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INDUCED ISOMORPHISMS OF CERTAIN TERNARY SEMIGROUPS

ANTONI CHRONOWSKI AND MIROSLAV NOVOTNÝ

Abstract. If $\mathbf{X}_1$, $\mathbf{Y}_1$ are relational structures of the same type, then the set of all ordered pairs (p, q) constitutes a ternary semigroup with a naturally defined operation where p denotes a homomorphism of $\mathbf{X}_1$ into $\mathbf{Y}_1$ and q is a homomorphism of $\mathbf{Y}_1$ into $\mathbf{X}_1$. If f_1 is an isomorphism of $\mathbf{X}_1$ onto a relational structure $\mathbf{X}_2$ and f_2 an isomorphism of $\mathbf{Y}_1$ onto a relational structure $\mathbf{Y}_2$, then the ordered pair (f_1, f_2) of isomorphisms defines an isomorphism of the ternary semigroup defined on the basis of $\mathbf{X}_1$ and $\mathbf{Y}_1$ onto the ternary semigroup defined on the basis of $\mathbf{X}_2$ and $\mathbf{Y}_2$; this isomorphism is said to be induced. We prove that there exist isomorphisms of ternary semigroups defined by pairs of relational structures that are not induced and formulate a criterion recognizing induced isomorphisms.

1. Introduction

Ternary semigroups provide natural examples of ternary algebras. In the present paper, we study ternary semigroups constructed on the basis of two relational structures of the same type. The carrier of the ternary semigroup is formed of all ordered pairs of homomorphisms where the first member of the pair is a homomorphism of the first structure into the second and the second member is a homomorphism of the second structure into the first. The ternary operation on the set of these pairs of homomorphisms is defined in a natural way using the composition of homomorphisms.

If $\mathbf{X}_1$, $\mathbf{X}_2$ are isomorphic relational structures and $\mathbf{Y}_1$, $\mathbf{Y}_2$ are isomorphic as well where we suppose that all structures are of the same type, then the ternary semigroup of homomorphisms formed on the basis of $\mathbf{X}_1$ and $\mathbf{Y}_1$ is isomomorphic

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to the ternary semigroup of homomorphisms formed on the basis of $\mathbf{X}_2$ and $\mathbf{Y}_2$. Our main problem consists in characterizing such isomorphisms that are called induced. This problem seems to be natural because there exist some relational structures $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ of the same type such that the ternary semigroup of homomorphisms formed on the basis of $\mathbf{X}_1$ and $\mathbf{Y}_1$ is isomorphic to the ternary semigroup of homomorphisms formed on the basis of $\mathbf{X}_2$ and $\mathbf{Y}_2$ while the corresponding isomorphism is not induced in the above mentioned sense.

We now present the details of our considerations.

2. DECOMPOSABLE MAPPINGS

Let X_1, X_2, Y_1, Y_2 be sets, f a mapping of the set $X_1 \times Y_1$ into the set $X_2 \times Y_2$. Suppose that there exists a mapping f_1 of X_1 into X_2 and a mapping f_2 of Y_1 into Y_2 such that $f(x_1, y_1) = (f_1(x_1), f_2(y_1))$ holds for any $(x_1, y_1) \in X_1 \times Y_1$. Then the mapping f is said to be *decomposable*; the mappings f_1, f_2 are called *components* of f. We write $f = f_1 \times f_2$. The reader must be warned that the symbol $\times$ does not mean a Cartesian product in this formula; we identify $((x_1, y_1), (x_2, y_2))$ with $((x_1, x_2), (y_1, y_2))$ where $((x_1, y_1), (x_2, y_2)) \in f$, $(x_1, x_2) \in f_1$, $(y_1, y_2) \in f_2$ and, hence, $((x_1, x_2), (y_1, y_2)) \in f_1 \times f_2$.

This is a slight generalization of the definition appearing in [6].

We see that the decomposability of f depends on the fixed decompositions of $X_1 \times Y_1$ and $X_2 \times Y_2$ into factors X_1, Y_1 and X_2, Y_2, respectively. If these factors are given, the components f_1, f_2 of f are defined in a unique way.

2.1. Lemma. *Let X_1, X_2, Y_1, Y_2 be sets, f a mapping of the set $X_1 \times Y_1$ into the set $X_2 \times Y_2$. If $f_1 \times f_2 = f = f_1' \times f_2'$, then $f_1 = f_1'$, $f_2 = f_2'$.*

Proof. If $(x, y) \in X_1 \times Y_1$ is arbitrary, then $(f_1(x), f_2(y)) = f(x, y) = (f_1'(x), f_2'(y))$ which implies $f_1(x) = f_1'(x)$, $f_2(y) = f_2'(y)$. □

The following result enables to recognize decomposable mappings.

2.2. Theorem. *Let X_1, X_2, Y_1, Y_2 be sets, f a mapping of the set $X_1 \times Y_1$ into $X_2 \times Y_2$. Then the following assertions are equivalent.*

(i) *The mapping f is decomposable.*

(ii) *For any $x_1 \in X_1$, $x_1' \in X_1$, $y_1 \in Y_1$, $y_1' \in Y_1$ there exist elements $x_2 \in X_2$, $x_2' \in X_2$, $y_2 \in Y_2$, $y_2' \in Y_2$ such that $f(x_1, y_1) = (x_2, y_2)$, $f(x_1, y_1') = (x_2, y_2')$, $f(x_1', y_1) = (x_2', y_2)$.*

Proof. If (i) holds and $x_1 \in X_1$, $x_1' \in X_1$, $y_1 \in Y_1$, $y_1' \in Y_1$ are arbitrary, we put $x_2 = f_1(x_1)$, $x_2' = f_1(x_1')$, $y_2 = f_2(y_1)$, $y_2' = f_2(y_1')$. Then $f(x_1, y_1) = (f_1(x_1), f_2(y_1)) = (x_2, y_2)$, $f(x_1, y_1') = (f_1(x_1), f_2(y_1')) = (x_2, y_2')$, $f(x_1', y_1) = (f_1(x_1'), f_2(y_1)) = (x_2', y_2)$. Thus, (ii) holds.

Let (ii) hold. Suppose that $x_1 \in X_1$, $y_1 \in Y_1$ are fixed elements. For any $x_1' \in X_1$ there exists exactly one $x_2' \in X_2$ such that $f(x_1', y_1) = (x_2', y_2)$ where $y_2 \in Y_2$. Thus, there exists a mapping f_1 of X_1 into X_2 such that $f(x_1', y_1) = (f_1(x_1'), y_2)$

for some $y_2 \in Y_2$. Similarly, there exists a mapping f_2 of Y_1 into Y_2 such that $f(x_1, y_1') = (x_2, f_2(y_1'))$ for some $x_2 \in X_2$.

By (ii) for $x_1' \in X_1$, $x_1 \in X_1$, $y_1' \in Y_1$, $y_1 \in Y_1$ there exist elements $u_2' \in X_2$, $u_2 \in X_2$, $v_2' \in Y_2$, $v_2 \in Y_2$ such that $f(x_1', y_1') = (u_2', v_2')$, $f(x_1', y_1) = (u_2', v_2)$, $f(x_1, y_1') = (u_2, v_2')$. We have obtained $u_2' = f_1(x_1')$, $v_2 = y_2$, $u_2 = x_2$, $v_2' = f_2(y_1')$. It follows that $f(x_1', y_1') = (f_1(x_1'), f_2(y_1'))$. Thus, (i) holds. □

2.3. Remark. Let X_1, X_2, Y_1, Y_2 be sets. It is easy to see that a bijection f of $X_1 \times Y_1$ onto $X_2 \times Y_2$ is decomposable if and only if there exists a bijection f_1 of X_1 onto X_2 and a bijection f_2 of Y_1 onto Y_2 such that $f = f_1 \times f_2$. □

3. TERNARY SEMIGROUPS

The fundamental notions of the theory of universal algebras can be easily found, e.g., in [3], Chapter 1.

If X is a set and $n \geq 1$ an integer, we write X^n for $X \times \cdots \times X$ where X appears n times.

A *ternary semigroup* (cf. [4], [7], [1], [2]) is an algebraic structure (A, f) such that A is a nonempty set and $f : A^3 \to A$ is a ternary operation satisfying the associative law:

$$f(f(x_1, x_2, x_3), x_4, x_5) = f(x_1, f(x_2, x_3, x_4), x_5) = f(x_1, x_2, f(x_3, x_4, x_5))$$

for any $x_1, \ldots, x_5$ in A.

Let $M \subseteq A$ be a closed subset of (A, f), i.e., a subset such that for any x_1, x_2, x_3 in M the condition $f(x_1, x_2, x_3) \in M$ holds. Then $f \cap (M^3 \times M)$ is a ternary operation on the set M; it is said to be the *restriction* of f to M.

3.1. Example. Let A be a nonempty set. For any $(x_1, x_2, x_3) \in A^3$ put $f(x_1, x_2, x_3) = x_1$. Then (A, f) is a ternary semigroup; an operation f defined in this way is said to be *trivial*.

If X, Y are nonempty sets, define $o((x_1, y_1), (x_2, y_2), (x_3, y_3)) = (x_1, y_1)$ for any (x_1, y_1), (x_2, y_2), (x_3, y_3) in $X \times Y$. Then $(X \times Y, o)$ is a ternary semigroup with a trivial operation. □

3.2. Lemma. *Let (A, f), (A', f') be ternary semigroups with trivial operations. Then the following assertions hold.*

(i) *Any mapping of A into A' is a homomorphism of (A, f) into (A', f').*
(ii) *Any bijection of A onto A' is an isomorphism of (A, f) onto (A', f').* □

Let (A, f) be a ternary semigroup. An element $x_0 \in A$ is said to be a *left zero* of (A, f) (cf. [1]) if $f(x_0, x_1, x_2) = x_0$ for any elements x_1, x_2 in A.

3.3. Lemma. *Let (A, f) be a ternary semigroup. Then the following assertions are equivalent.*

(i) *The operation f is trivial.*
(ii) *Any element in A is a left zero of (A, f).*

This is an immediate consequence of the definitions. □

3.4. Lemma. *Let (A, f) be a ternary semigroup, (M, f') its ternary subsemigroup, and $x_0 \in M$ an element. If x_0 is a left zero of (A, f), then it is a left zero of (M, f').*

This follows directly from the definition of a left zero. □

Let X, Y be nonempty sets. We denote by $T(X, Y)$ the set of all mappings of X into Y. Furthermore, we put $T[X, Y] = T(X, Y) \times T(Y, X)$. For any (p_1, q_1), (p_2, q_2), (p_3, q_3) in $T[X, Y]$ we set $O((p_1, q_1), (p_2, q_2), (p_3, q_3)) = (p_1 \circ q_2 \circ p_3, q_1 \circ p_2 \circ q_3)$. Then $(T[X, Y], O)$ is a ternary semigroup. The ternary semigroup $(T[X, Y], O)$ is called the *ternary semigroup of mappings of sets X and Y*. If $X \cap Y = \emptyset$, then $(T[X, Y], O)$ is called the *disjoint* ternary semigroup of mappings of sets X and Y.

It is easy to check that the ternary semigroups $(T[X, Y], O)$ and $(T[Y, X], O)$ are isomorphic.

A slightly modified argument applied in the proof of Theorem 3 in [4] yields the following theorem.

3.5. Theorem. *Every ternary semigroup (A, f) is embeddable into a disjoint ternary semigroup $(T[X, Y], O)$ of mappings of sets X and Y.* □

We denote by $C(X, Y)$ the set of all constant mappings of X into Y and put $C[X, Y] = C(X, Y) \times C(Y, X)$. Then $C[X, Y] \subseteq T[X, Y]$ and $O((p_1, q_1), (p_2, q_2), (p_3, q_3)) \in C[X, Y]$ for any (p_1, q_1), (p_2, q_2), and (p_3, q_3) in $C[X, Y]$. Hence, the set $C[X, Y]$ is closed in the ternary semigroup $(T[X, Y], O)$. Thus, if we denote by O'' the restriction of O to $C[X, Y]$, we obtain a ternary semigroup $(C[X, Y], O'')$.

In the same way as Lemma 4.1 in [1], we prove

3.6. Lemma. *Let X, Y be nonempty sets, $(p, q) \in T[X, Y]$ an arbitrary element. Then $(p, q) \in C[X, Y]$ holds if and only if (p, q) is a left zero of $(T[X, Y], O)$.* □

3.7. Lemma. *Let X, Y be nonempty sets. Then the following assertions hold.*

(i) *Any $(p, q) \in C[X, Y]$ is a left zero of $(C[X, Y], O'')$.*
(ii) *The operation O'' of $(C[X, Y], O'')$ is trivial.*

Proof. (i) follows from 3.6 and 3.4, (ii) is a consequence of (i) and 3.3. □

Let X, Y be nonempty sets. A constant mapping p of X into Y with the value $y \in Y$ will be denoted by p_y. A constant mapping q of Y into X with the value $x \in X$ will be denoted by q_x.

3.8. Lemma. *Let X, Y be nonempty sets. For any $x \in X$ put $b_1(x) = q_x$, for any $y \in Y$ define $b_2(y) = p_y$. Put $b = b_2 \times b_1$. Then b is an isomorphism of the* *ternary semigroup* $(Y \times X, o)$ *onto* $(C[X,Y], O'')$.

Proof. Clearly, b is a bijection of $Y \times X$ onto $C[X,Y]$. Since o, O'' are trivial operations by 3.1 and 3.7, b is an isomorphism of $(Y \times X, o)$ onto $(C[X,Y], O'')$ by 3.2. $\square$

3.9. Lemma. *Let* X, Y *be nonempty sets,* x_0 *in* X, *and* u, u', u'' *in* $T(X,Y)$. *Then* $u \circ q_{x_0} \circ u' = u''$ *holds if and only if* $u'' = p_{u(x_0)}$.

Proof. If $x \in X$ is arbitrary, then $(u \circ q_{x_0} \circ u')(x) = u(q_{x_0}(u'(x))) = u(x_0)$ and, hence, $u \circ q_{x_0} \circ u' = p_{u(x_0)}$. Thus $u'' = u \circ q_{x_0} \circ u'$ holds if and only if $u'' = p_{u(x_0)}$. $\square$

3.10. Corollary. *Let* X, Y *be nonempty sets. Let* (S,f) *be a ternary subsemigroup of* $(T[X,Y], O)$ *such that* $C[X,Y] \subseteq S$. *Then* $(u,v) \in S$ *is a left zero of* (S,f) *if and only if* $(u,v) \in C[X,Y]$.

Proof. If $(u,v) \in C[X,Y]$, then by 3.6 (u,v) is a left zero of $(T[X,Y], O)$. Since $C[X,Y] \subseteq S$, it follows from 3.4 that (u,v) is a left zero of (S,f).

Conversely, suppose that (u,v) is a left zero of (S,f). Let $x_0 \in X$ and $y_0 \in Y$ be fixed elements and $(u',v') \in S$. We have $O((u,v),(p_{y_0}, q_{x_0}),(u',v')) = (u,v)$. This implies that $u \circ q_{x_0} \circ u' = u$ and $v \circ p_{y_0} \circ v' = v$. By 3.9 we obtain $u = p_{u(x_0)}$ and, similarly, $v = q_{v(y_0)}$. Hence $(u,v) \in C[X,Y]$. $\square$

3.11. Corollary. *Let* X, Y *be nonempty sets,* $x_0 \in X$, $y_0 \in Y$, $u \in T(X,Y)$ *arbitrary elements. Then* $u(x_0) = y_0$ *holds if and only if* $u \circ q_{x_0} \circ p_{y_0} = p_{y_0}$.

Proof. By 3.9 the last equality is equivalent to $p_{y_0} = p_{u(x_0)}$ which means $y_0 = u(x_0)$. $\square$

3.12. Corollary. *Let* X, Y *be nonempty sets,* x_0, x_0' *in* X, y_0, y_0' *in* Y, *and* $(u,v) \in T[X,Y]$. *Then* $u(x_0) = y_0$, $v(y_0') = x_0'$ *hold if and only if* $O((u,v),(p_{y_0'}, q_{x_0}),(p_{y_0}, q_{x_0'})) = (p_{y_0}, q_{x_0'})$.

Proof. By definition of O the last equality is equivalent to $u \circ q_{x_0} \circ p_{y_0} = p_{y_0}$, $v \circ p_{y_0'} \circ q_{x_0'} = q_{x_0'}$ which means $u(x_0) = y_0$, $v(y_0') = x_0'$ by 3.11. $\square$

4. MONO-n-ARY RELATIONAL STRUCTURES

If X is a nonempty set, n a positive integer, and $r \subseteq X^n$, then the ordered pair $\mathbf{X} = (X, r)$ is said to be a *mono-n-ary relational structure*. The structure is said to be *reflexive* if for any $x \in X$ the condition $(x, \dots, x) \in r$ holds where x appears n times.

Let $\mathbf{X} = (X, r)$, $\mathbf{Y} = (Y, s)$ be mono-n-ary relational structures.

By a *cardinal product* of $\mathbf{X}$ and $\mathbf{Y}$, which will be denoted by $\mathbf{X} \times \mathbf{Y}$, we mean the set $X \times Y$ with the n-ary relation $r \times s$ where for any $(x_1, y_1), \dots, (x_n, y_n)$ in $X \times Y$ the condition $((x_1, y_1), \dots, (x_n, y_n)) \in r \times s$ holds if and only if $(x_1, \dots, x_n) \in r$, $(y_1, \dots, y_n) \in s$. The symbol $\times$ in the formula $r \times s$ does not mean a Cartesian product; we identify $((x_1, y_1), \ldots, (x_n, y_n))$ with $((x_1, \ldots, x_n), (y_1, \ldots, y_n))$. Clearly, $\mathbf{X} \times \mathbf{Y} = (X \times Y, r \times s)$ is a mono-n-ary relational structure. Cf [3] p.164.

Let h be a mapping of X into Y. The mapping h is said to be a *structure homomorphism* (abbreviated s-homomorphism) if for any $(x_1, \ldots, x_n) \in r$ the condition $(h(x_1), \ldots, h(x_n)) \in s$ holds. A bijection b of X onto Y is said to be an *s-isomorphism* of $\mathbf{X}$ onto $\mathbf{Y}$ if it is an s-homomorphism of $\mathbf{X}$ onto $\mathbf{Y}$ and if b^{-1} is an s-homomorphism of $\mathbf{Y}$ onto $\mathbf{X}$.

It is easy to notice that a mapping b of X into Y is an s-isomorphism of $\mathbf{X}$ onto $\mathbf{Y}$ if and only if the following conditions are satisfied.

(i) b is a bijection of X onto Y.
(ii) $(x_1, \ldots, x_n) \in r$ holds if and only if $(b(x_1), \ldots, b(x_n)) \in s$ for any $(x_1, \ldots, x_n) \in X^n$.

Clearly, s-isomorphisms are particular cases of strong homomorphisms in the sense of [5].

4.1. Lemma. *Let $\mathbf{X}_1 = (X_1, r_1)$, $\mathbf{X}_2 = (X_2, r_2)$, $\mathbf{Y}_1 = (Y_1, s_1)$, $\mathbf{Y}_2 = (Y_2, s_2)$ be reflexive mono-n-ary structures and $f_1 : X_1 \to X_2$, $f_2 : Y_1 \to Y_2$ be bijections. The bijection $f_1 \times f_2$ is an s-isomorphism of $\mathbf{X}_1 \times \mathbf{Y}_1$ onto $\mathbf{X}_2 \times \mathbf{Y}_2$ if and only if f_1 is an s-isomorphism of $\mathbf{X}_1$ onto $\mathbf{X}_2$ and f_2 is an s-isomorphism of $\mathbf{Y}_1$ onto $\mathbf{Y}_2$.*

Proof. Let f_1, f_2 be s-isomorphisms. Suppose that $x_1, \ldots, x_n$ are in X_1 and $y_1, \ldots, y_n$ in Y_1. Then any two consecutive conditions in the following sequence are equivalent.

(a) $((x_1, y_1), \ldots, (x_n, y_n)) \in r_1 \times s_1$;
(b) $(x_1, \ldots, x_n) \in r_1$, $(y_1, \ldots, y_n) \in s_1$;
(c) $(f_1(x_1), \ldots, f_1(x_n)) \in r_2$, $(f_2(y_1), \ldots, f_2(y_n)) \in s_2$;
(d) $((f_1(x_1), f_2(y_1)), \ldots, (f_1(x_n), f_2(y_n))) \in r_2 \times s_2$;
(e) $((f_1 \times f_2)(x_1, y_1), \ldots, (f_1 \times f_2)(x_n, y_n)) \in r_2 \times s_2$.

The equivalence of (a) and (e) implies that $f_1 \times f_2$ is an s-isomorphism of $\mathbf{X}_1 \times \mathbf{Y}_1$ onto $\mathbf{X}_2 \times \mathbf{Y}_2$.

Let $f_1 \times f_2$ be an s-isomorphism of $\mathbf{X}_1 \times \mathbf{Y}_1$ onto $\mathbf{X}_2 \times \mathbf{Y}_2$. Suppose that $x_1, \ldots, x_n$ are in X_1. Let $y \in Y_1$ be arbitrary. Then any two consecutive conditions in the following sequence are equivalent.

(f) $(x_1, \ldots, x_n) \in r_1$;
(g) $(x_1, \ldots, x_n) \in r_1$, $(y, \ldots, y) \in s_1$;
(h) $((x_1, y), \ldots, (x_n, y)) \in r_1 \times s_1$;
(k) $((f_1 \times f_2)(x_1, y), \ldots, (f_1 \times f_2)(x_n, y)) \in r_2 \times s_2$;
(l) $((f_1(x_1), f_2(y)), \ldots, (f_1(x_n), f_2(y))) \in r_2 \times s_2$;
(m) $(f_1(x_1), \ldots, f_1(x_n)) \in r_2$, $(f_2(y), \ldots, f_2(y)) \in s_2$;
(n) $(f_1(x_1), \ldots, f_1(x_n)) \in r_2$.

The equivalence of (f) and (n) implies that f_1 is an s-isomorphism of $\mathbf{X}_1$ onto $\mathbf{X}_2$. Similarly, we prove that f_2 is an s-isomorphism of $\mathbf{Y}_1$ onto $\mathbf{Y}_2$. □

4.2. Lemma. *Let $\mathbf{X} = (X, r)$, $\mathbf{Y} = (Y, s)$ be reflexive mono-n-ary relational* *structures. Then any constant mapping of X into Y is an s-homomorphism of* $\mathbf{X}$ *into* $\mathbf{Y}$.

Proof. If p_y is a constant mapping of X into Y, then for any $(x_1, \ldots, x_n) \in r$, we obtain $(p_y(x_1), \ldots, p_y(x_n)) = (y, \ldots, y) \in s$. □

Let $\mathbf{X} = (X, r)$, $\mathbf{Y} = (Y, s)$ be reflexive mono-n-ary relational structures. We denote by $H(\mathbf{X}, \mathbf{Y})$ the set of all s-homomorphisms of $\mathbf{X}$ into $\mathbf{Y}$. Furthermore, we put $H[\mathbf{X}, \mathbf{Y}] = H(\mathbf{X}, \mathbf{Y}) \times H(\mathbf{Y}, \mathbf{X})$. By 4.2, we have $C[X, Y] \subseteq H[\mathbf{X}, \mathbf{Y}] \subseteq T[X, Y]$. Since the superposition of s-homomorphisms is an s-homomorphism, the restriction O' of the ternary operation O to $H[\mathbf{X}, \mathbf{Y}]$ defines a ternary semigroup $(H[\mathbf{X}, \mathbf{Y}], O')$ on $H[\mathbf{X}, \mathbf{Y}]$.

As a consequence of 3.10 we obtain

4.3. Lemma. *Let $\mathbf{X} = (X, r)$, $\mathbf{Y} = (Y, s)$ be reflexive mono-n-ary relational structures, $(p, q) \in H[\mathbf{X}, \mathbf{Y}]$ an arbitrary element. Then (p, q) is a left zero of $(H[\mathbf{X}, \mathbf{Y}], O')$ if and only if $(p, q) \in C[X, Y]$.* □

4.4. Lemma. *Let $\mathbf{X}_1 = (X_1, r_1)$, $\mathbf{X}_2 = (X_2, r_2)$, $\mathbf{Y}_1 = (Y_1, s_1)$, $\mathbf{Y}_2 = (Y_2, s_2)$ be reflexive mono-n-ary relational structures. If F is an isomorphism of the ternary semigroup $(H[\mathbf{X}_1, \mathbf{Y}_1], O'_1)$ onto $(H[\mathbf{X}_2, \mathbf{Y}_2], O'_2)$, then the restriction G of F to $C[X_1, Y_1]$ is an isomorphism of the ternary semigroup $(C[X_1, Y_1], O''_1)$ onto $(C[X_2, Y_2], O''_2)$.*

Proof. Clearly, F assigns a left zero of $(H[\mathbf{X}_2, \mathbf{Y}_2], O'_2)$ to a left zero of $(H[\mathbf{X}_1, \mathbf{Y}_1], O'_1)$ and F^{-1} assigns a left zero of $(H[\mathbf{X}_1, \mathbf{Y}_1], O'_1)$ to any left zero of $(H[\mathbf{X}_2, \mathbf{Y}_2], O'_2)$. By 4.3 the restriction G of F to $C[X_1, Y_1]$ is a bijection of $C[X_1, Y_1]$ onto $C[X_2, Y_2]$. By 3.7 the operations O''_1, O''_2 are trivial. Thus, G is an isomorphism of $(C[X_1, Y_1], O''_1)$ onto $(C[X_2, Y_2], O''_2)$ by 3.2. □

5. INDUCED ISOMORPHISMS

5.1. Lemma. *Let $\mathbf{X}_1 = (X_1, r_1)$, $\mathbf{X}_2 = (X_2, r_2)$, $\mathbf{Y}_1 = (Y_1, s_1)$, $\mathbf{Y}_2 = (Y_2, s_2)$ be reflexive mono-n-ary relational structures, f_1 an s-isomorphism of $\mathbf{X}_1$ onto $\mathbf{X}_2$, and f_2 an s-isomorphism of $\mathbf{Y}_1$ onto $\mathbf{Y}_2$. For any $(p, q) \in H[\mathbf{X}_1, \mathbf{Y}_1]$ put $F(p, q) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1})$. Then F is an isomorphism of the ternary semigroup $(H[\mathbf{X}_1, \mathbf{Y}_1], O'_1)$ onto $(H[\mathbf{X}_2, \mathbf{Y}_2], O'_2)$.*

Proof. Since a composite of s-homomorphisms is an s-homomorphism, we obtain $f_2 \circ p \circ f_1^{-1} \in H(\mathbf{X}_2, \mathbf{Y}_2)$, $f_1 \circ q \circ f_2^{-1} \in H(\mathbf{Y}_2, \mathbf{X}_2)$ and, hence, $F(p, q) \in H(\mathbf{X}_2, \mathbf{Y}_2) \times H(\mathbf{Y}_2, \mathbf{X}_2) = H[\mathbf{X}_2, \mathbf{Y}_2]$. Thus, F is a mapping of $H[\mathbf{X}_1, \mathbf{Y}_1]$ into $H[\mathbf{X}_2, \mathbf{Y}_2]$.

Furthermore, if (p_1, q_1), (p_2, q_2), (p_3, q_3) are in $H[\mathbf{X}_1, \mathbf{Y}_1]$, then we obtain $O'_2(F(p_1, q_1), F(p_2, q_2), F(p_3, q_3)) = O'_2((f_2 \circ p_1 \circ f_1^{-1}, f_1 \circ q_1 \circ f_2^{-1}), (f_2 \circ p_2 \circ f_1^{-1}, f_1 \circ q_2 \circ f_2^{-1}), (f_2 \circ p_3 \circ f_1^{-1}, f_1 \circ q_3 \circ f_2^{-1})) = (f_2 \circ p_1 \circ q_2 \circ p_3 \circ f_1^{-1},$

$f_1 \circ q_1 \circ p_2 \circ q_3 \circ f_2^{-1}) = F(O_1'((p_1, q_1), (p_2, q_2), (p_3, q_3)))$ and, hence, F is a homomorphism of $(H[\mathbf{X}_1, \mathbf{Y}_1], O_1')$ into $(H[\mathbf{X}_2, \mathbf{Y}_2], O_2')$.

If $(p, q) \in H[\mathbf{X}_1, \mathbf{Y}_1]$, $(p', q') \in H[\mathbf{X}_1, \mathbf{Y}_1]$ are such that $F(p, q) = F(p', q')$, i.e. $f_2 \circ p \circ f_1^{-1} = f_2 \circ p' \circ f_1^{-1}$, $f_1 \circ q \circ f_2^{-1} = f_1 \circ q' \circ f_2^{-1}$, we have $p = p'$, $q = q'$. Thus $(p, q) = (p', q')$. Consequently F is injective.

If $(u, v) \in H[\mathbf{X}_2, \mathbf{Y}_2]$, put $p = f_2^{-1} \circ u \circ f_1$, $q = f_1^{-1} \circ v \circ f_2$. Similarly as above, we state that $(p, q) \in H[\mathbf{X}_1, \mathbf{Y}_1]$ and it follows that $F(p, q) = (u, v)$. Hence F is surjective.

Thus F is an isomorphism of the ternary semigroup $(H[\mathbf{X}_1, \mathbf{Y}_1], O_1')$ onto $(H[\mathbf{X}_2, \mathbf{Y}_2], O_2')$. □

Let $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ be reflexive mono-n-ary relational structures, f_1 an s-isomorphism of $\mathbf{X}_1$ onto $\mathbf{X}_2$ and f_2 an s-isomorphism of $\mathbf{Y}_1$ onto $\mathbf{Y}_2$. For any $(p, q) \in H[\mathbf{X}_1, \mathbf{Y}_1]$ put $F(p, q) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1})$. By 5.1, this mapping F is an isomorphism of the ternary semigroup $(H[\mathbf{X}_1, \mathbf{Y}_1], O_1')$ onto $(H[\mathbf{X}_2, \mathbf{Y}_2], O_2')$. It will be called the *isomorphism induced by the pair (f_1, f_2) of s-isomorphisms.*

There exist examples of reflexive mono-n-ary relational structures $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ and of isomorphisms of $(H[\mathbf{X}_1, \mathbf{Y}_1], O_1')$ onto $(H[\mathbf{X}_2, \mathbf{Y}_2], O_2')$ that are not induced by any pair of s-isomorphisms (cf. [1]). Thus, we have the following

5.2. Problem. *If $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ are reflexive mono-n-ary relational structures and F an isomorphism of $(H[\mathbf{X}_1, \mathbf{Y}_1], O_1')$ onto $(H[\mathbf{X}_2, \mathbf{Y}_2], O_2')$, formulate necessary and sufficient conditions for F to be induced by a pair of s-isomorphisms.*

Let $\mathbf{X}_1 = (X_1, r_1)$, $\mathbf{X}_2 = (X_2, r_2)$, $\mathbf{Y}_1 = (Y_1, s_1)$, $\mathbf{Y}_2 = (Y_2, s_2)$ be reflexive mono-n-ary relational structures, F an isomorphism of the ternary semigroup $(H[\mathbf{X}_1, \mathbf{Y}_1], O_1')$ onto $(H[\mathbf{X}_2, \mathbf{Y}_2], O_2')$. By 4.4, the restriction G of F to the set $C[X_1, Y_1]$ is an isomorphism of the ternary semigroup $(C[X_1, Y_1], O_1'')$ onto $(C[X_2, Y_2], O_2'')$. Similarly as in 3.8 we denote by p_{y_1} the constant mapping of X_1 into Y_1 with the value y_1, by q_{x_1} the constant mapping of Y_1 into X_1 with the value x_1, by u_{y_2} the constant mapping of X_2 into Y_2 with the value y_2, and by v_{x_2} the constant mapping of Y_2 into X_2 with the value x_2. Furthermore, put $b_{11}(x_1) = q_{x_1}$, $b_{12}(y_1) = p_{y_1}$ for any $(x_1, y_1) \in X_1 \times Y_1$ and define $b_1 = b_{12} \times b_{11}$. By 3.8, b_1 is an isomorphism of the ternary semigroup $(Y_1 \times X_1, o_1)$ onto $(C[X_1, Y_1], O_1'')$. Similarly, we put $b_{21}(x_2) = v_{x_2}$, $b_{22}(y_2) = u_{y_2}$ for any $(x_2, y_2) \in X_2 \times Y_2$ and define $b_2 = b_{22} \times b_{21}$. Then b_2 is an isomorphism of the ternary semigroup $(Y_2 \times X_2, o_2)$ onto $(C[X_2, Y_2], O_2'')$. It follows that $f = b_2^{-1} \circ G \circ b_1$ is an isomorphism of the ternary semigroup $(Y_1 \times X_1, o_1)$ onto $(Y_2 \times X_2, o_2)$. This mapping f will be said to be the *trace* of F.

5.3. Main Theorem. *Let $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ be reflexive mono-n-ary relational structures, F an isomorphism of the ternary semigroup $(H[\mathbf{X}_1, \mathbf{Y}_1], O_1')$ onto $(H[\mathbf{X}_2, \mathbf{Y}_2], O_2')$. Then the following assertions are equivalent.*

(i) *There exist s-isomorphisms $f_1 : \mathbf{X}_1 \to \mathbf{X}_2$ and $f_2 : \mathbf{Y}_1 \to \mathbf{Y}_2$ such that F is induced by the pair (f_1, f_2).*

(ii) *The trace f of F is a decomposable s-isomorphism of the cardinal product $\mathbf{Y}_1 \times \mathbf{X}_1$ onto the cardinal product $\mathbf{Y}_2 \times \mathbf{X}_2$.*

Proof. We put $\mathbf{X}_i = (X_i, r_i)$, $\mathbf{Y}_i = (Y_i, s_i)$ for $i = 1,\ 2$. Furthermore, we denote - similarly as above - by G the restriction of F to $C[X_1, Y_1]$, by $b_1 = b_{12} \times b_{11}$ the isomorphism of $(Y_1 \times X_1, o_1)$ onto $(C[X_1, Y_1], O_1'')$, and by $b_2 = b_{22} \times b_{21}$ the isomorphism of $(Y_2 \times X_2, o_2)$ onto $(C[X_2, Y_2], O_2'')$ defined in 3.8. Let $f = b_2^{-1} \circ G \circ b_1$ be the trace of F. We know that f is an isomorphism of the ternary semigroup $(Y_1 \times X_1, o_1)$ onto $(Y_2 \times X_2, o_2)$.

Let (i) hold. Then $F(p, q) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1})$ for any element $(p, q) \in H[\mathbf{X}_1, \mathbf{Y}_1]$. Particularly, if $(x_1, y_1) \in X_1 \times Y_1$ is arbitrary, we obtain $F(p_{y_1}, q_{x_1})(x_2, y_2) = (f_2 \circ p_{y_1} \circ f_1^{-1}, f_1 \circ q_{x_1} \circ f_2^{-1})(x_2, y_2) = (f_2(y_1), f_1(x_1))$ for any $(x_2, y_2) \in X_2 \times Y_2$ which implies that $F(p_{y_1}, q_{x_1}) = (u_{f_2(y_1)}, v_{f_1(x_1)})$. Since $b_1(y_1, x_1) = (b_{12}(y_1), b_{11}(x_1)) = (p_{y_1}, q_{x_1})$, $b_2(y_2, x_2) = (b_{22}(y_2), b_{21}(x_2)) = (u_{y_2}, v_{x_2})$, we obtain $(G \circ b_1)(y_1, x_1) = G(p_{y_1}, q_{x_1}) = F(p_{y_1}, q_{x_1}) = (u_{f_2(y_1)}, v_{f_1(x_1)})$ $= b_2(f_2(y_1), f_1(x_1))$ which means that $(b_2^{-1} \circ G \circ b_1)(y_1, x_1) = (f_2 \times f_1)(y_1, x_1)$. Thus, the trace $f = b_2^{-1} \circ G \circ b_1$ of F is decomposable and its components $f_1 : \mathbf{X}_1 \to \mathbf{X}_2$; $f_2 : \mathbf{Y}_1 \to \mathbf{Y}_2$ are s-isomorphisms. It follows that $f = f_2 \times f_1$ is an s-isomorphism of $\mathbf{Y}_1 \times \mathbf{X}_1$ onto $\mathbf{Y}_2 \times \mathbf{X}_2$ by 4.1. Thus (ii) holds.

Suppose that (ii) holds. Then f is an s-isomorphism of $\mathbf{Y}_1 \times \mathbf{X}_1$ onto $\mathbf{Y}_2 \times \mathbf{X}_2$ and is decomposable, i.e., $f = f_2 \times f_1$ where f_1 is a mapping of X_1 into X_2 and f_2 is a mapping of Y_1 into Y_2. By 2.3, f_1 is a bijection of X_1 onto X_2 and f_2 is a bijection of Y_1 onto Y_2. By 4.1, f_1 is an s-isomorphism of $\mathbf{X}_1$ onto $\mathbf{X}_2$ and f_2 is an s-isomorphism of $\mathbf{Y}_1$ onto $\mathbf{Y}_2$. We must prove that $F(p, q) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1})$ holds for any $(p, q) \in H[\mathbf{X}_1, \mathbf{Y}_1]$.

Let $(p, q) \in H[\mathbf{X}_1, \mathbf{Y}_1]$ be arbitrary. Put $(u, v) = F(p, q)$, let $x_2 \in X_2$, $y_2' \in Y_2$ be arbitrarily chosen elements. We define

$$(1) \qquad y_2 = u(x_2),\ x_2' = v(y_2'),\ y_1 = f_2^{-1}(y_2),\ x_1 = f_1^{-1}(x_2),\ x_1' = f_1^{-1}(x_2'),\ y_1' = f_2^{-1}(y_2').$$

Put

$$(2) \qquad y_1'' = p(x_1),\ x_1'' = q(y_1').$$

By 3.12 we obtain $O_1'((p, q), (p_{y_1'}, q_{x_1}), (p_{y_1''}, q_{x_1''})) = (p_{y_1''}, q_{x_1''})$. Since F is an isomorphism of $(H[\mathbf{X}_1, \mathbf{Y}_1], O_1')$ onto $(H[\mathbf{X}_2, \mathbf{Y}_2], O_2')$, we obtain

$$(3) \qquad G(b_1(y_1'', x_1'')) = G(p_{y_1''}, q_{x_1''}) = F(p_{y_1''}, q_{x_1''}) =$$

$$O_2'(F(p, q), F(p_{y_1'}, q_{x_1}), F(p_{y_1''}, q_{x_1''})) = O_2'((u, v), G(p_{y_1'}, q_{x_1}), G(p_{y_1''}, q_{x_1''})).$$

By 4.4 there exist x_2'', x_2''' in X_2 and y_2'', y_2''' in Y_2 such that

$$(4) \qquad G(p_{y_1'}, q_{x_1}) = (u_{y_2''}, v_{x_2''}),\ G(p_{y_1''}, q_{x_1''}) = (u_{y_2'''}, v_{x_2'''}).$$

We obtain

$$(5) \qquad b_2(y_2'', x_2'') = (u_{y_2''}, v_{x_2''}) = G(p_{y_1'}, q_{x_1}) = G(b_1(y_1', x_1)),$$

$$(6) \qquad b_2(y_2''', x_2''') = (u_{y_2'''}, v_{x_2'''}) = G(p_{y_1''}, q_{x_1''}) = G(b_1(y_1'', x_1''))$$

which implies that

$$(7) \qquad (y_2'', x_2'') = (b_2^{-1} \circ G \circ b_1)(y_1', x_1) = (f_2 \times f_1)(y_1', x_1) = (f_2(y_1'), f_1(x_1)),$$

$$(8) \qquad (y_2''', x_2''') = (b_2^{-1} \circ G \circ b_1)(y_1'', x_1'') = (f_2 \times f_1)(y_1'', x_1'') = (f_2(y_1''), f_1(x_1'')).$$

These conditions imply

$$(9) \qquad y_2'' = f_2(y_1'),\ x_2'' = f_1(x_1),\ y_2''' = f_2(y_1''),\ x_2''' = f_1(x_1'').$$

Taking (1) into account, we have

$$(10) \qquad y_2'' = y_2',\ x_2'' = x_2.$$

By (3), (4), we obtain

$$(11) \qquad (u_{y_2'''}, v_{x_2'''}) = O_2'((u,v), (u_{y_2''}, v_{x_2''}), (u_{y_2'''}, v_{x_2'''})).$$

By 3.12 we have

$$(12) \qquad u(x_2'') = y_2''',\ v(y_2'') = x_2'''$$

and (10) implies

$$(13) \qquad u(x_2) = y_2''',\ v(y_2') = x_2'''.$$

By (13), (9), (2), (1), we obtain

$$(14) \qquad u(x_2) = y_2''' = f_2(y_1'') = f_2(p(x_1)) = f_2(p(f_1^{-1}(x_2))),$$

$$(15) \qquad v(y_2') = x_2''' = f_1(x_1'') = f_1(q(y_1')) = f_1(q(f_2^{-1}(y_2'))).$$

Thus, $u(x_2) = (f_2 \circ p \circ f_1^{-1})(x_2)$, $v(y_2') = (f_1 \circ q \circ f_2^{-1})(y_2')$ for an arbitrary element $(x_2, y_2') \in X_2 \times Y_2$. Hence $F(p,q) = (u,v) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1})$ for any $(p,q) \in H[\mathbf{X}_1, \mathbf{Y}_1]$. Thus (i) holds. □

6. EXAMPLES

6.1. Example. Put $X_1 = \{x_{11}, x_{12}\}$, $X_2 = \{x_{21}, x_{22}\}$, $Y_1 = \{y_1\}$, $Y_2 = \{y_2\}$, $r_1 = \{(x_{11}, x_{11}), (x_{11}, x_{12}), (x_{12}, x_{12})\}$, $r_2 = \{(x_{21}, x_{21}), (x_{22}, x_{22})\}$, $s_1 = \{(y_1, y_1)\}$, $s_2 = \{(y_2, y_2)\}$, $\mathbf{X}_1 = (X_1, r_1)$, $\mathbf{X}_2 = (X_2, r_2)$, $\mathbf{Y}_1 = (Y_1, s_1)$, $\mathbf{Y}_2 = (Y_2, s_2)$. Suppose that the elements x_{11}, x_{12}, x_{21}, x_{22}, y_1, y_2 are mutually different. Then $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{Y}_1$, $\mathbf{Y}_2$ are reflexive mono-2-ary relational structures. Clearly, $H(\mathbf{X}_1, \mathbf{Y}_1) = \{p_{y_1}\}$, $H(\mathbf{Y}_1, \mathbf{X}_1) = \{q_{x_{11}}, q_{x_{12}}\}$, $H(\mathbf{X}_2, \mathbf{Y}_2) = \{u_{y_2}\}$, $H(\mathbf{Y}_2, \mathbf{X}_2) = \{v_{x_{21}}, v_{x_{22}}\}$. Hence $H[\mathbf{X}_1, \mathbf{Y}_1] = \{(p_{y_1}, q_{x_{11}}), (p_{y_1}, q_{x_{12}})\}, H[\mathbf{X}_2, \mathbf{Y}_2] = \{(u_{y_2}, v_{x_{21}}), (u_{y_2}, v_{x_{22}})\}$. Thus $C[X_1, Y_1] = H[\mathbf{X}_1, \mathbf{Y}_1]$, $C[X_2, Y_2] = H[\mathbf{X}_2, \mathbf{Y}_2]$.

Put $F(p_{y_1}, q_{x_{11}}) = (u_{y_2}, v_{x_{21}})$, $F(p_{y_1}, q_{x_{12}}) = (u_{y_2}, v_{x_{22}})$. By 3.2 and 3.7, F is an isomorphism of $(H[\mathbf{X}_1, \mathbf{Y}_1], O_1')$ onto $(H[\mathbf{X}_2, \mathbf{Y}_2], O_2')$. Its trace f is defined by $f(y_1, x_{11}) = (y_2, x_{21})$, $f(y_1, x_{12}) = (y_2, x_{22})$. Clearly, f is a decomposable bijection of $Y_1 \times X_1$ onto $Y_2 \times X_2$. We have $f = f_2 \times f_1$ where $f_2(y_1) = y_2$, $f_1(x_{11}) = x_{21}$, $f_1(x_{12}) = x_{22}$. Since $(x_{11}, x_{12}) \in r_1$, $(x_{21}, x_{22}) \notin r_2$, the bijection f_1 is no s-isomorphism of $\mathbf{X}_1$ onto $\mathbf{X}_2$. Thus, f is no s-isomorphism of $\mathbf{Y}_1 \times \mathbf{X}_1$ onto $\mathbf{Y}_2 \times \mathbf{X}_2$ by 4.1. By 5.3, F is not induced by any pair of s-isomorphisms. □

6.2. Example. Put $X_1 = \{x_{11}, x_{12}\}$, $X_2 = \{x_{21}, x_{22}\}$, $Y_1 = \{y_{11}, y_{12}\}$, $Y_2 = \{y_{21}, y_{22}\}$, $r_1 = \{(x_{11}, x_{11}), (x_{12}, x_{12})\}$, $r_2 = \{(x_{21}, x_{21}), (x_{22}, x_{22})\}$, $s_1 = \{(y_{11}, y_{11}),$

$(y_{12}, y_{12})\}$, $s_2 = \{(y_{21}, y_{21}), (y_{22}, y_{22})\}$, $\mathbf{X}_1 = (X_1, r_1)$, $\mathbf{X}_2 = (X_2, r_2), \mathbf{Y}_1 = (Y_1, s_1)$, $\mathbf{Y}_2 = (Y_2, s_2)$. Suppose that the elements x_{11}, x_{12}, x_{21}, x_{22}, y_{11}, y_{12}, y_{21}, y_{22} are mutually different. Clearly $H[\mathbf{X}_1, \mathbf{Y}_1] = T[X_1, Y_1]$ and $H[\mathbf{X}_2, \mathbf{Y}_2] = T[X_2, Y_2]$. Put $h_1(x_{11}) = y_{21}$, $h_1(x_{12}) = y_{22}$, $h_2(y_{11}) = x_{21}$, $h_2(y_{12}) = x_{22}$, $h = h_2 \times h_1$. For any $(p, q) \in H[\mathbf{X}_1, \mathbf{Y}_1]$ put $F'(p, q) = (h_2 \circ p \circ h_1^{-1}, h_1 \circ q \circ h_2^{-1})$.

Since h_1 is an s-isomorphism of $\mathbf{X}_1$ onto $\mathbf{Y}_2$ and h_2 is an s-isomorphism of $\mathbf{Y}_1$ onto $\mathbf{X}_2$, F' is an isomorphism of the ternary semigroup $(H[\mathbf{X}_1, \mathbf{Y}_1], O'_1)$ onto $(H[\mathbf{Y}_2, \mathbf{X}_2], O'_3)$ by 5.3 where we have $O'_3((v_1, u_1), (v_2, u_2), (v_3, u_3)) = (v_1 \circ u_2 \circ v_3, u_1 \circ v_2 \circ u_3)$ for any (v_1, u_1), (v_2, u_2), (v_3, u_3) in $H[\mathbf{Y}_2, \mathbf{X}_2]$. For any $(v, u) \in H[\mathbf{Y}_2, \mathbf{X}_2]$ put $F''(v, u) = (u, v)$. If (v_1, u_1), (v_2, u_2), (v_3, u_3) are arbitrary elements in $H[\mathbf{Y}_2, \mathbf{X}_2]$, we have $F''(O'_3((v_1, u_1), (v_2, u_2), (v_3, u_3))) = F''(v_1 \circ u_2 \circ v_3, u_1 \circ v_2 \circ u_3) = (u_1 \circ v_2 \circ u_3, v_1 \circ u_2 \circ v_3) = O'_2((u_1, v_1), (u_2, v_2), (u_3, v_3)) = O'_2(F''(v_1, u_1), F''(v_2, u_2), F''(v_3, u_3))$ which implies that F'' is a homomorphism of $(H[\mathbf{Y}_2, \mathbf{X}_2], O'_3)$ into $(H[\mathbf{X}_2, \mathbf{Y}_2], O'_2)$. Since F'' is a bijection, it is an isomorphism. It follows that $F'' \circ F'$ is an isomorphism of the ternary semigroup $(H[\mathbf{X}_1, \mathbf{Y}_1], O'_1)$ onto $(H[\mathbf{X}_2, \mathbf{Y}_2], O'_2)$ assigning the ordered pair $(h_1 \circ q \circ h_2^{-1}, h_2 \circ p \circ h_1^{-1}) \in H[\mathbf{X}_2, \mathbf{Y}_2]$ to any $(p, q) \in H[\mathbf{X}_1, \mathbf{Y}_1]$. Put $F = F'' \circ F'$.

The restriction G of F to the set $C[X_1, Y_1]$ has the following properties: $G(p_{y_{11}}, q_{x_{11}}) = F(p_{y_{11}}, q_{x_{11}}) = F''(F'(p_{y_{11}}, q_{x_{11}})) = F''(h_2 \circ p_{y_{11}} \circ h_1^{-1}, h_1 \circ q_{x_{11}} \circ h_2^{-1}) = F''(v_{x_{21}}, u_{y_{21}}) = (u_{y_{21}}, v_{x_{21}})$ and, similarly, $G(p_{y_{12}}, q_{x_{11}}) = (u_{y_{21}}, v_{x_{22}})$, $G(p_{y_{11}}, q_{x_{12}}) = (u_{y_{22}}, v_{x_{21}})$, $G(p_{y_{12}}, q_{x_{12}}) = (u_{y_{22}}, v_{x_{22}})$. Thus, the trace f of F defined by $f = b_2^{-1} \circ G \circ b_1$ satisfies the following conditions: $f(y_{11}, x_{11}) = (y_{21}, x_{21})$, $f(y_{12}, x_{11}) = (y_{21}, x_{22})$, $f(y_{11}, x_{12}) = (y_{22}, x_{21})$, $f(y_{12}, x_{12}) = (y_{22}, x_{22})$. This mapping f is no decomposable mapping of $\mathbf{Y}_1 \times \mathbf{X}_1$ onto $\mathbf{Y}_2 \times \mathbf{X}_2$. Indeed, if $f = f_2 \times f_1$, then $f(y_{11}, x_{11}) = (y_{21}, x_{21})$ implies that $f_2(y_{11}) = y_{21}$, $f_1(x_{11}) = x_{21}$ which entails that $f(y_{11}, x_{12}) = (f_2(y_{11}), f_1(x_{12})) = (y_{21}, f_1(x_{12}))$. But we have $f(y_{11}, x_{12}) = (y_{22}, x_{21})$ which implies that $y_{21} = y_{22}$; this is a contradiction. Thus, the trace of F is not decomposable and, therefore, F is not induced. □

6.3. Example. Let X_1, X_2, Y_1, Y_2 be nonempty sets, suppose that f_1 is a bijection of X_1 onto X_2, f_2 a bijection of Y_1 onto Y_2. Put $r_1 = X_1 \times X_1, r_2 = X_2 \times X_2$, $s_1 = Y_1 \times Y_1$, $s_2 = Y_2 \times Y_2$, $\mathbf{X}_1 = (X_1, r_1)$, $\mathbf{X}_2 = (X_2, r_2)$, $\mathbf{Y}_1 = (Y_1, s_1)$, $\mathbf{Y}_2 = (Y_2, s_2)$. Then, clearly, $H[\mathbf{X}_1, \mathbf{Y}_1] = T[X_1, Y_1]$, $H[\mathbf{X}_2, \mathbf{Y}_2] = T[X_2, Y_2]$, f_1 is an s-isomorphism of $\mathbf{X}_1$ onto $\mathbf{X}_2$, and f_2 is an s-isomorphism of $\mathbf{Y}_1$ onto $\mathbf{Y}_2$. Put $F(p, q) = (f_2 \circ p \circ f_1^{-1}, f_1 \circ q \circ f_2^{-1})$ for any $(p, q) \in H[\mathbf{X}_1, \mathbf{Y}_1]$. Then F is an isomorphism of $H[\mathbf{X}_1, \mathbf{Y}_1]$ onto $H[\mathbf{X}_2, \mathbf{Y}_2]$ that is induced by the pair (f_1, f_2) of s-isomorphisms. □

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Antoni Chronowski
Wyższa Szkoła Pedagogiczna
Instytut Matematyki
ul. Podchorążych 2
30-084 Kraków, POLAND

Miroslav Novotný
Faculty of Computer Science
Masaryk University
Bureova 20
602 00 Brno, CZECH REPUBLIC