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A COMMUTATIVITY THEOREM FOR ASSOCIATIVE RINGS

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Abstract. Let $m > 1, s \geq 1$ be fixed positive integers, and let R be a ring with unity 1 in which for every x in R there exist integers $p = p(x) \geq 0, q = q(x) \geq 0, n = n(x) \geq 0, r = r(x) \geq 0$ such that either $x^p[x^n, y]x^q = x^r[x, y^m]y^s$ or $x^p[x^n, y]x^q = y^s[x, y^m]x^r$ for all $y \in R$. In the present paper it is shown that R is commutative if it satisfies the property $Q(m)$ (i.e. for all $x, y \in R, m[x, y] = 0$ implies $[x, y] = 0$).

1. Introduction

Throughout the present paper R will denote an associative ring with unity $1, Z(R)$ the center of $R, N(R)$ the set of nilpotent elements of R, and $C(R)$ the commutator ideal of R . For any $x, y \in R$, set $[x, y] = xy - yx$. As usual $\mathbb{Z}[X]$ is the totality of polynomials in X with coefficients in $\mathbb{Z}$, the ring of integers. For fixed non-negative integers $m > 1, s \geq 1$, consider the following ring properties:

- $(*)$: For each x in R there exist integers $p = p(x) \geq 0, q = q(x) \geq 0, n = n(x) \geq 0, r = r(x) \geq 0$ such that $x^p[x^n, y]x^q = x^r[x, y^m]y^s$ for all $y \in R$.
- $(*)'$: For each x in R there exist integers $p = p(x) \geq 0, q = q(x) \geq 0, n = n(x) \geq 0, r = r(x) \geq 0$ such that $x^p[x^n, y]x^q = y^s[x, y^m]x^r$ for all $y \in R$.
- $Q(d)$: For all $x, y \in R, d[x, y] = 0$ implies that $[x, y] = 0$, where d is some positive integers.

It is easy to see that every d-torsion free ring has the property $Q(d)$ and every ring has the property $Q(1)$.

Recently several authors (cf.[1], [2], [4], [5], [7], [11], [13] and [16] etc.)have studied commutativity of rings satisfying various special cases of the property $(*)$ and $(*)'$. Particularly, in most of the cases, the exponents in the above conditions have been considered "global". Till now a very few attempts (cf.[3], [10] etc.) have been made to establish commutativity of rings, when the exponents in the underlying conditions are "local" i.e. they are depending on ring's elements for

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their values. In the present paper, our objective is to investigate commutativity of rings satisfying either of the properties $(*)$ or $(*)'$.

2. Main Result

Theorem. *Let $m > 1, s \geq 1$ be fixed positive integers for which R satisfies either of the properties $(*)$ or $(*)'$. Then R is commutative.*

In order to develop the proof of the above theorem we begin with the following lemmas, which are essentially proved in [8,p.221], [9,Theorem] and [6,Theorem 1] respectively. Although Lemma 2.4 is proved in [14] for a fixed exponent n, but with a slight modification in the proof, it can be established for variable exponent n.

Lemma 2.1. *Let x, y be elements in a ring R (may be without unity 1). If $[x,[x,y]] = 0$, then $[x^k, y] = kx^{k-1}[x,y]$ for all integers $k \geq 1$.*

Lemma 2.2. *Let f be a polynomial in n non-commuting indeterminates $x_1, x_2, \cdots, \cdots, x_n$ with relatively prime integer coefficients. Then the following are equivalent:*

(i) For every ring satisfying $f = 0, C(R)$ is a nil ideal.

(ii) For every prime $p, (GF(p))_2$ fails to satisfy $f = 0$.

Lemma 2.3. *Let R be a ring (may be without unity 1), and suppose that for each $x, y \in R$, there exists a polynomial $f(X) \in X\mathbb{Z}[X]$, depending on x and y for which $[x,y] = [x,y]f(x)$. Then R is commutative.*

Lemma 2.4. *Let f be a polynomial function of two variables on R with the property that $f(x+1,y) = f(x,y)$ for all $x, y \in R$. Suppose that for all $x, y \in R$ there exists integer n such that $x^n f(x,y) = 0$ or $f(x,y)x^n = 0$, then necessarily $f(x,y) = 0$.*

Proof. Suppose that $x^n f(x,y) = 0$. Choose a positive integer $n_1 = n(x+1, y)$ such that $(x+1)^{n_1} f(x,y) = 0$. If $N = max\{n, n_1\}$ then it follows that $x^N f(x,y) = 0$ and $(x+1)^N f(x,y) = 0$. We have $f(x,y) = \{(x+1) - x\}^{2N+1} f(x,y)$. On expanding the expression on the right hand side by binomial theorem, we find that $f(x,y) = 0$. A similar proof is valid in case, if R satisfies $f(x,y)x^n = 0$. $\square$

Now we shall prove the following:

Lemma 2.5. *Let R be a ring satisfying either of the properties $(*)$ or $(*)'$. Moreover, if R has the property $Q(m)$, then $N(R) \subseteq Z(R)$.*

Proof. Suppose that R satisfies the property $(*)$. Let $a \in N(R)$. Then there exists a positive integer t such that

$$a^k \in Z(R), \quad \text{for all} \quad k \geq t \quad \text{and} \quad t \quad \text{minimal}. \tag{2.1}$$

If $t = 1$, then for each such a, result is obvious. Therefore assume that $t > 1$. Now replace y by a^{t-1} in $(*)$, to get $x^p[x^n, a^{t-1}]x^q = x^r[x,(a^{t-1})^m](a^{t-1})^s$. Thus in view of (2.1) and the fact that $(t-1)m \geq t$ for $m > 1$, we find that

$$x^p[x^n, a^{t-1}]x^q = 0, \quad \text{for all} \quad x \text{ in } R. \tag{2.2}$$

Further replace y by $1 + a^{t-1}$ in $(*)$ and use (2.2), to get

$$0 = x^p[x^n, a^{t-1}]x^q = x^p[x^n, 1 + a^{t-1}]x^q = x^r[x, (1 + a^{t-1})^m](1 + a^{t-1})^s$$

Since, $1 + a^{t-1}$ is invertible, the last equation implies that $x^r[x, (1 + a^{t-1})^m] = 0$. Now application of Lemma 2.4, yields that

$$[x, (1+a^{t-1})^m] = 0, \quad \text{for all} \quad x \in R. \tag{2.3}$$

Combine (2.1) and (2.3), to get

$$0 = [x, (1 + a^{t-1})^m] = [x, 1 + ma^{t-1}] = m[x, a^{t-1}].$$

Now using property $Q(m)$, we find that $a^{t-1} \in Z(R)$. This contradicts the minimality of t in (2.1), and hence $t = 1$ i.e. $a \in Z(R)$.

Similar arguments may be used to get the required result, if R satisfies the property $(*)'$. □

Lemma 2.6. *Let $m > 1, s \geq 1$ be fixed positive integers for which R satisfies either of the properties $(*)$ or $(*)'$. Then $C(R) \subseteq N(R)$.*

Proof. Let R satisfy $(*)$. Replacement of y by $1+y$ in $(*)$, yields that $x^p[x^n, y]x^q = x^r[x, (1 + y)^m](1 + y)^s$. This gives that $x^r\{[x, y^m]y^s - [x, (1 + y)^m](1 + y)^s\} = 0$. Now apply Lemma 2.4, to get $[x, y^m]y^s - [x, (1 + y)^m](1 + y)^s = 0$. This is a polynomial identity and we see that $x = e_{11} + e_{12}, y = e_{11}$ fail to satisfy this equality in the ring of 2×2 matrices over $GF(p), p$ a prime. Hence by Lemma 2.2, $C(R) \subseteq N(R)$.

On the other hand if R satisfies $(*)'$, then by using similar techniques as above, with the choice of $x = e_{11} + e_{21}, y = e_{11}$, we get the required result. □

Proof of the Theorem. We sall prove the theorem for the property $(*)$. Proof for the property $(*)'$ follows similarly. In view of Lemmas 2.5 and 2.6, we have

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{2.4}$$

If $n = 0$, then we find that $x^r[x, y^m]y^s = 0$.Now application of (2.4) and Lemma 2.1, yields that $mx^r[x, y]y^{m+s-1} = 0$. Now apply Lemma 2.4, to get $m[x, y] = 0$, and in view of $Q(m)$,this yields the required result. Therefore, assume that $n > 0$. Now replace y by $1 + y$, to get $x^p[x^n, y]x^q = x^r[x, (1 + y)^m](1 + y)^s$. This gives that $x^r\{[x, y^m]y^s - [x, (1 + y)^m](1 + y)^s\} = 0$, and by Lemma 2.4, we find that $[x, y^m]y^s = [x, (1+y)^m](1+y)^s$, for all $x, y \in R$. In view of Lemma 2.1 and $Q(m)$, the last equation reduces to $[x, y]\{(1 + y)^{m+s-1} - y^{m+s-1}\} = 0$, for all $x, y \in R$. This is a polynomial identity and can be rewritten in the form $[x, y] = [x, y]yg(y)$, for some $g(X) \in \mathbb{Z}[X]$. Hence by Lemma 2.3, R is commutative. □

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