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AN INTEGRAL CONDITION OF OSCILLATION FOR EQUATION $\mathbf{y''' + p(t)y' + q(t)y = 0}$ WITH NONNEGATIVE COEFFICIENTS

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Abstract. Our aim in this paper is to obtain a new oscillation criterion for equation

$$(*) \qquad y''' + p(t)y' + q(t)y = 0$$

with a nonnegative coefficients which extends and improves some oscillation criteria for this equation. In the special case of equation (*), namely, for equation $y''' + q(t)y = 0$, our results solve the open question of *Chanturiya*.

1. Introduction

Consider the differential equation

$$y''' + p(t)y' + q(t)y = 0, \tag{1}$$

where $p,\ q,\ p' : I \longrightarrow \mathbb{R},\ \ I = [a, \infty) \subset (0, \infty),\ \mathbb{R} = (-\infty, \infty)$ are continuous. In the sequel we suppose that $p(t) \geq 0$, $q(t) \geq 0$, and $\sup\{q(s);\ s \geq t\} > 0$, $t \geq a$.

We consider only nontrivial solutions of equation (1). Such solution of (1) is called oscillatory on I if it has arbitrarily large zeros, otherwise it is called nonoscillatory on I. Concerning nonoscillatory solutions of (1) without loss of generality we can restrict our attention only to positive ones. Equation (1) is called oscillatory if it has at least one oscillatory solution.

In the particular case of (1) when $p(t) \equiv 0$, $t \in I$ equation (1) becomes to

$$y''' + q(t)y = 0. \tag{2}$$

Following Kiguradze [10], also [2, Definition 1.1], equation (2) is said to have property A if every solution y of (2) is either oscillatory or

$$\lim_{t\to\infty} y^{(i)}(t) = 0,\ (i = 0, 1, 2),$$

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monotonically. It is well-known that equation (2) is oscillatory iff it has property A, see e.g. [2, Lemma 2.8']. From the other hand equation (2) has property A if

$$\int^{\infty} t^{2-\varepsilon}\, q(t)\, dt = \infty, \quad \text{for any } \varepsilon \in (0,2], \tag{3}$$

see e.g. [2], [5], [16].

Recently Chanturiya improved condition (3) of oscillation of (2).

Theorem A. *([2, Theorem 2.12]). If*

$$\liminf_{t\to\infty} t \int_t^{\infty} sq(s)\, ds > \frac{2\sqrt{3}}{9},$$

then equation (2) has property A.

For analogous result for third-order functional differential equation the reader is referred to [3].

In the same book (see [2, Problem 1.14, p. 48, n = 3]), Chanturiya introduced the following question:

Question. Is the condition

$$\int_1^{\infty} t^2 \left(q(t) - \frac{2\sqrt{3}}{9}\, t^{-3} \right) dt = \infty$$

sufficient for equation (2) to have property A?

From our results the answer to this question follows immediately.

For $p(t) \not\equiv 0$, $t \in I$, there is a large literature on the oscillation of equation (1). About oscillation criteria of Kneser-type the reader is reffered to [6], [8] and [12], see also the books [2], [5], [16], and papers [1] and [4]. From others results we present

Theorem B. *([6, Theorem 5.12]) Let $p(t) > 0$, $q(t) > 0$, and $q(t) > p'(t)$ in (α, ∞), $\alpha > 0$. If equation*

$$u'' + p(t)u = 0, \tag{5}$$

is nonoscillatory, and if

$$\int_\alpha^{\infty} t[q(t) - p'(t)]\, dt = \infty \tag{6}$$

then equation (1) is oscillatory.

For nontrivial solution y of (1) we denote

$$F[y(t)] = 2y(t)y''(t) - y'^2(t) + p(t)y^2(t). \tag{7}$$

Theorem C. *([11, Theorem 3.1]) If $2q(t) - p'(t) \geq 0$ and not identically zero in any subinterval of I and there exists a number $m < \frac{1}{2}$ such that second-order differential equation*

$$u'' + (p(t) + mtq(t))\, u = 0, \tag{8}$$

is oscillatory, then equation (1) is also oscillatory. In fact, if y is any nonzero solution of (1) with $F[y(c)] \leq 0$ ($c \geq a$) then y is oscillatory.

The aim of this paper is to establish some integral criterion for oscillation of (1) for the case when Theorem B and Therem C fail. The next theorem gives sufficient conditions under which nonoscillatory solutions of (1) tend to zero as t tends to infinity. These our result for $p(t) \equiv 0$, that is equation (2), are the affirmative answer to the question of Chanturiya.

2. Preliminaries

In this section we present some lemmas requsite to proofs of main results.

The following lemma is proved in [11, Lemma 3.2], for nonlinear equation see also [14] and [15].

Lemma 1. *If $2q(t) - p'(t) \geq 0$ and not identically zero in any subinterval of I and y is a nonoscilatory solution of (1) which is eventually nonnegative with $F[y(c)] \leq 0$ (see (7), $c \in I$ arbitrary) then there exists a number $d \geq c$ such that $y(t) > 0$, $y'(t) > 0$, $y''(t) > 0$, and $y'''(t) \leq 0$, for $t \geq d$.*

Remark 1. Any solution y with a zero, that is $y(t^*) = 0$, satisfies $F[y(t^*)] \leq 0$.

Remark 2. Let hypothesis of Lemma 1 hold. Considering additional assumptions we eliminate positive increasing solutions and so we obtain oscillation criterion. Since for $t^2p(t) > \frac{1}{4}$, $t > 0$ equation (8) is oscillatory by Sturm comparison theorem (see Theorem 1.1 in [16]) and Kneser criterion (see [16, p. 45]), so Theorem C is aplicable. Therefore we will interested with the case $t^2p(t) \leq \frac{1}{4}$, $t > 0$.

It is easy to verify that the following inequality is fullfiled for all $t > 0$,

$$tp(t) - \frac{2}{3\sqrt{3}t}\left(1 - t^2p(t)\right)^{\frac{3}{2}} \leq 0 \quad \text{for} \quad t^2p(t) \leq \frac{1}{4}, \tag{9}$$

since $4t^6p^3(t) + 15t^4p^2(t) + 12t^2p(t) - 4 = (4t^2p(t) - 1)(t^2p(t) + 2)^2$.

Using this inequality we obtain the assertion needed to proof of main results. Proof of this assertion is elementary.

Lemma 2. *Let $0 \leq t^2p(t) \leq \frac{1}{4}$ for all $t > 0$. Let P be the polynomial in the variable z,*

$$P(z) = z^3 - 3z^2 + \left(2 + t^2p(t)\right)z + t^3q(t), \; t > 0.$$

Then

$$P(z) \geq t^3 q(t) + t^2 p(t) - \frac{2}{3\sqrt{3}}\big(1 - t^2 p(t)\big)^{\frac{3}{2}}, \quad t > 0 \tag{10}$$

for all $z \geq 1 - 2\sqrt{\frac{1-t^2p(t)}{3}}$.

Remark 3. The right-hand side of (10) is the local minimum of P in the point $z_0 = 1 + \sqrt{\frac{1-t^2p(t)}{3}}$.

3. Main results

For the proof of our oscillation result we use the similar method like in the paper [13], see also [11, Theorem 3.1].

Theorem 1. *Let hypotheses of Lemma 1 hold, and in addition* $t^2 p(t) \leq \frac{1}{4}$ *for all all* $t > 0$. *If*

$$\int^{\infty} \left(t^2 q(t) + t p(t) - \frac{2}{3\sqrt{3}t}\big(1 - t^2 p(t)\big)^{\frac{3}{2}}\right) \, dt = \infty, \tag{11}$$

then equation (1) is oscillatory. In fact, any solution y *which satisfies* $F[y(t^*)] \leq 0$ *for some* $t^* > a$, *is oscillatory.*

Proof. Let y be a solution of (1) which satisfies $F[y(t_0)] \leq 0$ for some $t_0 > a$. Then by Lemma 1, y is oscillatory or $y(t)y'(t) > 0$ for all sufficiently large t. Suppose without loss of generality that $y(t) > 0$, $y'(t) > 0$ for all $t \geq b \geq t_0$. Now, we denote

$$z(t) = \frac{t y'(t)}{y(t)}, \quad t \geq b.$$

So $z(t) > 0$ and it is easy to verify that z satisfies the second-order Riccati equation

$$\Big((tz)' + \frac{3}{2}z^2 - 4z\Big)' + \frac{1}{t}\big(z^3 - 3z^2 + (2 + t^2 p(t))z + t^3 q(t)\big) = 0. \tag{12}$$

Substituting the estimate (10) to (12) we have

$$\Big((tz)' + \frac{3}{2}z^2 - 4z\Big)' \leq -\frac{1}{t}\big(t^3 q(t) + t^2 p(t) - \frac{2}{3\sqrt{3}}(1 - t^2 p(t))^{\frac{3}{2}}\big) = -Q(t),$$

for all $t \geq b$. Integrating the above inequality from b to $t \geq b$ we get

$$(tz(t))' + \frac{3}{2}z^2(t) - 4z(t) \leq K_0 - \int_b^t Q(s)\, ds,$$

where K_0 is a constant. Since $\frac{3}{2}z^2(t) - 4z(t) \geq -\frac{8}{3}$, an integration of the above inequality from b to $t \geq b$ yields

$$tz(t) \leq K_2 + K_1 t - \int_b^t \int_b^s Q(u)\, du ds, \tag{13}$$

where $K_1 = K_0 + \frac{8}{3}$, and $K_2 = b(z(b) - K_1)$. So it follows from (11) and (13) that $z(t) < 0$ for sufficiently large t, which contradicts positivity of z. So equation (1) cannot have any solution with property $y(t)y'(t) > 0$ for all large t and by Lemma 1 equation (1) is oscillatory.

The next theorem describes asymptotic behavior of nonoscillatory solutions of (1).

Theorem 2. *Let $0 \le t^2 p(t) \le \frac{1}{4}$, and $q(t) > 0$, $t \in I$. If (11) is satisfied, then any nonoscilatory solution of (1) has property $\lim_{t\to\infty} y(t) = 0$.*

Proof. Since $t^2p(t) \le \frac{1}{4}$, $t \in I$ from Kneser comparison theorem it follows that equation (5) is nonoscillatory. So by Theorem 3.6 in [7] it follows that there exists $d \ge a$ such that either $y(t)y'(t) \ge 0$ or $y(t)y'(t) < 0$ for all $t \ge d$. Let y be a nonoscillatory solution, and suppose that $y(t) > 0$, $y'(t) \ge 0$ for all $t \ge d$. We again denote $z(t) = \frac{ty'(t)}{y(t)}$, $t \ge d$. So $z(t) \ge 0$. Let hypothesis (11) hold. The same process as in the proof of Theorem 1 shows (by (13)) that z becomes negative for sufficiently large t, a contradiction. Let $y(t) > 0$, $y'(t) < 0$ for $t \ge d$. Hence $\lim_{t\to\infty} y(t) = L \ge 0$ exists. Let $L > 0$. Multiplying equation (1) by t^2 and integration from d to $t \ge d$ yields

$$t^2y''(t) - 2ty'(t) + \frac{9}{4}y(t) \le K - L\int_d^t s^2 q(s)\,ds,$$

where K is some constant. By (9) and condition (11) we have $\int_d^\infty t^2 q(t)dt = \infty$. From this and from the last inequality, for all sufficiently large t, we have $y''(t) < 0$, which contradicts $y(t) > 0$, $y'(t) < 0$. The proof is complete.

Now we consider equation (2). Let y be a nonoscillatory solution of (2). Without loss of generality we may suppose that y is a positive one. Then according a lemma of Kiguradze [9, Lemma 3], see also [2, Lemma 1.1], there is a number $t_1 \ge a$ such that

$$y(t) > 0,\ \ y'(t) < 0,\ \ y''(t) > 0,\ \ y'''(t) \le 0,$$

or

$$y(t) > 0,\ \ y'(t) > 0,\ \ y''(t) > 0,\ \ y'''(t) \le 0,$$

for all $t \ge t_1$. So, from Theorem 1 and Theorem 2 we have

Corollary 1. *Let*

$$\int_1^\infty t^2\left(q(t) - \frac{2}{3\sqrt{3}}t^{-3}\right)dt = \infty.$$

Then equation (2) has property A.

Remark 4. Corollary 1 is the affirmative answer to the question of Chanturiya.

4. Comparisons and examples

To show that Theorem 1 can be applied even in the case when Theorem B and Theorem C are not applicable, let us consider the following equation

$$u''' + p_0 t^{\beta} u' + q_0 t^{-3} u = 0, \quad t > 0, \tag{14}$$

where $\beta \leq -2,\ p_0 > 0,\ q_0 > \frac{2}{3\sqrt{3}}$, and $p_0 < \frac{1}{4}$ if $\beta = -2$; β, p_0 and q_0 are some constants.

Directly we see that Theorem B is not applicable to equation (14). For $\beta = -2$ equation (14) becomes to the Euler equation. The neccesary and sufficient condition for oscillation of Euler equation (14) is

$$q_0 + p_0 - \frac{2}{3\sqrt{3}}(1 - p_0)^{\frac{3}{2}} > 0. \tag{15}$$

It is easy to check that condition (15) is equivalent to condition (11) of Theorem 1. To compare our result to Theorem C we note that equation (8) in this case, that is $\beta = -2$, and $p_0 < \frac{1}{4}$ becomes to the Euler equation,

$$v'' + (p_0 + mq_0)t^{-2} v = 0. \tag{8'}$$

Equation (8') is oscillatory iff $p_0 + mq_0 > \frac{1}{4}$ for some $m < \frac{1}{2}$, that is $2p_0 + q_0 > 2p_0 + 2mq_0 > \frac{1}{2}$. So it is easy to check that inequality $q_0 + p_0 > \frac{1}{2} - p_0 > \frac{2}{3\sqrt{3}}(1 - p_0)^{\frac{3}{2}}$ holds for some $q_0 > 0$ and every $0 < p_0 < \frac{1}{4}$. From this it follows that Theorem 1 is better than Theorem C in this case, e.g. for $p_0 = 0.06$, $q_0 = 0.3$ condition (15) is fullfiled, while (8') is nonoscillatory.

Let $\beta < -2$. So there is a number $\delta > 0$ such that $\beta = -2 - \delta$ and hence $t^2 p(t) = p_0 t^{-\delta} \leq \frac{1}{4}$ for $t \geq a_0 = (4p_0)^{\frac{1}{\delta}}$. If we denote $x = p_0 t^{-\delta}$ for $t \geq a_0$ then the function $f(x) = x - \frac{2}{3\sqrt{3}}(1 - x)^{\frac{3}{2}}$ is increasing and so for $0 \leq x \leq \frac{1}{4}$ we have $f(x) \geq f(0) = -\frac{2}{3\sqrt{3}}$. Therefore

$$\int_{a_0}^{\infty} \frac{1}{t}\big(q_0 + p_0 t^{-\delta} - \frac{2}{3\sqrt{3}}(1 - p_0 t^{-\delta})^{\frac{3}{2}}\big)\, dt \geq (q_0 - \frac{2}{3\sqrt{3}}) \int_{a_0}^{\infty} \frac{dt}{t}.$$

So we see by Theorem 1 that for $p_0 > 0$, $q_0 > \frac{2}{3\sqrt{3}}$ and $\beta < -2$, equation (14) is oscillatory. On the other hand, by Theorem C equation (14) is oscillatory only if $q_0 > 0.5$.

Remark 5. About comparison Theorem A to Corollary 1 the reader is referred to [13].

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