ARCHIVUM MATHEMATICUM (BRNO)
Tomus 31 (1995), 113 – 120

A COMPARISON THEOREM FOR LINEAR DELAY DIFFERENTIAL EQUATIONS

Jozef Džurina

Abstract. In this paper property (A) of the linear delay differential equation

$$L_n u(t) + p(t)u(\tau(t)) = 0,$$

is to deduce from the oscillation of a set of the first order delay differential equations.

Let us consider the delay differential equation

$$L_n u(t) + p(t)u(\tau(t)) = 0, \tag{1}$$

where $n \geq 2$ and

$$L_n u(t) = \left(\frac{1}{r_{n-1}(t)} \left(\cdots \left(\frac{1}{r_1(t)} u'(t) \right)' \cdots \right)' \right)'. \tag{2}$$

We always assume that $r_i(t)$, $1 \leq i \leq n-1$, $\tau(t)$ and $p(t)$ are continuous on $[t_0, \infty)$, $r_i(t) > 0$, $\tau(t) < t$ and $\tau(t) \to \infty$ as $t \to \infty$. Moreover, in the sequel we assume that

$$\int^{\infty} r_i(s)\, ds = \infty \quad \text{for} \quad 1 \leq i \leq n-1. \tag{3}$$

For convenience we introduce the following notation:

$$\begin{aligned} L_0 u(t) &= u(t), \\ L_i u(t) &= \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1} u(t), \quad 1 \leq i \leq n-1. \\ L_n u(t) &= \frac{d}{dt} L_{n-1} u(t). \end{aligned}$$

1991 *Mathematics Subject Classification*: Primary 34C10.
Key words and phrases: Comparison theorem, property (A)...
Received May 16, 1994.

The domain $\mathcal{D}(L_n)$ of L_n is defined to be the set of all functions $u : [T_u, \infty) \to \mathbb{R}$ such that $L_i u(t)$, $0 \le i \le n$ exist and are continuous on $[T_u, \infty)$. A nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

The asymptotic behavior of the solutions of (1) is described in the following lemma which is a generalization of a lemma of Kiguradze [4, Lemma 3].

Lemma 1. *Let $u(t)$ be a nonoscillatory solution of (1) then there is an integer ℓ, $\ell \in \{0, 1, \cdots, n-1\}$ with $n + \ell$ odd and $t_1 \ge t_0$ such that*

$$\begin{aligned} u(t)L_i u(t) &> 0, \quad 1 \le i \le \ell, \\ (-1)^{i-\ell} u(t) L_i u(t) &> 0, \quad \ell \le i \le n. \end{aligned} \tag{4}$$

for all $t \ge t_1$.

A function $u(t)$ satisfying (4) is said to be a function of degree ℓ. The set of all nonoscillatory solutions of degree ℓ of (1) is denoted by $\mathcal{N}_\ell$. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (1), then

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_{n-1} \qquad \text{for } n \text{ odd;} \\ \mathcal{N} &= \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_{n-1} \qquad \text{for } n \text{ even.} \end{aligned}$$

We are interested in the following extreme situation described in the definition:

Definition 1. Equation (1) is said to have property (A) if for n odd $\mathcal{N} = \mathcal{N}_0$ and for n even $\mathcal{N} = \emptyset$ (i.e. (1) is oscillatory).

There is much literature regarding property (A). The objectives of this paper is to compare equation (1) with the set of the first order delay equations

$$y'(t) + q_i(t) y(\tau(t)) = 0. \tag{E_i}$$

We present the relationship between property (A) of equation (1) and oscillation of the equations (E_i).

We begin by formulating some preparatory results which are needed in the sequel.

Let $i_k \in \{1, 2, \cdots, n-1\}$, $1 \le k \le n-1$ and $t, s \in [t_0, \infty)$, we define

$$\begin{aligned} I_0 &= 1, \\ I_k(t, s; r_{i_k}, \cdots, r_{i_1}) &= \int_s^t r_{i_k}(x) I_{k-1}(x, s; r_{i_{k-1}}, \cdots, r_{i_1})\, dx. \end{aligned}$$

It is easy to verify that for $1 \le k \le n-1$

$$\begin{aligned} I_k(t, s; r_{i_k}, \cdots, r_{i_1}) &= (-1)^k I_k(s, t; r_{i_1}, \cdots, r_{i_k}), \\ I_k(t, s; r_{i_k}, \cdots, r_{i_1}) &= \int_s^t r_{i_1}(x) I_{k-1}(t, x; r_{i_k}, \cdots, r_{i_2})\, dx. \end{aligned} \tag{5}$$

Lemma 2. *If $u \in \mathcal{D}(L_n)$ then the following formula holds for $0 \le i \le k \le n-1$ and $t, s \in [T_u, \infty)$:*

$$L_i u(t) = \sum_{j=i}^{k} (-1)^{j-i} L_j u(s) I_{j-i}(s, t; r_j, \cdots, r_{i+1})$$
$$+ (-1)^{k-i+1} \int_t^s I_{k-i}(x, t; r_k, \cdots, r_{i+1})\, r_{k+1}(x) L_{k+1} u(x)\, dx. \tag{6}$$

This lemma is a generalization of Taylor's formula, where we have formally put $r_n(t) \equiv t$. The proof is immediate.

For convenience we make use the following notations:

$$q_i(t) = r_{i+1}(t) \int_t^\infty p(s) I_{n-i-2}(s, t; r_{n-1}, \cdots, r_{i+2})$$
$$\times \int_{t_1}^{\tau(t)} I_{i-1}(\tau(s), x; r_1, \cdots, r_{i-1})\, r_i(x)\, dx ds,$$

$i = 1, 2, \cdots, n-2,$

$$q_{n-1}(t) = p(t) \int_{t_1}^{\tau(t)} I_{n-2}(\tau(t), x; r_1, \cdots, r_{n-2})\, r_{n-1}(x)\, dx,$$

for sufficiently large t_1 with $\tau(t) > t_1$.

Theorem 1. *Let*

$$\tau(t) \quad \textit{be nondecreasing.} \tag{7}$$

Assume that for $i = 1, 2, \cdots, n-1$ with $n+i$ odd, the linear differential inequalities

$$y'(t) + q_i(t) y(\tau(t)) \le 0 \tag{$\widetilde{E}_i$}$$

have no eventually positive solutions. Then equation (1) has property (A).

Proof. We assume that $u(t)$ is a positive solution of (1) on $[t_0, \infty)$ which does not belong to the class $\mathcal{N}_0$. According to Lemma 1 there exist an integer $\ell \in \{1, 2, \cdots, n-1\}$, with $n + \ell$ odd and a t_1 such that (4) holds for $t \ge t_1$.

Let $\ell < n-1$. Putting $i = \ell + 1$, $k = n-1$ and $s \ge t \ge t_1$ in (6), we have in view of (4) and (5)

$$-L_{\ell+1} u(t) \ge -\int_t^s I_{n-\ell-2}(x, t; r_{n-1}, \cdots, r_{\ell+2})\, L_n u(x)\, dx,$$

letting $s \to \infty$ and using (1), we obtain

$$-L_{\ell+1} u(t) \ge \int_t^\infty I_{n-\ell-2}(x, t; r_{n-1}, \cdots, r_{\ell+2})\, p(x) u(\tau(x))\, dx, \tag{8}$$

for $t \geq t_1$. Now by Lemma 2, with $i = 0$, $k = \ell - 1$ and $t \geq s = t_1$ taking (4) and (5) into account, one gets

$$L_0 u(t) \geq \int_{t_1}^{t} I_{\ell-1}(t, x; r_1, \cdots, r_{\ell-1})\, r_\ell(x) L_\ell u(x)\, dx \tag{9}$$

for $t \geq t_1$. Combining (8) with (9) we have with respect to (7)

$$\begin{aligned}
-L_{\ell+1} u(t) &\geq \int_{t}^{\infty} p(s) I_{n-\ell-2}(s, t; r_{n-1}, \cdots, r_{\ell+2}) \\
&\quad \times \int_{t_1}^{\tau(s)} I_{\ell-1}(\tau(s), x; r_1, \cdots, r_{\ell-1})\, r_\ell(x) L_\ell u(x)\, dx ds \\
&\geq \int_{t}^{\infty} p(s) I_{n-\ell-2}(s, t; r_{n-1}, \cdots, r_{\ell+2}) \\
&\quad \times \int_{t_1}^{\tau(t)} I_{\ell-1}(\tau(s), x; r_1, \cdots, r_{\ell-1})\, r_\ell(x) L_\ell u(x)\, dx ds.
\end{aligned}$$

Since $L_\ell u(t)$ is decreasing we obtain from the last inequality

$$-L_{\ell+1} u(t) \geq L_\ell u(\tau(t)) \frac{q_\ell(t)}{r_{\ell+1}(t)}, \tag{10}$$

for $t \geq t_1$. Let $y(t)$ be given by

$$y(t) \equiv L_\ell u(t),$$

then $y(t) > 0$ on $[t_1, \infty)$ and $y'(t) = r_{\ell+1}(t) L_{\ell+1} u(t)$ and by (10) we have

$$y'(t) + q_\ell(t) y(\tau(t)) \leq 0, \quad t \geq t_1\,,$$

which contradicts with the fact that differential inequality $(\widetilde{E}_\ell)$ has no positive solutions.

Let $\ell = n - 1$. Setting $i = 0$, $k = n - 2$ and $t \geq s = t_1$ in (6) and noting (4), we have

$$L_0 u(t) \geq \int_{t_1}^{t} I_{n-2}(t, x; r_1, \cdots, r_{n-2})\, r_{n-1}(x) L_{n-1} u(x)\, dx \tag{11}$$

for $t \geq t_1$. Since $-L_n u(t) = p(t) u(\tau(t))$ we obtain by (11)

$$-L_n u(t) \geq p(t) \int_{t_1}^{\tau(t)} I_{n-2}(\tau(t), x; r_1, \cdots, r_{n-2})\, r_{n-1}(x) L_{n-1} u(x)\, dx \tag{12}$$

for $t \geq t_1$. As $L_{n-1} u(t)$ is decreasing it follows from (12) that

$$-L_n u(t) \geq L_{n-1} u(\tau(t)) q_{n-1}(t). \tag{13}$$

Let $y(t)$ be given by

$$y(t) \equiv L_{n-1} u(t),$$

then $y(t) > 0$ on $[t_1, \infty)$ and $y'(t) = r_n(t) L_n u(t)$ and we have by (13)

$$y'(t) + q_{n-1}(t) y(\tau(t)) \leq 0, \quad t \geq t_1,$$

which contradicts the assumption that differential inequality $(\widetilde{E}_{n-1})$ has no positive solutions. The proof is complete.

Remark 1. One should note that the condition (7) is not necessary when proving that the class $\mathcal{N}_{n-1} = \emptyset$.

Theorem 2. *Let (7) hold. Assume that for all* $i = 1, 2, \cdots, n-1$ *with* $n + i$ *odd*

$$\liminf_{t \to \infty} \int_{\tau(t)}^{t} q_i(s)\, ds > \frac{1}{\mathrm{e}}. \tag{14}$$

Then equation (1) has property (A).

Proof. It is known (see e.g. [1, Theorem 1] or [6, Theorem 2.1.1]) that conditions (14) are sufficient for differential inequalities $(\widetilde{E}_i)$ to have no positive solutions. Our assertion follows by Theorem 1.

In the case when $\lim\limits_{t \to \infty} \int_{\tau(t)}^{t} q_i(s)\, ds$ does not exists, we still have the following result.

Theorem 3. *Let (7) hold. Assume that for all* $i = 1, 2, \cdots, n-1$ *with* $n + i$ *odd*

$$\limsup_{t \to \infty} \int_{\tau(t)}^{t} q_i(s)\, ds > 1. \tag{15}$$

Then equation (1) has property (A).

Proof. Theorem 2.1.3 in [6] insures that conditions (15) together with (7) are sufficient for differential inequalities $(\widetilde{E}_i)$ to have no positive solutions. Our assertion follows by Theorem 1.

Lemma 3. *Suppose that* $q(t) \in C([t_0, \infty))$ *is positive. Equation*

$$y'(t) + q(t) y(\tau(t)) = 0$$

has a positive solution if and only if so does the differential inequality

$$y'(t) + q(t) y(\tau(t)) \leq 0.$$

This lemma can be found in [3, Corollary 3.2.2].

Theorem 4. *Let (7) hold. Assume that for all $i = 1, 2, \cdots, n-1$ with $n+i$ odd, the delay differential equations*

$$y'(t) + q_i(t) y(\tau(t)) = 0 \tag{E_i}$$

are oscillatory. Then equation (1) has property (A).

Proof. This theorem immediately follows from Theorem 1 and Lemma 3.

Consider the special case of the equation (1), namely the third order equation

$$\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}u'\right)'\right)' + p(t)u(\tau(t)) = 0. \tag{16}$$

Let us denote

$$R_1(t) = \int_{t_0}^{t} r_1(s)\, ds.$$

Then we have taking Remark 1 into account:

Corollary 1. *Suppose that the equation*

$$y'(t) + \left(p(t)\int_{t_1}^{\tau(t)} r_2(x)\left[R_1(\tau(t)) - R_1(x)\right] dx\right) y(\tau(t)) = 0$$

is oscillatory. Then equation (16) has property (A).

Corollary 2. *Suppose that either*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s) \int_{t_1}^{\tau(s)} r_2(x)\left[R_1(\tau(s)) - R_1(x)\right] dx ds > \frac{1}{\mathrm{e}}$$

or

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s) \int_{t_1}^{\tau(s)} r_2(x)\left[R_1(\tau(s)) - R_1(x)\right] dx ds > 1.$$

Then equation (16) has property (A).

Example 1. Let us consider the equation

$$y'''(t) + p(t)y(\tau(t)) = 0, \quad t \ge t_0, \tag{17}$$

where the functions $p(t)$ and $\tau(t)$ are as in equation (1). Applying Corollary 2 we conclude that equation (17) has property (A) if either

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \tau^2(s)p(s)\, ds > \frac{2}{\mathrm{e}}$$

or

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \tau^2(s)p(s)\,ds > 2.$$

According to the generalization of a result of Hille [2, Theorem 11] equation (17) has property (A) if

$$\liminf_{t\to\infty} \tau^2(t) \int_t^{\infty} p(s)\,ds > \frac{1}{3\sqrt{3}}. \tag{18}$$

On the other hand by the criterion due to Kusano and Naito (see [5]) equation (17) has property (A) if

$$\liminf_{t\to\infty} \tau(t) \int_t^{\infty} [\tau(s) - \tau(t)]\, p(s)\,ds > \frac{1}{4}. \tag{19}$$

The following illustrative example is intended to show that Corollary 2 is not included in the above-mentioned results.

Example 2. Let us consider the equation

$$y'''(t) + \frac{a}{t^3} y(\lambda t) = 0, \quad t \geq 1, \quad a > 0, \quad 0 < \lambda < 1. \tag{20}$$

By Corollary 2 equation (20) has property (A) if

$$a\lambda^2 \ln\left(\frac{1}{\lambda}\right) > \frac{2}{\mathrm{e}}.$$

Note that condition (18) takes for equation (20) the form

$$a\lambda^2 > \frac{2}{3\sqrt{3}}.$$

It is easy to see that Corollary 2 provides for equation (20) better result than Theorem 11 in [2] if

$$0 < \lambda < \mathrm{e}^{-\frac{3\sqrt{3}}{\mathrm{e}}}.$$

On the other hand, since condition (19) takes for equation (20) the form

$$a\lambda^2 > \frac{1}{2},$$

Corollary 2 gives better result for (20) than Kusano and Naito's criterion if

$$0 < \lambda < \mathrm{e}^{-\frac{4}{\mathrm{e}}}.$$

The purpose of the following theorem is to relax condition of monotonicity imposed on the function $\tau(t)$ in Theorems 1-4.

Let us consider another differential equation

$$L_n u(t) + p(t) u(Q(t)) = 0, \tag{21}$$

where L_n and $p(t)$ are defined as in equation (1) and $Q(t) \in C([t_0, \infty))$ and $Q(t) \to \infty$ as $t \to \infty$.

Theorem 5. *Assume that (7) holds and*

$$Q(t) \geq \tau(t), \quad t \geq t_0.$$

If for all $i = 1, 2, \cdots, n-1$ with $n+i$ odd, differential inequalities $(\widetilde{E}_i)$ have no eventually positive solutions, then equation (21) has property (A).

Proof. By Theorem 1 equation (1) has property (A), and then by Theorem 1 in [5] equation (21) has property (A).

References

[1] Chanturia, T. A., Koplatadze, R. G., *On the oscillatory and monotone solutions of the first order differential equations with deviating argument*, Dif. Uravnenija **18** (1982), 1463–1465. (Russian)

[2] Džurina, J., *Comparison theorems for ODEs*, Math. Slovaca **42** (1992), 299–315.

[3] Győri, I., Ladas, G., *Oscillation theory of delay differential equations*, Clarendon press, Oxford, 1991.

[4] Kiguradze, I. T., *On the oscillation of solutions of the equation $d^m u/dt^m + a(t)|u|^n sign\, u = 0$*, Mat. Sb **65** (1964), 172–187. (Russian)

[5] Kusano, T., Naito, M., *Comparison theorems for functional differential equations with deviating arguments*, J. Math. Soc. Japan **3** (1981), 509–532.

[6] Ladde, G. S., Lakshmikantham, V., Zhang, B. G., *Oscillation theory of differential equations with deviating arguments*, Dekker, New York, 1987.

[7] Mihalíková, B., Šoltés, P., *Oscillations of differential equation with retarded argument*, Math. Slovaca **38** (1985), 295–303.

Jozef Džurina
Department of Mathematical Analysis
Šafárik University
Jesenná 5
041 54 Košice, SLOVAKIA