ARCHIVUM MATHEMATICUM (BRNO)
Tomus 31 (1995), 85 – 96

GENERALIZED RECIPROCITY FOR SELF–ADJOINT LINEAR DIFFERENTIAL EQUATIONS

Ondřej Došlý

Abstract. Let $L(y) = y^{(n)} + q_{n-1}(t)y^{(n-1)} + \cdots + q_0(t)y$, $t \in [a, b)$, be an n-th order differential operator, L^* be its adjoint and p, w be positive functions. It is proved that the self-adjoint equation $L^*\big(p(t)L(y)\big) = w(t)y$ is nonoscillatory at b if and only if the equation $L\big(w^{-1}(t)L^*(y)\big) = p^{-1}(t)y$ is nonoscillatory at b. Using this result a new necessary condition for property BD of the self-adjoint differential operators with middle terms is obtained.

1. INTRODUCTION.

Consider the self-adjoint, two term, even order, linear differential equation

$$(-1)^n(p(t)y^{(n)})^{(n)} = w(t)y, \tag{1.1}$$

where $t \in I = [a, b)$, $-\infty < a < b \le \infty$ and $p, w \in C^n(I)$ are positive real-valued functions. It is known that equation (1.1) is nonoscillatory at b if and only if the so-called reciprocal equation

$$(-1)^n(\frac{1}{w(t)}y^{(n)})^{(n)} = \frac{1}{p(t)}y \tag{1.2}$$

is nonoscillatory at b, see [1] (for necessary definitions and terminology see Section 2). This statement is a consequence of a more general result concerning linear Hamiltonian systems (further LHS) which states that a LHS

$$u' = A(t)u + B(t)v, \quad v' = -C(t)u - A^T(t)v \tag{1.3}$$

with nonnegative definite $n \times n$ matrices B, C is nonoscillatory at b if and only if the reciprocal system

$$\tilde{u}' = -A^T(t)\tilde{u} + C(t)\tilde{v}, \quad \tilde{v}' = -B(t)\tilde{u} + A(t)\tilde{v} \tag{1.4}$$

1991 *Mathematics Subject Classification*: 34C10.
Key words and phrases: Self-adjoint equation, reciprocal equation, property BD, principal solution, minimal differential operator.
Supported by the Grant No. 201/93/0452 of the Czech Grant Agency.
Received October 8, 1993.

is nonoscillatory at b, see [18,19]. The last statement may be viewed as follows. The transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} H(t) & M(t) \\ K(t) & N(t) \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \tag{1.5}$$

where H, K, M, N are $n \times n$ matrices of differentiable functions, transforms (1.3) into another LHS if and only if the matrix

$$\mathcal{R} = \begin{pmatrix} H & M \\ K & N \end{pmatrix}$$

is $\mathcal{J} - unitary$, i. e.,

$$\mathcal{R}(t)\mathcal{J}\mathcal{R}^T(t) = \mathcal{R}^T(t)\mathcal{J}\mathcal{R}(t) = \mathcal{J}, \tag{1.6}$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

I being the $n \times n$ identity matrix. System (1.4) results from (1.3) upon transformation (1.5) with $\mathcal{R}(t) = \mathcal{J}$ and the relationship between oscillation of (1.3) and (1.4) shows that if the matrices B, C are nonnegative definite then this transformation preserves oscillation behaviour of transformed systems.

Recently, the author extended this result by proving that under an additional assumption (which reduces to nonnegativity of B, C if $\mathcal{R} = \mathcal{J}$) transformation (1.5) preserves oscillation behaviour provided the matrix M is nonsingular, see [8]. Here we use this result in order to assign to a self-adjoint equation

$$\sum_{k=0}^{n} (-1)^k (p_k(t) y^{(k)})^{(k)} = w(t) y, \tag{1.8}$$

where $p_k \in C^k(I)$, $p_n > 0$, an equation of the same form which is nonoscillatory if and only if (1.8) is nonoscillatory. Particularly, if $p_k \equiv 0$, $k = 0, \dots, n-1$, our result complies with the relationship between oscillation of (1.1) and (1.2). This generalized reciprocity is then used, similar to [4,7], for the investigation of spectral properties of singular differential operators generated by the differential expression on the left-hand-side of (1.8). In the last section we give an alternative proof of the statement that transformation (1.5) preserves oscillation behaviour if the matrix $M(t)$ is nonsingular.

2. AUXILIARY RESULTS.

First recall some basic properties of LHS and their relationship to self-adjoint, even order, differential equations. Let y be a solution of the self-adjoint equation

$$\sum_{k=0}^{n} (-1)^k (p_k(t) y^{(k)})^{(k)} = 0. \tag{2.1}$$

Set $u = (y, ..., y^{(n-1)})$, $v_n = p_n(t)y^{(n)}$, $v_{n-k} = -v'_{n-k+1} + p_{n-k}y^{(n-k)}$, $k = 1, \ldots, n-1$. Then (u, v) is a solution of LHS (1.3) with

$$
\begin{aligned}
B(t) &= \operatorname{diag}\{0, \ldots, 0, p_n^{-1}(t)\}, \\
C(t) &= -\operatorname{diag}\{p_0(t), \ldots, p_{n-1}(t)\},
\end{aligned}
\tag{2.2}
$$

$$A = A_{i,j} = \begin{cases} 1, & \text{for } j = i+1, \quad i = 1, \ldots, n-1, \\ 0, & \text{elsewhere.} \end{cases}$$

In this case we say that the solution (u, v) is generated by y. Simultaneously with (1.3) consider its matrix analogy

$$U' = A(t)U + B(t)V, \quad V' = -C(t)U - A^T(t)V, \tag{1.3_M}$$

where U, V are $n \times n$ matrices. A solution (U, V) of (1.3_M) is said to be *isotropic* if $U^T(t)V(t) - V^T(t)U(t) \equiv 0$. An isotropic solution (U_b, V_b) of (2.3) is said to be *principal at b* if U_b is nonsingular in some left neighbourhood of b and

$$\lim_{t \to b-} \left(\int^t U_b^{-1}(s)B(s)U_b^{T-1}(s)ds \right)^{-1} = 0.$$

The principal solution of (1.3_M) at b is determined uniquely up to a right multiple by a constant nonsingular $n \times n$ matrix.

Let $y_1, \ldots, y_n$ be solutions of (2.1). If the columns of the solution (U, V) of the LHS (1.3_M) with A, B, C given by (2.2) are generated by $y_1, \ldots, y_n$ and (U, V) is principal solution at b, the solutions $y_1, \ldots, y_n$ are said to form the *principal sysem* of solutions of (2.1) at b. System (1.3) is said to be *identically normal* on I whenever the trivial solution $(u, v) \equiv (0, 0)$ is the only solution for which $u(t) \equiv 0$ on a nondegenerate subinterval I.

Two points $t_1, t_2 \in I$ are said to be *conjugate* relative to (1.3) if there exists a solution (u, v) such that $u(t_1) = 0 = u(t_2)$ whereby $u(t) \not\equiv 0$ between t_1, t_2 and they are said to be conjugate relative to (2.1) if $y^{(k)}(t_1) = 0 = y^{(k)}(t_2)$, $k = 0, \ldots, n-1$, for some nontrivial solution y of (2.1). Obviously, $t_1, t_2 \in I$ are conjugate relative to (2.1) if and only if they are conjugate relative to (1.3) with A, B, C given by (2.2). Equation (2.1) and system (1.3) are said to be *disconjugate* on $I_0 \subseteq I$ if there exists no pair of points of I_0 which are conjugate relative to (2.1) and (1.3) respectively, these equation and system are said to be *nonoscillatory at b* if there exists $c \in I$ such that they are disconjugate on (c, b).

If (U, V) is an isotropic solution of (1.3_M) with U nonsingular on $I_0 \subseteq I$ then $W = VU^{-1}$ is the solution of the Riccati matrix equation

$$W' + A^T(t)W + WA(t) + WB(t)W + C(t) = 0. \tag{2.3}$$

Theorem A. [1,8]. *Suppose that the matrix* $\mathcal{R} = \begin{pmatrix} H & M \\ K & N \end{pmatrix}$ *is* $\mathcal{J}$*-unitary. Then transformation (1.5) transforms (1.3) into LHS*

$$\tilde{u}' = \bar{A}(t)\tilde{u} + \bar{B}(t)\tilde{v}, \quad \tilde{v}' = -\bar{C}(t)\tilde{u} - \bar{A}^T(t)\tilde{v} \tag{2.4}$$

and the matrices $\bar{A}, \bar{B}, \bar{C}$ *are given by*

$$\begin{aligned} \bar{A} &= N^T(-H' + AH + BK) + M^T(K' + CH + A^T K), \\ \bar{B} &= N^T(-M' + AM + BN) + M^T(N' + CM + A^T N), \\ \bar{C} &= H^T(K' + CH + A^T K) + K^T(-H' + AH + BK). \end{aligned} \tag{2.5}$$

Moreover, if (1.3) is identically normal, the matrices $B, \bar{B}$ *are nonnegative definite and* M *is nonsingular near* b *then (1.3) is nonoscillatory at* b *if and only if (2.4) is nonoscillatory at* b*.*

Now suppose that equation (2.1) is disconjugate on an interval $I_0 \subseteq I$. Then there exists a symmetric solution $W = (W_{i,j})$ of the Riccati equation (2.3) with A, B, C given by (2.2). Denote $q_i(t) := -W_{i+1,n}(t),\ i = 0, \dots, n-1$, and consider the n-th order differential operator

$$L(y) = y^{(n)} + q_{n-1}(t)y^{(n-1)} + \cdots + q_0(t)y. \tag{2.6$_1$}$$

The adjoint operator is of the form

$$L^*(y) = (-1)^n y^{(n)} + (-1)^{n-1}(q_{n-1}(t)y)^{(n-1)} + \cdots - (q_1(t)y)' + q_0(t)y. \tag{2.6$_2$}$$

Using these operators we have the following statement concerning factorization of (2.1).

Theorem B. [5, Chap. II]. *Let (2.1) be disconjugate on* $I_0 \subseteq I$*. Then for any* $y \in C^{2n}(I_0)$ *we have*

$$\sum_{k=0}^{n}(-1)^k(p_k(t)y^{(k)})^{(k)} = L^*(p_n(t)L(y)), \tag{2.7}$$

where L, L^* *are given by (2.6).*

3. RECIPROCITY OF GENERAL SELF-ADJOINT EQUATIONS.

Theorem 1. *Suppose that equation (2.1) is nonoscillatory at* b *and let (2.7) be its factorization near* b*. The equation*

$$\sum_{k=0}^{n}(-1)^k(p_k(t)y^{(k)})^{(k)} = w(t)y, \tag{3.1}$$

i. e., the equation $L^*(p_n(t)L(y)) = w(t)y$, is nonoscillatory at b if and only if the equation

$$L(\frac{1}{w(t)}L^*(y)) = \frac{1}{p_n(t)}y \tag{3.2}$$

is nonoscillatory at b.

Proof. Let W be a symmetric solution of Riccati equation (2.3) with A, B, C given by (2.2) which exists near b and consider the transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & W(t) \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}. \tag{3.3}$$

This transformation transforms the LHS corresponding to (3.1), i. e. the LHS with A, B, given by (2.2) and

$$C = -\ \operatorname{diag}\{-w + p_0, p_1, \dots, p_{n-1}\}$$

into the LHS

$$\begin{aligned} \tilde{u}' &= -(A + BW)^T \tilde{u} + \operatorname{diag}\{w, 0, \dots, 0\}\tilde{v}, \\ \tilde{v}' &= -\operatorname{diag}\{0, \dots, 0, p_n^{-1}\}\tilde{u} + (A + BW)\tilde{v}, \end{aligned} \tag{3.4}$$

as follows from Theorem A. By a direct computation one may verify that (3.4) is the LHS which corresponds to the self-adjiont equation

$$L(w^{-1}(t)L^*(y)) = p_n^{-1}(t)y \tag{3.5}$$

for $y = \tilde{u}_n$, where $\tilde{u}_n$ is the last component of $\tilde{u}$. Since $M(t) \equiv I$ is nonsingular, LHS (1.3) with A, B, C given by (2.2) is identically normal and $\operatorname{diag}\{w, 0, \dots, 0\}$, $\operatorname{diag}\{0, \dots, 0, p_n^{-1}\}$ are nonnegative definite, by the second part of Theorem A system (3.4) is nonoscillatory at b if and only if LHS corresonding to (3.1) is nonoscillatory at b . To finish the proof, it remains to prove that (3.4) is nonoscillatory if and only if (3.5) is nonoscillatory. The first equation in (3.4) reads

$$\begin{aligned} \tilde{u}_1' &= -p_n^{-1}W_{1,n}\tilde{u}_1 + w\tilde{v}_1, \\ \tilde{u}_2' &= -\tilde{u}_1 - p_n^{-1}W_{2,n}\tilde{u}_2, \\ &\vdots \\ \tilde{u}_{n-1}' &= -\tilde{u}_{n-2} - p_n^{-1}W_{n-1,n}\tilde{u}_{n-1}, \\ \tilde{u}_n' &= -\tilde{u}_{n-1} - p_n^{-1}W_{n,n}\tilde{u}_n, \end{aligned}$$

where $W_{i,j}$ are entries of W. If t_1, t_2 are conjugate relative to (3.4), i. e. $\tilde{u}_i(t_1) = 0 = \tilde{u}_i(t_2), i = 1, \dots, n$, from the above equations we get $\tilde{u}_n^{(j)}(t_1) = 0 = \tilde{u}_n^{(j)}(t_2)$,

$j = 0, \dots, n-1$, hence t_1, t_2 are conjugate relative to (3.4) if and only if they are conjugate relative to (3.5) what we needed to prove. □

If $p_0 \equiv 0, \dots, p_{n-1} \equiv 0$ in (2.1) then the Riccati equation associated with this equation is of the form

$$W' + A^T W + WA + WB(t)W = 0 \tag{3.6}$$

which has symetric solution $W \equiv 0$. Hence, in this case the transformation matrix in (3.3) equals $\mathcal{J}$, $L = \frac{d^n}{dt^n}$ and Theorem 1 complies with relationship between (1.1) and (1.2) mentioned in Section 1. Of course, one may chose another symmetric solution of (3.6) which exists up to b. For example, such a solution is

$$W(t) = \left[\int^t H^{-1}(s)\,\mathrm{diag}\{0, \dots, 0, p_n^{-1}(s)\}\, H^{T^{-1}}(s) ds\right]^{-1}, \tag{3.7}$$

where H is the solution of $H' = AH$ satisfying $H(0) = I$, i. e.

$$H_{ij}(t) = \begin{cases} \frac{t^{(j-i)}}{(j-i)!} & \text{for } j \geq i, \\ 0 & \text{for } i > j. \end{cases} \tag{3.8}$$

Applying this idea to the fourth order equation (with $p_2 \equiv 1$) we have the following statement.

Corollary 1. *Suppose that $b = \infty$. Equation $y^{(iv)} = w(t)y$ is nonoscillatory at ∞ if and only if the equation*

$$(w^{-1}y'')'' - \big((\frac{w'}{tw^2} + \frac{4}{t^2 w})y'\big)' + \big(\frac{4w'^2}{t^2w^3} - \frac{2w''}{t^2w^2} + \frac{12 - 2tw'}{t^4 w} - 1\big)y = 0 \tag{3.9}$$

is nonoscillatory at ∞.

Proof. Computing (3.6), (3.7) for $n = 2$ and $B = \mathrm{diag}\{0, 1\}$, we have

$$W(t) = \begin{pmatrix} 12t^{-3} & -6t^{-2} \\ -6t^{-2} & 4t^{-1} \end{pmatrix},$$

hence $L(y) = y'' - 4t^{-1}y' + 6t^{-2}y$, $L^*(y) = y'' + 4t^{-1}y + 2t^{-2}y$ and a direct computation gives that $L\big(w^{-1}L^*(y)\big) = y$ complies with (3.9). □

In order to apply Theorem 1 in particular cases, one needs to know the factorization of the operator

$$\sum_{k=0}^{n} (-1)^k (p_k(t)y^{(k)})^{(k)} =: M(y),$$

which is the same, in general, as to know the solutions of the equation $M(y) = 0$. Besides of the most frequent case $M(y) = (-1)^n (p_n(t) y^{(n)})^{(n)}$, the solutions of the equation $M(y) = 0$ can be computed for example if $M(y)$ is the Kneser operator

$$M(y) = y^{(2n)} - \frac{\mu_n}{t^{2n}} y,$$

where $\mu_n = P\left(\frac{2n-1}{2}\right)$ is the so-called Kneser constant and $P(\lambda) = \lambda(\lambda-1)\dots(\lambda-2n+1)$. Oscillation criteria for the equation $M(y) = p(t)y$, which may be used for the reciprocal type criteria along the line suggested by Theorem 1 were investigated in [10].

4. BD CRITERIA FOR SINGULAR DIFFERENTIAL OPERATORS.

Oscillation theory od self-adjoint equations is closely related to the spectral theory of singular differential operators. Recall briefly some basic concepts of this theory (for a more comprehensive introduction the reader is referred to the monographs [11,13,16]).

Let $w(t)$ be a positive continuous weight function and consider the formally self-adjoint differential operator

$$l(y) = \frac{1}{w(t)} \sum_{k=0}^{n} (-1)^k (p_k(t) y^{(k)})^{(k)}, \ t \in I = [a, b).$$

We suppose that a is the regular and b the singular point of l. This operators generates the so-called minimal differential operator l_0 in the Hilbert space $\mathcal{L}^2(I, w)$ consisting of all functions y for which $\int_a^b w(t) y^2(t)\, dt < \infty$. Following [14], the operator l is said to possess property BD if every self-adjoint extension of l_0 has spectrum discrete and bounded below. The following result relates oscillation and spectral theory od singular differential operators.

Theorem C. [13]. *A necessary and sufficient condition that l has property BD is that the equation $l(y) = \lambda$ be nonoscillatory at b for all $\lambda \in \mathbb{R}$.*

It is known that the one term differential operator $(-1)^n (p(t) y^{(n)})^{(n)}$ has property BD in $\mathcal{L}^2(I)$ with $b = \infty$ (i. e. $w \equiv 1$) if and only if

$$\lim_{t \to \infty} t^{2n-1} \int_t^\infty p^{-1}(s)\, ds = 0. \tag{4.2}$$

The sufficiency of this criterion was established by Tkachenko [13] and the necessity by Lewis [15] using certain oscillation criterion for (1.2) with $w \equiv 1$ and the relationship between oscillation of (1.1) and (1.2). This idea has been extended in various directions in [4,7,14,16]. Here we present another generalization which is based on Theorem 1 and the following oscillation criterion for (3.1) given in [7].

Theorem D. [7]. *Let $y_1, \dots, y_n$ be the principal system of solutions of (2.1) at b. If there exists $c = (c_1, \dots, c_n)^T \in \mathbb{R}^n$ such that*

$$\limsup_{t\to b-} \frac{\int_t^b w(s)(c_1 y_1(s) + \cdots + c_n y_n(s))^2 ds}{c^T \left(\int^t U^{-1}(s)B(s)U^{T-1}(s)\right)^{-1} c} > 1, \tag{4.3}$$

where U is the Wronski matrix of $y_1, \dots, y_n$, $B = diag\{0, \dots, 0, p_n^{-1}(t)\}$, then (3.1) is oscillatory at b.

Theorem 2. *Let $M(y) = \sum_{k=0}^n (-1)^k (p_k(t)y^{(k)})^{(k)}$. Suppose that the equation $M(y) = 0$ is nonoscillatory at b, $M(y) = L^*\big(p_n L(y)\big)$ is its any factorization near b and $y_1, \dots, y_n$ is a principal system of solutions at b of the equation $L\big(w^{-1}L^*(y)\big) = 0$. If the operator (4.1) has property BD in the weightened space $\mathcal{L}^2(I, w)$, then for any $c = (c_1, \dots, c_n)^T \in \mathbb{R}^n$ we have*

$$\lim_{t\to b-} \frac{\int_t^b p_n^{-1}(s)(c_1 y_1(s) + \cdots + c_n y_n(s))^2 ds}{c^T \left(\int^t U^{-1}(s)\bar{B}(s)U^{T-1}(s)\right)^{-1} c} = 0, \tag{4.4}$$

where U is the Wronski matrix of $y_1, \dots, y_n$ and $\bar{B} = diag\{0, \dots, 0, w^{-1}(t)\}$,

Proof. If the operator l has property BD, by Theorem C the equation $M(y) = \lambda w(t)y$ is nonoscillatory at b for any $\lambda \in \mathbb{R}$. Suppose that (4.4) fails to hold, i. e.

$$\limsup_{t\to b-} \frac{\int_t^b p_n^{-1}(s)(c_1 y_1(s) + \cdots + c_n y_n(s))^2 ds}{c^T \left(\int^t U^{-1}(s)\bar{B}(s)U^{T-1}(s)\right)^{-1} c} = \varepsilon > 0,$$

for some $c = (c_1, \dots, c_n)^T \in \mathbb{R}^n$. It follows, by Theorem D, that for $\lambda = \frac{1}{2\varepsilon}$ the equation $L\big(w^{-1}L^*(y)\big) = \lambda p_n^{-1} y$ is oscillatory at b. Hence, in view of Theorem 1, the equation $L^*\big(p_n(L(y)\big) = \lambda w y$ is also oscillatory at b, but this is a contradiction since the equation $L^*\big(p_n L(y)\big) = M(y) = \lambda w y$ is nonoscillatory at b for any $\lambda \in \mathbb{R}$. $\square$

Remarks. i) If $b = \infty$, $w \equiv 1$ and $L = \frac{d^n}{dt^n}$ then one may directly verify that $y_1 = 1, y_2 = t, \dots, y_n = t^{n-1}/(n-1)!$ form the principal system of solutions at ∞ of $L^*\big(w^{-1}L(y)\big) = (-1)^n y^{(2n)} = 0$ with the Wronski matrix $U(t) = H(t)$, $H(t)$ being given by (3.7). In this case for $c = e_n = (0, \dots, 0, 1)^T$ we have

$$c^T \Big(\int^t H^{-1}(s)\,\mathrm{diag}\{0, \dots, 0, 1\} H^{T-1}(s)\, ds\Big)^{-1} c = K t^{-(2n-1)},$$

(K being a real constant whose value may be computed explicitely, but for our considerations its value is immaterial) and hence (4.4) reduces to (4.2).

ii) If $L = \frac{d^n}{dt^n}$ Theorem 2 complies with necessary condition for property BD of the one term operator

$$M(y) = (-1)^n \frac{1}{w(t)}(p(t)y^{(n)})^{(n)} \tag{4.5}$$

given in [7].

iii) In [14] the following sufficient condition for property BD of (4.5) was introduced:

Theorem E. *Let the positive functions $M_1, M_2, \ldots, M_n$ satisfy $M_1' \geq w,\ M_2 = M_1^2/M_1', \ldots, M_n = M_{n-1}^2/M_{n-1}'$. If*

$$\lim_{t\to\infty} M_n(t) \int_t^\infty \frac{1}{p(s)} ds = 0 \tag{4.6}$$

than M has property BD in $L^2((a,\infty), w))$.

For $w(t) = t^\alpha$, $\alpha \in \mathbb{R}$, condition (4.6) was shown to be also necessary for property BD of M, see [4,7]. It would be interesting to find what is the "gap" between sufficient condition (4.6) and the necessary one given by Theorem 2 for general weight functions w.

5. AN ALTERNATIVE PROOF OF THEOREM A.

Recall the main idea of the proof of the second part of Theorem A which states that transformation (1.5) with M nonsingular preserves oscillation properties if the matrices $B, \bar{B}$ are nonnegative definite. This proof is based on the so-called *trigonometric transformation* of LHS introduced in [6]. A trigonometric transformation is the transformation (1.5) with $M(t) \equiv 0$ (hence it trivially preserves oscillation behaviour – see definition of conjugate points) which transforms (1.3) into the so-called trigonometric system

$$\tilde{u}' = Q(t)\tilde{v}, \quad \tilde{v}' = -Q(t)\tilde{u}, \tag{5.1$_1$}$$

where Q is a nonnegative definite $n \times n$ matrix. System (2.4) which results from (1.3) upon transformation (1.5) can be trasformed into another trigonometric system

$$\tilde{u}' = \bar{Q}(t)\tilde{v}, \quad \tilde{v}' = -\bar{Q}(t)\tilde{u}. \tag{5.1$_2$}$$

It is known (c. f. [18, Chap. V.10]) that trigonometric system (5.1_1) is nonoscillatory at b if and only if $\int^b \operatorname{Tr} Q(t)\, dt < \infty$ ($\operatorname{Tr}(\cdot)$ stands for the trace of a matrix). In [8] it was proved that nonsingularity of $M(t)$ in (1.5) implies $|\int^b \operatorname{Tr}(Q(t) - \bar{Q}(t))dt| \leq m\pi$ for some $m \in \mathbb{N}$. This together with nonnegativity of B and $\bar{B}$ imply that (5.1_1), (5.2_2) are both oscillatory or nonoscillatory. The statement then follows from the fact that trigonometric transformations of (1.3) and (2.4) into (5.1_1), (5.2_2) respectively, preserve oscillation behaviour of transformed systems.

Here we give an alternative proof which is based on relationship between oscillation of reciprocal systems (1.3), (1.4) and certain factorization of $\mathcal{J}$-unitary matrices which is given by the following theorem. For its easier formulation we will adopt the following terminology: A $\mathcal{J}$-unitary $2n \times 2n$ matrix $\mathcal{R}(t)$ consisting of $n \times n$ matrices

$$\mathcal{R}(t) = \begin{pmatrix} H(t) & M(t) \\ K(t) & N(t) \end{pmatrix} \tag{5.2}$$

is said to be *triangular* if $M(t) \equiv 0$.

Theorem 3. *Let $\mathcal{R}(t)$ be a $\mathcal{J}$-unitary $2n \times 2n$ matrix of the form (5.2) with M nonsingular. Then there exist triangular $\mathcal{J}$-unitary $2n \times 2n$ matrices $\mathcal{R}_1(t)$, $\mathcal{R}_2(t)$ such that*

$$\mathcal{R}(t) = \mathcal{R}_2(t)\mathcal{J}\mathcal{R}_1(t). \tag{5.3}$$

Moreover, if the matrix B in (1.3) is nonnegative definite and (1.5) transforms this LHS into (2.4) with $\bar{B}$ nonnegative definite, the matrix $\mathcal{R}_2(t)$ can be chosen in such a way that (1.5) with transformation matrix $\mathcal{R}_2(t)$ transforms (1.3) into a system with both $\bar{B}, \bar{C}$ nonnegative definite.

Proof. Directly one may verify that $2n \times 2n$ matrix $\mathcal{R}$ of the form (5.2) is $\mathcal{J}$-unitary if and only if

$$H^T K = K^T H, \quad M^T N = N^T M, \quad H^T N - K^T M = I \tag{5.4_1}$$

which is equivalent to

$$HM^T = MH^T, \quad KN^T = NK^T, \quad HN^T - MK^T = I. \tag{5.4_2}$$

Set

$$\mathcal{R}_2 = \begin{pmatrix} I & 0 \\ M^{-1}N & I \end{pmatrix}, \quad \mathcal{R}_1 = \begin{pmatrix} M^{T-1} & 0 \\ N & M \end{pmatrix}. \tag{5.5}$$

Using (5.4) it is easy to see that $\mathcal{R}_1, \mathcal{R}_2$ are $\mathcal{J}$-unitary and (5.3) holds. It remains to prove the statement concerning the choise of the matrix $\mathcal{R}_2$ if B and $\bar{B} = N^T(-M' + AM + BN) - M^T(-N' - CM - A^T N)$ are nonnegative definite.

Since by (2.5) transformation (1.5) with the transformation matrix $\mathcal{R}_2$ gives $\bar{B} = B$, it suffices to prove that

$$\begin{aligned} \bar{C} &= -(NM^{-1})^T(A + BNM^{-1}) - (NM^{-1})' - C - A^T NM^{-1} \\ &= M^{T-1}\left[N^T(AM + BN) - M^T(N' - NM^{-1}M' - CM - A^T N)\right]M^{-1} \\ &= M^{T-1}\left[N^T(-M' + AM + BN) - M^T(-N' - CM - A^T N)\right]M^{-1} \end{aligned}$$

is nonnegative definite, but this follows from nonnegativity of $\bar{B}$ upon (1.5) with the transformation matrix $\mathcal{R}$ since the matrix in the last equality is $M^{T-1}\bar{B}M^{-1}$.□

Having proved Theorem 3, the second part of Theorem A now easily follows from relationship between mutually reciprocal systems (1.3), (1.4) with B, C nonnegative definite and the fact that transformation (1.5) with triangular matrices $\mathcal{R}_1$, $\mathcal{R}_2$ preserves oscillation behaviour of transformed systems.

Remarks. i) To prove that a $\mathcal{J}$-unitary matrix (5.2) with M nonsingular may be expressed in the form (5.3) with $\mathcal{R}_1$, $\mathcal{R}_2$ triangular is relatively easy. Substantial in this factorization is the second part of Theorem 3 which states that under the assumption of nonnegativity of B and $N^T(-M'+AM+BN)-M^T(-N'-CM-A^TN)$ the "middle" transformation with the transformation matrix $\mathcal{J}$ is applied to a system with both B, C nonnegative definite, hence this transformation preserved oscillation behaviour.

ii) The factorization of a $\mathcal{J}$-unitary matrix $\mathcal{R}$ (5.2) with M nonsingular given by (5.3) yields also an alternative proof of the main statement of [9]. In this paper the conditions are given under which transformation (1.5) transforms principal and nonprincipal solutions of (1.3_M) into the principal and nonprincipal solutions of (2.4). Using Theorem 3 this result may be inferred from [2], where this problem has been investigated in case $\mathcal{R}=\mathcal{J}$.

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Ondřej Došlý
Faculty of Science, Masaryk University
Department of Mathematical Analysis
Janáčkovo nám. 2a
662 95 Brno, CZECH REPUBLIC