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LIFTINGS OF 1-FORMS TO THE LINEAR r-TANGENT BUNDLE

W. M. Mikulski

Abstract. Let r, n be fixed natural numbers. We prove that for n-manifolds the set of all linear natural operators $T^* \to T^*T^{(r)}$ is a finitely dimensional vector space over $\mathbf{R}$. We construct explicitly the bases of the vector spaces. As a corollary we find all linear natural operators $T^* \to T^{r*}$.

All manifolds and maps are assumed to be infinitely differentiable.

0. Let r, n be fixed natural numbers. Given a manifold M we denote by $T^{r*}M = J^r(M, \mathbf{R})_0$ the space of all r-jets of maps $M \to \mathbf{R}$ with target 0. This is a vector bundle over M with the source projection. The dual vector bundle $(T^{r*}M)^*$ of $T^{r*}M$ is denoted by $T^{(r)}M$ and called the linear r-tangent bundle of M. We denote the fibre of $T^{r*}M$ and $T^{(r)}M$ over x by T^{r*}_xM and $T^{(r)}_xM$ respectively. Every embedding $\varphi : M \to N$ of two n-manifolds induces a vector bundle homomorphism $T^{r*}\varphi : T^{r*}M \to T^{r*}N$ over φ defined by $T^{r*}\varphi(j^r_x\gamma) = j^r_{\varphi(x)}(\gamma \circ \varphi^{-1})$ for any $\gamma : M \to \mathbf{R}$ and any $x \in M$ with $\gamma(x) = 0$, where by $j^r_x\gamma$ we denote the r-jet of γ at x. This embedding induces also a vector bundle homomorphism $T^{(r)}\varphi : T^{(r)}M \to T^{(r)}N$ over φ dual to $T^{r*}\varphi^{-1}$, i.e. $T^{(r)}\varphi(\Theta)(j^r_{\varphi(x)}\eta) = \Theta(j^r_x(\eta \circ \varphi))$ for any $\Theta \in T^{(r)}_xM$, any $x \in M$ and any $j^r_{\varphi(x)}\eta \in T^{r*}_{\varphi(x)}N$, cf. [4].

In this paper we study the problem how a 1-form ω on a manifold M can induce a 1-form on $T^{(r)}M$ and a section of $T^{r*}M \to M$. This problem is reflected in the concept of linear natural operators $T^* \to T^*T^{(r)}$ and $T^* \to T^{r*}$, cf. [4]. In the fundamental monograph [4] there is a very general definition of natural operators. We restrict ourselves to the following one.

Definition 0.1. Let r, n be fixed natural numbers. Let $\Omega^1(M)$ denotes the vector space of all 1-forms on M and $\Gamma T^{r*}M$ denotes the vector space of all sections of

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$T^{r*}M \to M$. *A linear natural operator* $A : T^* \to T^*T^{(r)}$ *(or* $A : T^* \to T^{r*}$*)* is a system of $\mathbf{R}$-linear functions

$$A_M : \Omega^1(M) \to \Omega^1(T^{(r)}M) \qquad (\text{or } A_M : \Omega^1(M) \to \Gamma T^{r*}M\ )\ ,$$

for any n-manifold M, such that A is invariant with respect to φ for any embedding $\varphi : M \to N$ of two n-manifolds, i.e. $A_M(\varphi^*\omega) = (T^{(r)}\varphi)^* A_N(\omega)$ (or $T^{r*}\varphi \circ A_M(\varphi^*\omega) = A_N(\omega) \circ \varphi$) for any $\omega \in \Omega^1(N)$, where $\varphi^*\omega \in \Omega^1(M)$, $(\varphi^*\omega)_x := \omega_{\varphi(x)} \circ T_x\varphi$, $x \in M$ is the pull-back of ω with respect to φ.

The set of all linear natural operators $T^* \to T^*T^{(r)}$ (or $T^* \to T^{r*}$) is a vector space over $\mathbf{R}$. If $A, B : T^* \to T^*T^{(r)}$ (or $A, B : T^* \to T^{r*}$) are two linear natural operators and $\alpha, \beta \in \mathbf{R}$, then $\alpha A + \beta B : T^* \to T^*T^{(r)}$ (or $\alpha A + \beta B : T^* \to T^{r*}$) is defined by $(\alpha A + \beta B)_M(\omega) = \alpha(A_M(\omega)) + \beta(B_M(\omega))$ for any n-manifold M and any $\omega \in \Omega^1(M)$.

This paper is dedicated to prove the following theorem.

Theorem 0.1. *Let* $r, n \in \mathbf{N}$. *For* n*-manifolds the vector space of all linear natural operators* $T^* \to T^*T^{(r)}$ *is of dimension 1, if* $r \geq 3$ *and* $n \geq 2$*, of dimension 2, if* $r = 2$ *or* $n = 1$*, and of dimension 3, if* $n \geq 2$ *and* $r = 1$.

In the proof of the theorem we construct explicitly the bases of the vector spaces (see Theorem 1.1).

Similar problems are studied in papers [1]-[3], [5]-[8].

In item 1 we describe main examples of linear natural operators $T^* \to T^*T^{(r)}$ and we formulate the main theorem. In item 2 we prove that the natural operators described in item 1 are linearly independent in the vector space of all linear natural operators $T^* \to T^*T^{(r)}$. In item 3 we prove a reducibility lemma. Some transformation rules are presented in item 4. The proof of the main theorem is given in item 5. In item 6, the following fact will be proved as a consequence of Theorem 0.1.

Corollary 0.1. *Let* $r, n \in \mathbf{N}$. *For* n*-manifolds the vector space of all linear natural operators* $T^* \to T^{r*}$ *is of dimension 0, if* $r \geq 3$ *and* $n \geq 2$*, and of dimension 1, if* $r \leq 2$ *or* $n = 1$.

1. In this item we describe main examples of linear natural operators $T^* \to T^*T^{(r)}$ and we formulate the main theorem.

From now on the usual coordinates on $\mathbf{R}^n$ are denoted by $x^1, ..., x^n$. The canonical 1-forms on $\mathbf{R}^n$ induced by $x^1, ..., x^n$ are denoted by $dx^1, ..., dx^n$.

We start with the following obvious example.

Example 1.1. The family of functions

$$\Omega^1(M) \ni \omega \to \pi_M^*(\omega) \in \Omega^1(T^{(r)}M)\ ,$$

where M is an n-manifold, $\pi_M : T^{(r)}M \to M$ is the projection and $\pi_M^*(\omega)$ is the pull-back of ω with respect to π_M, is a linear natural operator $T^* \to T^*T^{(r)}$ in the sense of Definition 0.1. This operator is denoted by π^*.

To give other examples of linear natural operators $T^* \to T^*T^{(r)}$ we need some preparations. In Example 1.2 we use the following two lemmas.

Lemma 1.1. *Let $r \in \mathbf{N}$. Then there exists one and only one linear natural operator $\tilde{A}^{(r)} : T^* \to T^{r*}$ such that*

$$(\tilde{A}^{(r)}_M(df))(x_o) = j^r_{x_o}(f - f(x_o)) \tag{1.1}$$

for any 1-manifold M, any $x_o \in M$ and any $f : M \to \mathbf{R}$, where $df \in \Omega^1(M)$ denotes the differential of f.

Proof. By the Poincare lemma any element of $(J^{r-1}T^*)_0\mathbf{R} = \{j^{r-1}_0\omega : \omega \in \Omega^1(\mathbf{R})\}$ is of the form $j^{r-1}_0(df)$, where $f : \mathbf{R} \to \mathbf{R}$, $f(0) = 0$. If $j^{r-1}_0(df) = j^{r-1}_0(dh)$, where $f, h : \mathbf{R} \to \mathbf{R}$, $f(0) = h(0) = 0$, then $j^{r-1}_0(f') = j^{r-1}_0(h')$, i.e. $j^r_0 f = j^r_0 h$. Thus the mapping

$$\alpha^r : (J^{r-1}T^*)_0\mathbf{R} \to T^{r*}_0\mathbf{R}\ , \qquad \alpha^r(j^{r-1}_0(df)) = j^r_0 f\ ,$$

where $f : \mathbf{R} \to \mathbf{R}$, $f(0) = 0$, is well-defined. Obviously α^r is linear. If $\varphi : \mathbf{R} \to \mathbf{R}$ is a diffeomorphism preserving 0, then

$$\begin{aligned}(\alpha^r \circ ((J^{r-1}T^*)_0\varphi^{-1}))(j^{r-1}_0(df)) &= \alpha^r(j^{r-1}_0(d(f \circ \varphi))) \\ &= j^r_0(f \circ \varphi) = ((T^{r*}\varphi^{-1})_0 \circ \alpha^r)(j^{r-1}_0(df))\ ,\end{aligned}$$

where $(J^{r-1}T^*)_0\varphi^{-1} : (J^{r-1}T^*)_0\mathbf{R} \to (J^{r-1}T^*)_0\mathbf{R}$ is given by $j^{r-1}_0\omega \to j^{r-1}_0(\varphi^*\omega)$ and $(T^{r*}\varphi^{-1})_0$ is the restriction of $T^{r*}\varphi^{-1}$ to the fibre over 0. Hence α^r is L^r_1-equivariant.

According to the general theory of natural operators, cf. [4], there exists one and only one linear natural operator $\tilde{A}^{(r)} : T^* \to T^{r*}$ corresponding to α^r, i.e such that $\tilde{A}^{(r)}_{\mathbf{R}}(\omega)(0) = \alpha^r(j^{(r-1)}_0\omega)$ for any $\omega \in \Omega^1(\mathbf{R})$. (Namely

$$(\tilde{A}^{(r)}_M(\omega))(x) = T^{r*}\varphi^{-1}(\alpha^r(j^{r-1}_0((\varphi^{-1})^*\omega)))$$

for any $\omega \in \Omega^1(M)$ and any $x \in M$, where φ is a chart on M such that $\varphi(x) = 0$. This definition is correct because of the equivariancy of α^r.)

We see that

$$(\tilde{A}^{(r)}_{\mathbf{R}}(dg))(0) = \alpha^r(j^{r-1}_0(dg)) = j^r_0(g - g(0))$$

for any $g : \mathbf{R} \to \mathbf{R}$. Now, using the invariancy of both sides of (1.1) with respect to charts we obtain (1.1) for any 1-manifold M, any $x_o \in M$ and any $f : M \to \mathbf{R}$. $\square$

Lemma 1.2. *Let $n \in \mathbf{N}$. There exists one and only one linear natural operator $\tilde{B}^{(n)} : T^* \to T^{2*}$ such that*

$$(\tilde{B}^{(n)}_M(f dg))(x_o) = \frac{1}{2} j^2_{x_o}((f + f(x_o))(g - g(x_o))) \tag{1.2}$$

for any n-manifold M, any $x_o \in M$ and any $f, g : M \to \mathbf{R}$.

Proof. The mapping $\beta^n : (J^1T^*)_0\mathbf{R}^n \to T^{2*}_0\mathbf{R}^n$ given by

$$\beta^n(j^1_0(\sum_{i=1}^n f_i dx^i)) = j^2_0(\sum_{i=1}^n f_i(0)x^i + \frac{1}{2}\sum_{i,k=1}^n \frac{\partial f_i}{\partial x^k}(0)x^i x^k)$$

is well-defined and linear. We are going to show that β^n is L^2_n-equivariant, i.e. that for any diffeomorphism $\varphi = (\varphi^1, ..., \varphi^n) : \mathbf{R}^n \to \mathbf{R}^n$ preserving $0 \in \mathbf{R}^n$ we have

$$\beta^n \circ ((J^1T^*)_0\varphi^{-1}) = (T^{2*}\varphi^{-1})_0 \circ \beta^n \ . \tag{1.3}$$

Since both sides of (1.3) are linear, then without loss of generality it is sufficient to verify equality (1.3) at $j^1_0(dx^1)$, $j^1_0(x^1dx^1)$ and $j^1_0(x^2dx^1)$. We have

$$\begin{aligned}
&(\beta^n \circ ((J^1T^*)_0\varphi^{-1}))(j^1_0(dx^1)) = \beta^n(j^1_0(d\varphi^1)) \\
&= \beta^n(j^1_0(\sum_{i=1}^n \frac{\partial\varphi^1}{\partial x^i}dx^i)) \\
&= j^2_0(\sum_{i=1}^n \frac{\partial\varphi^1}{\partial x^i}(0)x^i + \frac{1}{2}\sum_{i,k=1}^n \frac{\partial^2\varphi^1}{\partial x^i\partial x^k}(0)x^i x^k)
\end{aligned}$$

and

$$\begin{aligned}
&((T^{2*}\varphi^{-1})_0 \circ \beta^n)(j^1_0(dx^1)) = (T^{2*}\varphi^{-1})_0(j^2_0(x^1)) = j^2_0(\varphi^1) \\
&= j^2_0(\sum_{i=1}^n \frac{\partial\varphi^1}{\partial x^i}(0)x^i + \frac{1}{2}\sum_{i,k=1}^n \frac{\partial^2\varphi^1}{\partial x^i\partial x^k}(0)x^i x^k) \ .
\end{aligned}$$

Similarly for $\tau = 1, 2$ we have

$$\begin{aligned}
&(\beta^n \circ ((J^1T^*)_0\varphi^{-1}))(j^1_0(x^\tau dx^1)) = \beta^n(j^1_0(\varphi^\tau d\varphi^1)) \\
&= \beta^n(j^1_0(\sum_{i=1}^n \varphi^\tau \frac{\partial\varphi^1}{\partial x^i}dx^i)) \\
&= j^2_0(\frac{1}{2}\sum_{k,i=1}^n \frac{\partial\varphi^\tau}{\partial x^k}(0)\frac{\partial\varphi^1}{\partial x^i}(0)x^i x^k)
\end{aligned}$$

and

$$
\begin{aligned}
((T^{2*}\varphi^{-1})_0 \circ \beta^n)(j_0^1(x^\tau dx^1)) &= (T^{2*}\varphi^{-1})_0(j_0^2(\frac{1}{2}x^\tau x^1)) \\
&= j_0^2(\frac{1}{2}\varphi^\tau \varphi^1) \\
&= j_0^2(\frac{1}{2}\sum_{k,i=1}^{n} \frac{\partial \varphi^\tau}{\partial x^k}(0)\frac{\partial \varphi^1}{\partial x^i}(0)x^i x^k) \ .
\end{aligned}
$$

Thus β^n is L_n^2-equivariant.

Let $\tilde{B}^{(n)} : T^* \to T^{2*}$ be the linear natural operator corresponding to β^n, i.e. such that $\tilde{B}^{(n)}_{\mathbf{R}^n}(\omega)(0) = \beta^n(j_0^1\omega)$ for any $\omega \in \Omega(\mathbf{R}^n)$. We will prove (1.2) for any n-manifold M, any $x_o \in M$ and any $f, g : M \to \mathbf{R}$. Using the invariancy of both sides of (1.2) with respect to embeddings and the linearity of both sides of (1.2) with respect to g we can assume that $M = \mathbf{R}^n$, $x_o = 0$ and $g = x^1$. Then

$$
\begin{aligned}
(\tilde{B}^{(n)}_{\mathbf{R}^n}(f dx^1))(0) = \beta^n(j_0^1(f dx^1)) &= j_0^2(f(0)x^1 + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(0)x^i x^1) \\
&= \frac{1}{2} j_0^2((f + f(0))x^1) \, .
\end{aligned}
$$

The lemma is proved. □

Remark 1.1. (a) For 1-manifolds we have $\tilde{B}^{(1)} = \tilde{A}^{(2)}$, where $\tilde{A}^{(2)}$ is described in Lemma 1.1 and $\tilde{B}^{(1)}$ is described in Lemma 1.2.

(b) Let $n, r \geq 2$ and $a \in \mathbf{R}$. We have $x^2 dx^1 = \frac{x^2}{1+x^1} d(x^1 + \frac{1}{2}(x^1)^2)$ near $(1, 1, 0, ..., 0) \in \mathbf{R}^n$ and

$$
j^r_{(1,1,0,\dots,0)}((x^2 + a)(x^1 - 1)) \neq j^r_{(1,1,0,\dots,0)}((\frac{x^2}{1+x^1} + \frac{a}{2})(x^1 + \frac{1}{2}(x^1)^2 - 1,5))
$$

if either $r \geq 3$ or $a \neq 1$. (It is sufficient to consider the coordinates of these jets corresponding to $\frac{\partial^2}{\partial (x^1)^2}$ and $\frac{\partial^3}{\partial x^2 (\partial x^1)^2}$.) Thus if there exists a linear natural operator $\tilde{B} : T^* \to T^{r*}$ such that

$$
\tilde{B}_M(f dg) = j^r_{x_o}((f + a f(x_o))(g - g(x_o)))
$$

for any n-manifold M,any $x_o \in M$ and any $f, g : M \to \mathbf{R}$, then $a = 1$ and $r = 2$.

We are now in position to give some examples of linear natural operators $T^* \to T^*T^{(r)}$.

Example 1.2. (I) Let $\tilde{A} : T^* \to T^{r*}$ be a linear natural operator. Then for any $\omega \in \Omega^1(M)$ we have a section $\tilde{A}_M(\omega) : M \to T^{r*}M$. This section can be interpreted as a mapping $\overline{A}_M(\omega) : T^{(r)}M \to \mathbf{R}$, $(\overline{A}_M(\omega))(\Theta) = \Theta((\tilde{A}_M(\omega))(x))$, where $\Theta \in T^{(r)}_x M$, $x \in M$. The family of functions

$$
(1.4) \qquad \Omega^1(M) \ni \omega \to A_M(\omega) := d(\overline{A}_M(\omega)) \in \Omega^1(T^{(r)}M) \ ,
$$

where M is an n-manifold, is a linear natural operator $T^* \to T^*T^{(r)}$ in the sense of Definition 0.1. We apply this general construction to some linear natural operators $T^* \to T^{r*}$.

(a) Applying this construction to $\tilde{A}^{(r)} : T^* \to T^{r*}$ described in Lemma 1.1 we get a linear natural operator $A^{(r)} : T^* \to T^*T^{(r)}$ for 1-manifolds.

(b) Similarly, applying this construction to $\tilde{B}^{(n)} : T^* \to T^{2*}$ described in Lemma 1.2 we get a linear natural operator $B^{(n)} : T^* \to T^*T^{(2)}$ for n-manifolds .

(c) Using the natural vector bundle isomorphism $i_M : T^{1*}M \to T^*M$, $i_M(j^1_x\gamma) := d_x\gamma$, we define a linear natural operator $\tilde{C}^{(n)} : T^* \to T^{1*}$, $\tilde{C}^{(n)}_M(\omega) := i^{-1}_M \circ \omega$, for any n-manifold M and any $\omega \in \Omega^1(M)$. Oving to the general construction described above this operator induces a linear natural operator $C^{(n)} : T^* \to T^*T^{(1)}$ for n-manifolds.

(II) Let $i^*_M : TM \to T^{(1)}M$ be the natural vector bundle isomorphism dual to $i_M : T^{1*}M \to T^*M$. We define a linear natural operator $D^{(n)} : T^* \to T^*T^{(1)}$ by

$$D^{(n)}_M(\omega) = ((i^*_M)^{-1})^*\omega^C - C^{(n)}_M(\omega)$$

for any n-manifold M and any $\omega \in \Omega^1(M)$, where $\omega^C \in \Omega^1(TM)$ denotes the complete lift of ω to TM in the sense of [9] and $C^{(n)}$ is described above.

The main result of this paper is the following theorem corresponding to Theorem 0.1.

Theorem 1.1. (a) *If $n \geq 2$ and $r \geq 3$, then the linear natural operator π^* described in Example 1.1 forms a basis of the vector space of all linear natural operators $T^* \to T^*T^{(r)}$ for n-manifolds.*

(b) *If $n \geq 2$, then the linear natural operators π^* and $B^{(n)}$ described in Examples 1.1 and 1.2 form a basis of the vector space of all linear natural operators $T^* \to T^*T^{(2)}$ for n-manifolds.*

(c) *If $n \geq 2$, then the linear natural operators $\pi^*, C^{(n)}$ and $D^{(n)}$ described in Examples 1.1 and 1.2 form a basis of the vector space of all linear natural operators $T^* \to T^*T^{(1)}$ for n-manifolds.*

(d) *If $r \geq 1$, then the linear natural operators π^* and $A^{(r)}$ described in Examples 1.1 and 1.2 form a basis of the vector space of all linear natural operators $T^* \to T^*T^{(r)}$ for 1-manifolds .*

2. The linear natural operators described in Examples 1.1 and 1.2 have the following properties.

Lemma 2.1. (a) *Let $r \geq 1$. Then $\pi*$ and $A^{(r)}$ are linearly independent in the vector space of all linear natural operators $T^* \to T^*T^{(r)}$ for 1-manifolds.*

(b) *Let $n \geq 2$. Then π^* and $B^{(n)}$ are linearly independent in the vector space of all linear natural operators $T^* \to T^*T^{(2)}$ for n-manifolds.*

(c) *Let $n \geq 2$. Then $\pi^*, C^{(n)}, D^{(n)}$ are linearly independent in the vector space of all linear natural operators $T^* \to T^*T^{(1)}$ for n-manifolds.*

Proof. At first we make some preparations. Let $VT^{(r)}M := ker(T\pi_M) \subset TT^{(r)}M$ be the vertical distribution. We see that:

(1) $A^{(r)}_{\mathbf{R}}(dx^1)(v) \neq 0$ for some $v \in VT^{(r)}\mathbf{R} \cap (TT^{(r)})_0\mathbf{R}$. For, $(\tilde{A}^{(r)}_{\mathbf{R}}(dx^1))(0) = j^r_0(x^1) \neq 0$, and hence $\overline{A}^{(r)}_{\mathbf{R}}(dx^1)|T^{(r)}_0\mathbf{R}$ is not constant.

(2) Similarly, $B^{(n)}_{\mathbf{R}^n}(dx^1)(v) \neq 0$ for some $v \in VT^{(2)}\mathbf{R}^n \cap (TT^{(2)})_0\mathbf{R}^n$, and $C^{(n)}_{\mathbf{R}^n}(dx^1)(v) \neq 0$ for some $v \in VT^{(1)}\mathbf{R}^n \cap (TT^{(1)})_0\mathbf{R}^n$.

(3) $\pi^*_M(\omega)(v) = 0$ for any n-manifold M, any $\omega \in \Omega^1(M)$ and any $v \in VT^{(r)}M$.

(4) $D^{(n)}_M(\omega)(v) = 0$ for any n-manifold M, any $\omega \in \Omega^1(M)$ and any $v \in VT^{(1)}M$.

For let us consider a vector field X on M, a point $y \in T_xM$, $x \in M$, such that $X^V(y) = v$, where X^V is the vertical lift of X to TM, cf. [9]. Then $\omega^C(X^V)(y) = (\omega(X))(x)$ and

$$(((i^*_M)^*(C^{(n)}_M(\omega)))(X^V))(y) = \frac{d}{dt}\omega(y + tX(x))|_{t=0} = (\omega(X))(x) \ .$$

Hence $D^{(n)}_M(\omega)(v) = 0$.

(5) $D^{(n)}_{\mathbf{R}^n}(x^2dx^1)(i^*_{\mathbf{R}^n}(v_o)) \neq 0$, where $v_o = \partial_1(0) \in T_0\mathbf{R}^n$ and $i^*_M : TM \to T^{(1)}M$ is the natural vector bundle isomorphism (see Example 1.2 (II)).
For, $((x^2dx^1)^C(\partial^C_2))(v_o) = ((x^2dx^1)(\partial_2))^C(v_o) = 0$ and

$$\begin{aligned}(((i^*_M)(C^{(n)}_{\mathbf{R}^n}(x^2dx^1)))(\partial^C_2))(v_o) &= \frac{d}{dt}((x^2dx^1)(T\tau_{(0,t,0,\dots,0)}(v_o)))|_{t=0} \\ &= \frac{d}{dt}(t)|_{t=0} = 1\,,\end{aligned}$$

where $\tau_{(0,t,0,\dots,0)} : \mathbf{R}^n \to \mathbf{R}^n$ is the translation by $(0,t,0,...,0)$ (the flow of ∂_2) and ∂^C_2 is the complete lift of ∂_2 to $T\mathbf{R}^n$, i.e. the vector field on $T\mathbf{R}^n$ generated by the flow $T\tau_{(0,t,0,\dots,0)}$. Hence $D^{(n)}_{\mathbf{R}^n}(x^2dx^1) \neq 0$ at $i^*_M(v_o)$.

(6) $\pi^*_{\mathbf{R}^n}(x^2dx^1) = 0$ on the fibre $T^{(1)}_0\mathbf{R}^n$.

Using these facts one can prove the lemma as follows. We prove only part (c). Suppose that $\alpha\pi^* + \beta C^{(n)} + \gamma D^{(n)} = 0$ for some $\alpha, \beta, \gamma \in \mathbf{R}$. Since $\pi^*_{\mathbf{R}^n}(dx^1)(v) = 0$, $D^{(n)}_{\mathbf{R}^n}(dx^1)(v) = 0$ and $C^{(n)}_{\mathbf{R}^n}(dx^1)(v) \neq 0$ for some $v \in VT^{(1)}\mathbf{R}^n \cap (TT^{(1)})_0\mathbf{R}^n$, then $\beta = 0$. Next, since $D^{(n)}_{\mathbf{R}^n}(x^2dx^1) \neq 0$ at $i^*_{\mathbf{R}^n}(v_o)$ and $\pi^*_{\mathbf{R}^n}(x^2dx^1) = 0$ at $i^*_{\mathbf{R}^n}(v_o)$, then $\gamma = 0$. Then, since $\pi^* \neq 0$, $\alpha = 0$. □

3. In this item we prove the following reducibility lemma.

Lemma 3.1. *Let $A, B : T^* \to T^*T^{(r)}$ be two linear natural operators for n-manifolds. Suppose that $A_{\mathbf{R}^n}(\omega_o) = B_{\mathbf{R}^n}(\omega_o)$, where $\omega_o = x^2dx^1$, if $n \geq 2$, and $\omega_o = dx^1$, if $n = 1$. Then $A = B$.*

Proof. . At first we assume that $n \geq 2$. Consider a mapping $f : \mathbf{R}^n \to \mathbf{R}$. Let $y \in \mathbf{R}^n$. There is $\tau \in \mathbf{R}$ such that $\frac{\partial(f+\tau x^2)}{\partial x^2}(y) \neq 0$. Using the assumption of the lemma and the invariancy of A and B with respect to the local diffeomorphism $(x^1, f + \tau x^2, x^3, ..., x^n)$ defined on some neighbourhood of y we deduce that $A_{\mathbf{R}^n}((f + \tau x^2)dx^1) = B_{\mathbf{R}^n}((f+\tau x^2)dx^1)$ over y. Then by the linearity of A and B we deduce that $A_{\mathbf{R}^n}(fdx^1) = B_{\mathbf{R}^n}(fdx^1)$ over y. Hence $A_{\mathbf{R}^n}(fdx^1) = B_{\mathbf{R}^n}(fdx^1)$ for any $f : \mathbf{R}^n \to \mathbf{R}$. Next using the invariancy of A and B with respect to diffeomorphisms which permute coordinates we deduce that $A_{\mathbf{R}^n}(fdx^i) = B_{\mathbf{R}^n}(fdx^i)$ for any $f : \mathbf{R}^n \to \mathbf{R}$ and any $i = 1, ..., n$. Thus by the linearity of A and B we obtain that $A_{\mathbf{R}^n}(\omega) = B_{\mathbf{R}^n}(\omega)$ for any $\omega \in \Omega^1(\mathbf{R}^n)$. Then $A = B$ because of the invariancy of A and B with respect to charts.

Now, let $n = 1$. Consider $\omega \in \Omega^1(\mathbf{R})$ and $y \in \mathbf{R}^n$. There is $\tau \in \mathbf{R}$ such that $(\omega + \tau\omega_o)(y) \neq 0$. There is a local diffeomorphism defined on some neighbourhood of y which transforms $germ_y(\omega + \tau\omega_o)$ into $germ_0(\omega_o)$. By the assumption of the lemma and the invariancy of A and B we deduce that $A_{\mathbf{R}}(\omega+\tau\omega_o) = B_{\mathbf{R}}(\omega+\tau\omega_o)$ over y. Then by the linearity of A and B we get $A_{\mathbf{R}}(\omega) = B_{\mathbf{R}}(\omega)$ over y. Hence $A = B$ because of the invariancy of A and B with respect to charts. □

Remark 3.1. This lemma is true for every natural bundle E instead of $T^{(r)}$.

4. Let $S = \{\alpha = (\alpha_1, ..., \alpha_n) \in (\mathbf{N} \bigcup \{0\})^n : 1 \leq |\alpha| \leq r\}$. On $T^{(r)}\mathbf{R}^n$ we have the coordinates (x^i, X^α), $i = 1, ..., n, \alpha \in S$, given by

$$x^i(\Theta) = x^i_o, \qquad X^\alpha(\Theta) = \Theta(j^r_{x_o}((x - x_o)^\alpha)) \ , \tag{4.0}$$

where $\Theta \in T^{(r)}_{x_o}\mathbf{R}^n$ and $x_o = (x^1_o, ..., x^n_o) \in \mathbf{R}^n$. We shall identify $\mathbf{R}^n \times \mathbf{R}^S$ with $T^{(r)}\mathbf{R}^n$ by these coordinates.

We prove the following lemma.

Lemma 4.1. *Let $n \geq 2$. Let G be a local diffeomorphism defined on some neighbourhood U of $0 \in \mathbf{R}^n$ by $G = ((x^1 - \frac{1}{2}(x^1)^2, \frac{x^2}{1-x^1}, x^3, ..., x^n)$. Then:*

$$X^{(1,0,...,0)} \circ T^{(1)}G = (1 - x^1)X^{(1,0,...,0)} \tag{4.1}$$

$$X^{(0,1,0,...,0)} \circ T^{(1)}G = \frac{x^2}{(1 - x^1)^2}X^{(1,0,...,0)} + \frac{1}{1 - x^1}X^{(0,1,0,...,0)} \tag{4.2}$$

over U, if $r = 1$;

$$X^{(1,1,0,...,0)} \circ T^{(2)}G = \frac{x^2}{1 - x^1}X^{(2,0,...,0)} + X^{(1,1,0,...,0)} \ , \tag{4.3}$$

$$X^{(1,0,\dots,0)} \circ T^{(2)}G = (1-x^1)X^{(1,0,\dots,0)} - \frac{1}{2}X^{(2,0,\dots,0)} \ , \tag{4.4}$$

$$\begin{aligned} X^{(0,1,0,\dots,0)} \circ T^{(2)}G = {} & \frac{x^2}{(1-x^1)^2}X^{(1,0,\dots,0)} + \frac{1}{1-x^1}X^{(0,1,0,\dots,0)} \\ & + \frac{x^2}{(1-x^1)^3}X^{(2,0,\dots,0)} + \frac{1}{(1-x^1)^2}X^{(1,1,0,\dots,0)} \end{aligned} \tag{4.5}$$

over U, if $r=2$; and

$$X^{(1,1,0,\dots,0)} \circ T^{(3)}G = X^{(1,1,0,\dots,0)} + \frac{1}{2}X^{(2,1,0,\dots,0)} \tag{4.6}$$

on the fibre over $0 \in \mathbf{R}^n$, if $r=3$.

Proof. Let $x_o = (x_o^1, \dots, x_o^n) \in U$ and $\Theta \in T^{(2)}_{x_o}\mathbf{R}^n$. We see that

$$j^2_{x_o}[G^1 - G^1(x_o)] = j^2_{x_o}[(1-x_o^1)(x^1-x_o^1) - \frac{1}{2}(x^1-x_o^1)^2] \text{ and} \tag{4.7}$$

$$\begin{aligned} j^2_{x_o}[G^2 - G^2(x_o)] = j^2_{x_o}[& \frac{x_o^2}{(1-x_o^1)^2}(x^1-x_o^1) + \frac{1}{1-x_o^1}(x^2-x_o^2) \\ & + \frac{x_o^2}{(1-x_o^1)^3}(x^1-x_o^1)^2 \\ & + \frac{1}{(1-x_o^1)^2}(x^1-x_o^1)(x^2-x_o^2)] \end{aligned} \tag{4.8}$$

$$\begin{aligned} & j^2_{x_o}[(G^1-G^1(x_o))(G^2-G^2(x_o))] \\ & \qquad = j^2_{x_o}[\frac{x_o^2}{1-x_o^1}(x^1-x_o^1)^2 + (x^1-x_o^1)(x^2-x_o^2)] \end{aligned} \tag{4.9}$$

$$j^3_0(G_1G_2) = j^3_0(x^1x^2 + \frac{1}{2}x^2(x^1)^2) \ , \tag{4.10}$$

where $G = (G^1, G^2, \dots, G^n)$. Then for $\Theta \in T^{(2)}_{x_o}\mathbf{R}^n$ we have

$$\begin{aligned} (X^{(1,1,0,\dots,0)} \circ T^{(2)}G)(\Theta) &= (T^{(2)}G(\Theta))(j^2_{G(x_o)}[(x^1-G^1(x_o))(x^2-G^2(x_o))]) \\ &= \Theta(j^2_{x_o}[(G^1-G^1(x_o))(G^2-G^2(x_o))]) \\ &= \Theta(j^2_{x_o}[\frac{x_o^2}{1-x_o^1}(x^1-x_o^1)^2 + (x^1-x_o^1)(x^2-x_o^2)]) \\ &= (\frac{x^2}{1-x^1}X^{(2,0,\dots,0)})(\Theta) + X^{(1,1,0,\dots,0)}(\Theta) \end{aligned}$$

because of (4.9). It implies (4.3). Similarly, from (4.7) and (4.8) it follows (4.1), (4.4) and (4.2), (4.5) respectively. From (4.10) it follows (4.6). □

5. Proof of Theorem 1.1. Let $A : T^* \to T^*T^{(r)}$ be a linear natural operator. After the identification $T^{(r)}\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^S$ (see Item 4) we can write

$$A_{\mathbf{R}^n}(\omega_o) = \sum_{i=1}^{n} f_i dx^i + \sum_{\alpha \in S} f_\alpha dX^\alpha \tag{5.0}$$

for some mappings $f_i, f_\alpha : \mathbf{R}^n \times \mathbf{R}^S \to \mathbf{R}$, where ω_o is as in Lemma 3.1. We consider two cases.

(I) At first we assume that $n \geq 2$. Using the invariancy of A with respect to the diffeomorphisms $t\ id$, $(x^1, tx^2, ..., tx^n)$, $(tx^1, x^2, tx^3, ..., tx^n)$, $(x^1, x^2, tx^3, ..., tx^n)$ and $(x^1 + 1, x^2, ..., x^n)$ and the linearity of A we obtain

$$t^2 f_\alpha(x^j, X^\beta) = t^{|\alpha|} f_\alpha(tx^j, t^{|\beta|} X^\beta)\ , \tag{5.1}$$

$$t f_\alpha(x^j, X^\beta) = t^{|\alpha| - \alpha_1} f_\alpha(t^{1-\delta_{1j}} x^j, t^{|\beta| - \beta_1} X^\beta)\ , \tag{5.2}$$

$$t f_\alpha(x^j, X^\beta) = t^{|\alpha| - \alpha_2} f_\alpha(t^{1-\delta_{2j}} x^j, t^{|\beta| - \beta_2} X^\beta)\ , \tag{5.3}$$

$$f_\alpha(x^j, X^\beta) = t^{|\alpha| - \alpha_1 - \alpha_2} f_\alpha(t^{1-\delta_{1j}-\delta_{2j}} x^j, t^{|\beta| - \beta_1 - \beta_2} X^\beta)\ , \tag{5.4}$$

$$f_\alpha(x^j, X^\beta) = f_\alpha(x^j + \delta_{1j}, X^\beta)\ , \tag{5.5}$$

$$t^2 f_i(x^j, X^\beta) = t f_i(tx^j, t^{|\beta|} X^\beta)\ , \tag{5.6}$$

$$t f_i(x^j, X^\beta) = t^{1-\delta_{1i}} f_i(t^{1-\delta_{1j}} x^j, t^{|\beta| - \beta_1} X^\beta)\ , \tag{5.7}$$

$$t f_i(x^j, X^\beta) = t^{1-\delta_{2i}} f_i(t^{1-\delta_{2j}} x^j, t^{|\beta| - \beta_2} X^\beta)\ , \tag{5.8}$$

$$f_i(x^j, X^\beta) = t^{1-\delta_{1i}-\delta_{2i}} f_i(t^{1-\delta_{1j}-\delta_{2j}} x^j, t^{|\beta| - \beta_1 - \beta_2} X^\beta)\ , \tag{5.9}$$

$$f_i(x^j, X^\beta) = f_i(x^j + \delta_{1j}, X^\beta)\ , \tag{5.10}$$

for any $t \in \mathbf{R} - \{0\}$, any $(x^j, X^\beta) = (x^j, X^\beta)_{j=1,\dots,n,\beta\in S} \in \mathbf{R}^n \times \mathbf{R}^S$, any $\alpha \in S$ and any $i = 1, ..., n$, where δ_{ij} is the Cronecker delta. (For example,

$$t^2 A_{\mathbf{R}^n}(\omega_o) = A_{\mathbf{R}^n}(t^2\omega_o) = A_{\mathbf{R}^n}((t\ id)^*\omega_o) = (T^{(r)}(t\ id))^*(A_{\mathbf{R}^n}(\omega_o))$$

and $X^\alpha \circ T^{(r)}(t\ id) = t^{|\alpha|}X^\alpha$ for any $\alpha \in S$ and any $t \in \mathbf{R} - \{0\}$. It implies (5.1) and (5.6). Similarly one can prove (5.2) – (5.5) and (5.7) – (5.10).)

From (5.1), (5.2), (5.3) and (5.4) it follows that $f_\alpha = 0$, if $|\alpha| \geq 3$ or $|\alpha| - \alpha_1 \geq 2$ or $|\alpha| - \alpha_2 \geq 2$ or $|\alpha| - \alpha_1 - \alpha_2 \geq 1$, i.e. if $\alpha \in S - \{(1,1,0,...,0), (1,0,...,0), (0,1,0,...,0)\}$.

From (5.1) we get that $f_{(1,1,0\dots,0)}$ does not depend on x^j and X^β for any $j = 1, ..., n$ and any $\beta \in S$.

By (5.1) and the homogeneous function theorem, cf. [4], we obtain that $f_{(1,0,\dots,0)}$ and $f_{(0,1,0,\dots,0)}$ are linear in x^j and X^β with $|\beta| = 1$ and they are independent of X^β with $|\beta| \geq 2$. Next, using (5.4) and (5.5) we see that they are independent of $x^1, x^3, ..., x^n$ and $X^{(0,0,1,0,\dots,0)}, \dots, X^{(0,\dots,0,1)}$. Using (5.3) we deduce that $f_{(1,0,\dots,0)}$ is independent of $X^{(1,0,\dots,0)}$. Similarly, $f_{(0,1,0,\dots,0)}$ is independent of $X^{(0,1,0,\dots,0)}$ and x^2, because of (5.2).

Using similar arguments it folows from (5.6) – (5.10) that $f_3 = ... = f_n = 0$, f_1 depends linearly on x^2 and $X^{(0,1,0,\dots,0)}$ and it is independent of the other x^j and X^β, and f_2 depends linearly on $X^{(1,0,\dots,0)}$ and it is independent of the other X^β and x^j.

Hence

$$\begin{aligned} A_{\mathbf{R}^n}(\omega_o) &= \mu_1 dX^{(1,1,0,\dots,0)} + (\mu_2 x^2 + \mu_3 X^{(0,1,0,\dots,0)})dX^{(1,0,\dots,0)} \\ &+ \mu_4 X^{(1,0,\dots,0)} dX^{(0,1,0,\dots,0)} + (\mu_5 x^2 + \mu_6 X^{(0,1,0,\dots,0)})dx^1 \\ &+ \mu_7 X^{(1,0,\dots,0)} dx^2 \end{aligned} \tag{5.11}$$

for some $\mu_1, ..., \mu_7 \in \mathbf{R}$. Obviously, if $r = 1$ then the term $\mu_1 dX^{(1,1,0,\dots,0)}$ does not exist.
It is clear that $\pi^*_{\mathbf{R}^n}(\omega_o) = x^2 dx^1$. Thus replacing A by $A - \mu_5\pi^*$ one can assume (in (5.11)) that $\mu_5 = 0$.

We consider four subcases:

(a) At first we assume that $r = 1$. The local diffeomorphism G described in Lemma 4.1 preserves $germ_0(\omega_o)$. Hence by the invariancy of A with respect to G

and by (4.1) and (4.2) from (5.11) we obtain

$$\begin{aligned}
&\mu_3 X^{(0,1,0,\dots,0)} dX^{(1,0,\dots,0)} + \mu_4 X^{(1,0,\dots,0)} dX^{(0,1,0,\dots,0)} \\
&\quad + \mu_6 X^{(0,1,0,\dots,0)} dx^1 + \mu_7 X^{(1,0,\dots,0)} dx^2 \\
&\quad = \mu_3 (X^{(0,1,0,\dots,0)} \circ T^{(1)}G) d(X^{(1,0,\dots,0)} \circ T^{(1)}G) \\
&\quad + \mu_4 (X^{(1,0,\dots,0)} \circ T^{(1)}G) d(X^{(0,1,0,\dots,0)} \circ T^{(1)}G) \\
&\quad + \mu_6 (X^{(0,1,0,\dots,0)} \circ T^{(1)}G) d(x^1 \circ T^{(1)}G) + \mu_7 (X^{(1,0,\dots,0)} \circ T^{(1)}G) d(x^2 \circ T^{(1)}G) \\
&\quad = \mu_3 X^{(0,1,0,\dots,0)} (dX^{(1,0,\dots,0)} - X^{(1,0,\dots,0)} dx^1) \\
&\quad + \mu_4 X^{(1,0,\dots,0)} (X^{(1,0,\dots,0)} dx^2 + dX^{(0,1,0,\dots,0)} + X^{(0,1,0,\dots,0)} dx^1) \\
&\quad + \mu_6 X^{(0,1,0,\dots,0)} dx^1 + \mu_7 X^{(1,0,\dots,0)} dx^2
\end{aligned}$$

on the fibre $(T^{(1)}\mathbf{R}^n)_0$ of $T^{(1)}\mathbf{R}^n$ over 0. (For example, by (4.1)

$$\begin{aligned}
d(X^{(1,0,\dots,0)} \circ T^{(1)}G) &= d((1-x^1)X^{(1,0,\dots,0)}) \\
&= -X^{(1,0,\dots,0)} dx^1 + (1-x^1) dX^{(1,0,\dots,0)} \\
&= dX^{(1,0,\dots,0)} - X^{(1,0,\dots,0)} dx^1
\end{aligned}$$

on the fibre $(T^{(1)}\mathbf{R}^n)_0$ as $x^1 = \dots = x^n = 0$. Similarly we analyse the other terms.) It implies that $\mu_3 = \mu_4 = 0$. (We analyse the coefficients corresponding to $(X^{(0,1,0,\dots,0)})^2\, dx^2$ and $X^{(0,1,\dots,0)} X^{(1,0,\dots,0)}\, dx^1$.)
Next using the invariancy of A with respect to G and (4.1) and (4.2) from (5.11) we have

$$\begin{aligned}
&\mu_2 x^2 dX^{(1,0,\dots,0)} + \mu_6 X^{(0,1,0,\dots,0)} dx^1 + \mu_7 X^{(1,0,\dots,0)} dx^2 \\
&\quad = \mu_2 \frac{x^2}{1-x^1} (-X^{(1,0,\dots,0)} dx^1 + (1-x^1) dX^{(1,0,\dots,0)}) \\
&\quad + \mu_6 \Big(\frac{x^2}{(1-x^1)^2} X^{(1,0,\dots,0)} + \frac{1}{1-x^1} X^{(0,1,0,\dots,0)}\Big)(1-x^1) dx^1 \\
&\quad + \mu_7 (1-x^1) X^{(1,0,\dots,0)} \Big(\frac{1}{1-x^1} dx^2 + \frac{x^2}{(1-x^1)^2} dx^1\Big)
\end{aligned}$$

over some neighbourhood of $0 \in \mathbf{R}^n$. It implies that $\mu_2 = \mu_6 + \mu_7$. (We multiply booth sides of the equality by $1 - x^1$ and next we analyse the coefficients on $x^2 X^{(1,0,\dots,0)}\, dx^1$.)

On the other hand by Lemma 3.1 the mapping $A \to A_{\mathbf{R}^n}(x^2 dx^1)$ is a linerar monomorphism. Hence the vector space of all linear natural operators $T^* \to T^*T^{(1)}$ has dimension ≤ 3. (This vector space depends eventually on μ_5, μ_6 and μ_7.) Since $\pi^*, C^{(n)}, D^{(n)} : T^* \to T^*T^{(1)}$ described in Examples 1.1 and 1.2 are linearly independent (see Lemma 2.1), we end the proof in this subcase.

(b) Now we assume that $r = 2$.

In general, for $r_1 \leq r_2$ we have a natural vector bundle monomorphism $I_M^{r_1,r_2} : T^{(r_1)}M \to T^{(r_2)}M$, $(I_M^{r_1,r_2}(\Theta))(j_x^{r_2}\gamma) := \Theta(j_x^{r_1}\gamma)$, where M is a manifold, $\Theta \in T_x^{(r_1)}M$, $x \in M$ and $j_x^{r_2}\gamma \in T_x^{r_2 *}M = (T_x^{(r_2)}M)^*$.

The natural vector bundle monomorphism $I = I^{1,2} : T^{(1)} \to T^{(2)}$ induces a linear natural operator $I^*A : T^* \to T^*T^{(1)}$, $(I^*A)_M(\omega) := I_M^*(A_M(\omega))$, for any n-manifold M and any $\omega \in \Omega^1(M)$. We see that $X^\alpha \circ I_{\mathbf{R}^n} = X^\alpha$ for any $\alpha \in S$ with $|\alpha| = 1$, and $X^\alpha \circ I_{\mathbf{R}^n} = 0$ for any $\alpha \in S$ with $|\alpha| = 2$. Therefore it follows from subcase (a) that we have the formula (5.11) with $\mu_4 = \mu_3 = \mu_5 = 0$ and $\mu_2 = \mu_6 + \mu_7$. By the invariancy of A with respect to G and by (4.3), (4.4) and (4.5) from (5.11) we obtain

$$\begin{aligned}
&\mu_1 dX^{(1,1,0,\dots,0)} + \mu_6 X^{(0,1,0,\dots,0)}dx^1 + \mu_7 X^{(1,0,\dots,0)}dx^2 \\
&\quad = \mu_1(X^{(2,0,\dots,0)}dx^2 + dX^{(1,1,0,\dots,0)}) \\
&\quad + \mu_6(X^{(0,1,0,\dots,0)} + X^{(1,1,0,\dots,0)})dx^1 + \mu_7(X^{(1,0,\dots,0)} - \frac{1}{2}X^{(2,0,\dots,0)})dx^2
\end{aligned}$$

on the fibre $(T^{(2)}\mathbf{R}^n)_0$. It implies that $\mu_6 = 0$ and $\mu_1 = \frac{1}{2}\mu_7$. (We consider the coefficients corresponding to $X^{(1,1,0,\dots,0)}\,dx^1$ and $X^{(2,0,\dots,0)}\,dx^2$.) Hence by Lemma 3.1 the vector space of all linear natural operators $T^* \to T^*T^{(2)}$ has dimension ≤ 2. Since $\pi^*, B^{(n)} : T^* \to T^*T^{(2)}$ described in Examples 1.1 and 1.2 are linearly independent (see Lemma 2.1), we end the proof in this subcase.

(c) Let $r = 3$. Using similar arguments as in subcase (b) with the natural vector bundle monomorphism $T^{(2)} \to T^{(3)}$ instead of $T^{(1)} \to T^{(2)}$ we have the formula (5.11) with $\mu_1 = \frac{1}{2}\mu_7$, $\mu_2 = \mu_7$ and $\mu_3 = \mu_4 = \mu_5 = \mu_6 = 0$. Using the invariancy of A with respect to G and (4.6) from (5.11) we obtain

$$\frac{1}{2}\mu_7 dX^{(1,1,0,\dots,0)} = \frac{1}{2}\mu_7(dX^{(1,1,0,\dots,0)} + \frac{1}{2}dX^{(2,1,0,\dots,0)})$$

on the vertical space $V_0(T^{(3)}\mathbf{R}^n)$ with $0 \in (T^{(3)}\mathbf{R}^n)_0$. Then $\mu_7 = 0$. Hence by Lemma 3.1 the vector space of all linear natural operators $T^* \to T^*T^{(3)}$ has dimension ≤ 1. Since $\pi^* : T^* \to T^*T^{(3)}$ described in Example 1.1 is not zero, we end the proof in this subcase.

(d) Let $r \geq 4$. Using similar arguments as in subcase (c) with the natural vector bundle monomorphism $T^{(3)} \to T^{(r)}$ instead of $T^{(2)} \to T^{(3)}$ we have the formula (5.11) with $\mu_1 = \dots = \mu_7 = 0$. Hence by Lemma 3.1 the vector space of all linear natural operators $T^* \to T^*T^{(r)}$ has dimension ≤ 1. Since $\pi^* : T^* \to T^*T^{(r)}$ described in Example 1.1 is not zero, we end the proof in this subcase.

(II) Now we assume that $n = 1$. Using the invariancy of A with respect to $t\,id$ and the linearity of A we deduce that

$$tf_\alpha(x^1, X^\alpha) = t^{|\alpha|}f_\alpha(tx^1, t^{|\beta|}X^\beta) \text{ and}$$

$$tf_1(x^1, X^\alpha) = tf_1(tx^1, t^{|\beta|}X^\beta)$$

for any $\alpha \in S$ and any $(x^1, X^\beta) \in \mathbf{R}^n \times \mathbf{R}^S$. Then $f_\alpha = 0$, if $|\alpha| \geq 2$, and $f_{(1)}, f_1$ are constants. Hence

$$A_{\mathbf{R}}(\omega_o) = \mu_1 dX^{(1)} + \mu_2 dx^1$$

for some $\mu_1, \mu_2 \in \mathbf{R}$. Thus the vector space of all linear natural operators $T^* \to T^*T^{(r)}$ has dimension ≤ 2. Since $\pi^*, A^{(r)} : T^* \to T^*T^{(r)}$ described in Examples 1.1 and 1.2 are linearly independent, we end the proof in this case. □

6. We have the following simple corollary of Theorem 1.1.

Corollary 6.1. (a) *If $n \geq 2$ and $r \geq 3$, then the vector space of all linear natural operators $T^* \to T^{r*}$ for n-manifolds is the zero vector space.*

(b) *If $n \geq 2$, then the vector space of all linear natural operators $T^* \to T^{2*}$ for n-manifolds is generated (over $\mathbf{R}$) by $\tilde{B}^{(n)} : T^* \to T^{2*}$ described in Lemma 1.2.*

(c) *If $n \geq 2$, then the vector space of all linear natural operators $T^* \to T^{1*}$ for n-manifolds is generated (over $\mathbf{R}$) by $\tilde{C}^{(n)} : T^* \to T^{1*}$ described in Example 1.2.*

(d) *If $r \geq 1$, then the vector space of all linear natural operators $T^* \to T^{r*}$ for 1-manifolds is generated (over $\mathbf{R}$) by $\tilde{A}^{(r)}$ described in Lemma 1.1.*

Proof. Let $r, n \in \mathbf{N}$. Let $\tilde{A} : T^* \to T^{r*}$ be a linear natural operator for n-manifolds. Using Example 1.2 we have the induced linear natural operator $A : T^* \to T^*T^{(r)}$ given by (1.4). We see that if $\tilde{A}', \tilde{A}" : T^* \to T^{r*}$ are two linear natural operators such that for the induced operators $A', A" : T^* \to T^*T^{(r)}$ we have $A'_M(\omega) = A"_M(\omega)$ on the vertical distribution $VT^{(r)}M$ for any n-manifold M and any $\omega \in \Omega^1(M)$, then $\tilde{A}' = \tilde{A}"$. On the other hand $\pi^*_M(\omega)$ and $D^{(n)}_M(\omega)$ are zero on $VT^{(r)}M$ (see Proof of Lemma 2.1). Using Theorem 1.1 we end the proof. □

References

[1] Doupovec, M., Kurek, J., *Liftings of covariant $(0,2)$-tensor fields to the bundle of k-dimensional 1-velocities*, Suppl. Rend. Circ. Mat. Palermo (in press).

[2] Gancarzewicz, J., Mahi, S., *Liftings of 1-forms to the tangent bundle of higher order*, Czech. Math. J. 40 (115) (1990), 397–407.

[3] Janyška, J., *Natural operations with projectable tangent valued forms on fibered manifolds*, Annali di Math. CLIX (1991), 171–184.

[4] Kolář, I., Michor, P. W., Slovák, J., *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin, 1993.

[5] Kurek, J., *On the first order natural operators transforming 1-forms on a manifold to linear frame bundle*, Demonstratio Math. 26 (1993), 287–293.

[6] Kurek, J., *On the first order natural operators transforming 1-forms on a manifold to the tangent bundle*, Ann. U.M.C.S. 43 (1989), 79–83.

[7] Mikulski, W. M., *The natural operators lifting 1-forms on manifolds to the bundles of A-velocities*, Mh. Math., 119 (1995), 63–77.

[8] Mikulski, W. M., *The geometrical constructions lifting tensor fields of type (0,2) on manifolds to the bundles of A-velocities*, Nagoya Math. J., 140 (1995) (in press).

[9] Yano, K., Ishihara, S., *Tangent and cotangent bundles*, Marcel Dekker, INC. , New York, 1973.

W. M. Mikulski
Institute of Mathematics
Jagellonian University
Reymonta 4
Krakow, POLAND