),x) \approx t(t(y,y),y) \in IdV_{PS}$. (Note that $t(x,y)$ means, the term is constructed only from the variables x and y.) Altogether, $V_{PS} \subseteq H_pMod\{F(F(x,y),z) \approx F(x,F(y,z)),\ F(x,x) \approx F(y,y),\ F(F(x,x),x) \approx F(F(y,y),y)\}$. On the other hand we see that the identities $x(yz) \approx (xy)z, xyxzxyx \approx xyzyx, x^2 \approx y^2, x^3 \approx y^3$ are satisfied in $H_pMod\{F(F(x,y),z) \approx F(x,F(y,z)),\ F(x,x) \approx F(y,y),\ F(F(x,x),x) \approx F(F(y,y),y)\}$. This shows the equality $V_{PS} = H_pMod\{F(F(x,y),z) \approx F(x,F(y,z)),\ F(x,x) \approx F(y,y),\ F(F(x,x),x) \approx F(F(y,y),y)\}$ and by Result 2.5 the variety V_{PS} is pre-solid. □

Theorem 3.6. *For every nontrivial pre-solid variety V of semigroups the following propositions are equivalent:*

(i) $V \subseteq V_{PS}$,
(ii) V *is not solid.*

Proof. (i) $\Rightarrow$ (ii): Since the equation $F(x,x) \approx F(y,y)$ is no hyperidentity in V the variety V is not solid.
(ii) $\Rightarrow$ (i): Since V is pre-solid and not solid by Lemma 3.4 the variety of rectangular bands is not included in V. Then there is an identity in IdV which is not outer-most. We conclude that there are natural numbers m, n and variables $u_0, \ldots, u_m, v_0, \ldots v_n \in X$ such that $u_0 \neq v_0$ or $u_m \neq v_n$ and $u_0 \ldots u_m \approx v_0 \ldots v_n \in IdV$. Without restriction of the generality assume that $u_0 \neq v_0$. Substituting $t(x,y) = x^2$ and $t(x,y) = x^3$ we obtain $u_0^2 \approx v_0^2 \in IdV$, respectively $u_0^3 \approx v_0^3 \in IdV$ and thus $V \subseteq Mod\{x^2 \approx y^2,\ x^3 \approx y^3\}$. Since V is pre-solid by Corollary 3.3, $V \subseteq V_{HS}$. Altogether we have $V \subseteq V_{PS}$. □

Theorem 3.6 shows that V_{PS} is the greatest pre-solid variety of semigroups which is not solid. It is very natural to ask for the least pre-solid variety of semigroups which is not solid. Let $Z = Mod\{x(yz) \approx (xy)z,\ xy \approx zt\}$ be the variety of all zero-semigroups. It is easy to see that Z is pre-solid but not solid.

Theorem 3.7. *Let V be a pre-solid variety of semigroups. If V is not solid then $Z \subseteq V$.*

Proof. Since V is not solid V cannot be trivial. Any nontrivial variety V of semigroups must include at least one of the atoms of the lattice of all semigroup varieties (listed for example in [8]). Since V is pre-solid by Corollary 3.3 we have $V \subseteq V_{HS}$ and every atom included in V must be hyperassociative. In [1] we determined all hyperassociative atoms in the lattice of all semigroup varieties: zero-semigroups ($xy \approx zt$), right semigroups ($xy \approx x$), left semigroups ($xy \approx y$), semilattices ($xy \approx yx,\ x^2 \approx x$), and 2-groups ($xy \approx yx, x^2y \approx y$). Since by Theorem 3.6 $V \subseteq V_{PS}$ the identities $x^2 \approx y^2$ and $x^3 \approx y^3$ must be satisfied in an atom included in V. These identities are satisfied only in the variety Z of all zero-semigroups, thus $Z \subseteq V$. □

Considering the set $S_p(V_{HS})$ we get:

Proposition 3.8. *The set $S_p(V_{HS}) \setminus S(V_{HS})$ forms a sublattice of $\mathcal{S}_p(V_{HS})$.*

Proof. Consider two varieties $V_1, V_2 \in S_p(V_{HS})\backslash S(V_{HS})$. Then $V_1 \vee V_2$ and $V_1 \wedge V_2$ are pre-solid by Result 2.6. Because of $V_1 \vee V_2, V_1 \wedge V_2 \subseteq V_{PS}$ the equation $x^2 \approx y^2$ is satisfied in $V_1 \vee V_2$ and in$V_1 \wedge V_2$. The equation $x^2 \approx y^2$ cannot be a hyperidentity in these varieties. This shows that both varieties are not solid. □

A solid variety V must contain the variety RB. Therefore the join of a solid variety V and the pre-solid variety Z can be written as $V \vee Z = V \vee (RB \vee Z)$. It is well-known that $RB \vee Z = Mod\{xyz \approx xz\}$ is solid ([9], [15], [3]) and that the join of two solid varieties is solid (Result 2.6). This example motivates the following question: Is the join of a solid and an arbitrary pre-solid variety solid ? To attack this question we use Theorem 2.7 and obtain:

Proposition 3.9. *Let V_1 be a nontrivial solid and let V_2 be a a pre-solid variety of semigroups. Then the variety $V_1 \vee V_2$ is solid.*

Proof. By Result 2.6 the variety $V_1 \vee V_2$ is pre-solid and because of $RB \subseteq V_1 \subseteq V_1 \vee V_2$ and Lemma 3.4 solid. □

Remark that the varieties RB and V_{PS} have certain "splitting " properties for the lattice $\mathcal{S}_p(V_{HS})$. Indeed, if the variety V is nontrivial, pre-solid and contains RB then V is solid. If the variety RB is not contained in V then V belongs to $S_p(V_{HS})\backslash S(V_{HS})$. If V is nontrivial, pre-solid and contained in V_{PS} then V belongs to $S_p(V_{HS}) \setminus S(V_{HS})$. If V is not contained in V_{PS} then V is solid.

4. Applications

Proposition 3.8 is useful to derive new solid varieties from given ones. We will consider a special case of this theorem.
An identity $t \approx t'$ is called normal (see e.g.[9]) if t and t' are the same variable or neither t nor t' are variables. By $N(\tau)$ we denote the set of all normal identities of type τ. For a variety V of type τ we set $N(V) := N(\tau) \cap IdV$. The variety $V_N := ModN(V)$ is called the normalization of V. It is easy to see that IdZ is the set of all normal identities of type $\tau = (2)$. Therefore, for a variety V of semigroups $V \vee Z$ is the normalization of V and we have:

Corollary 4.1. *The normalization of a pre-solid variety of semigroups is pre-solid.*

Proof. Let V be a pre-solid but not solid variety of semigroups, then by Theorem 3.7, $V_N = V = V \vee Z$. If V is solid then by Theorem 3.8 V_N is solid and thus pre-solid. □

(For solid varieties Corollary 4.1 was shown in [4]).

One could ask whether the solidity of $V \vee Z$ implies the solidity of V.

To formulate the following Theorem we need all solid varieties of bands. There are exactly the following solid varieties of bands ([15]):

1. $RB = Mod\{x(yz) \approx (xy)z \approx xz,\ x^2 \approx x\}$ – rectangular bands,
2. $RegB = Mod\{x(yz) \approx (xy)z,\ x^2 \approx x,\ xyxzx \approx xyzx\}$ – regular bands,
3. $NB = Mod\{x(yz) \approx (xy)z,\ x^2 \approx x,\ xyzt \approx xzyt\}$ – normal bands.

Theorem 4.2. *Let V be a nontrivial variety of semigroups. Then the following is equivalent:*

(i) *V is solid,*
(ii) *$V \vee Z$ is solid and for V there holds $V \supseteq Z$* or *$V \in \{RB, NB, RegB\}$*

Proof. Let V be solid, then $V \vee Z$ is solid by Corollary 4.1. Assume that $x^2 \approx x \notin IdV$. We will show that IdV consists only of normal identities. At first we check identities of the form $x^m \approx x$ with $m > 2$. Let $k > 2$ be the least natural number with $x^k \approx x \in IdV$. From $F(x^{k-1}, x) \approx x$ we obtain by the hypersubstitution $\sigma : F \mapsto x$ the identity $x^{k-1} \approx x \in IdV$ in contradiction of the minimality of k. Assume there is an identity $u \approx v \in IdV$ which is not normal and so that the sets of variables in u and in v are different. Each of the terms u and v contains at most two variables since otherwise $x^m \approx x \in IdV$ with $m \geq 3$. Therefore $u \approx v$ is one of the following identities: $yz \approx x$, or $xz \approx x$, or $zx \approx x$. These identities cannot be hyperidentities in V in contradiction to the solidity of V. This shows $Z \subseteq V$. Assume now that $x^2 \approx x \in IdV$, that is, V is a variety of bands. Then $V \in \{RB, RegB, NB\}$.
If $Z \subseteq V$ then from the solidity of $V \vee Z$ follows the solidity of $V = V \vee Z$. The solidity of $RB, RegB$, and NB is clear. □

5. Examples for Pre-solid Varieties of Semigroups

In the introduction we remarked that there is no nontrivial solid variety of commutative semigroups. In [5] we determined all pre-solid varieties of commutative semigroups. Clearly, the pre-hyperequational class defined by the associative and the commutative law must be the greatest pre-solid variety of commutative semigroups (Result 2.5). In [5] we determined an equational basis for this variety. There holds:
$H_pMod\{F(F(x,y),z) \approx F(x,F(y,z)),\ F(x,y) \approx F(y,x)\} = Mod\{(xy)z \approx x(yz),\ xy \approx yx,\ xy^2 \approx x^2y,\ x^2 \approx y^2\} =: V_{PC}$.
We fix the following denotations:
$p_n : x_0x_1 \dots x_n \approx y_0y_1 \dots y_n$,
$I_n = \{(xy)z \approx x(yz),\ xy \approx yx,\ xy^2 \approx x^2y, x^2 \approx y^2,\ p_n\}$
$P_n = ModI_n$ for every natural number n.
Then we have:

Result 5.1. ([5]) *The varieties $P_n, n \in N$ and V_{PC} are all pre-solid varieties of commutative semigroups. They form a lattice, namely the subvariety lattice of V_{PC} (which is a chain).* □

It is a well-known fact ([15]) that in every variety of medial semigroups the medial law is a hyperidentity. The greatest solid variety of medial semigroups is the hyperequational class defined by the medial and the associative law, $HMod\{F(F(x,y),F(z,t)) \approx F(F(x,z),F(y,t)),\ F(F(x,y),z) \approx F(x,F(y,z))\} = Mod\{x(yz) \approx (xy)z,\ xyzt \approx xzyt,\ x^2 \approx x^4,\ x^3y^2zx \approx xy^2zx\} = HM$. This variety is pre-solid. We ask for pre-solid varieties of medial semigroups which are not solid.

Lemma 5.2. *The variety* $V_{MPS} := Mod\{(xy)z \approx x(yz),\ xyzt \approx xzyt,\ x^2 \approx y^2,\ x^3 \approx y^3\}$ *is pre-solid.*

Proof. We show that V_{MPS} is a pre-hyperequational class. The inclusion $V_{MPS} \subseteq HM$ implies that V_{MPS} satisfies the associative and the medial law as hyperidentities. These equations are outer-most and therefore pre-hyperidentities in V_{MPS}. Further, $V_{MPS} \subseteq V_{PS}$ since V_{PS} is the greatest pre-solid variety of semigroups which is not solid. Since the equations $F(x,x) \approx F(y,y)$ and $F(F(x,x),x) \approx F(F(y,y),y)$ are pre-hyperidentities in V_{PS} they are pre-hyperidentities in $V_{MPS} \subseteq V_{PS}$. This shows: $V_{MPS} \subseteq H_pMod\{F(F(x,y),z) \approx F(x,F(y,z)),$
$F(F(x,y),F(z,t)) \approx F(F(x,z),F(y,t)),\ F(x,x) \approx F(y,y),\ F(F(x,x),x) \approx F(F(y,y),y)\}$. On the other hand, it is easy to show that this pre-hyperequational class is included in V_{MPS}. □

Theorem 5.3. *For every pre-solid variety* V *of semigroups the following is equivalent:*

(i) V *is medial but not solid,*
(ii) $V \subseteq V_{MPS}$.

Proof. (i) $\Rightarrow$ (ii): Since V is pre-solid the associative law is a pre-hyperidentity in V and hence it is a hyperidentity in V. The variety V is medial and therefore, $V \subseteq HM = Mod\{x(yz) \approx (xy)z,\ xyzt \approx xzyt,\ x^2 \approx x^4,\ x^3y^2zx \approx xy^2zx\}$. Further, we have $V \subseteq V_{PS}$ since V is pre-solid but not solid (Theorem 3.6). Consequently, $V \subseteq HM \cap V_{PS} \subseteq Mod\{x(yz) \approx (xy)z,\ xyzt \approx xzyt,\ x^2 \approx y^2,\ x^3 \approx y^3\}$.
(ii) $\Rightarrow$ (i): Obviously V is medial and $V \subseteq V_{MPS} \subseteq V_{PS}$. Then by Theorem 3.6 V is not solid. □

Theorem 5.3 means that V_{MPS} is the greatest pre-solid variety of medial semigroups which is not solid.

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