ARCHIVUM MATHEMATICUM (BRNO)
Tomus 31 (1995), 147 – 153

TERNARY SEMIGROUPS OF MORPHISMS OF OBJECTS IN CATEGORIES

Antoni Chronowski and Miroslav Novotný

Abstract. In this paper the notion of a ternary semigroup of morphisms of objects in a category is introduced. The connection between an isomorphism of categories and an isomorphism of ternary semigroups of morphisms of suitable objects in these categories is considered. Finally, the results obtained for general categories are applied to the categories $\mathbf{REL}n+1$ and $\mathbf{ALG}n$ which were studied in [5].

1. Introduction

Isomorphisms of various categories were used to solve algebraic problems. Since the construction of all homomorphisms of a mono-unary algebra into another one is known, the isomorphism of a category of binary structures and a category of mono-unary algebras was used to construct all strong homomorphisms of a binary structure into another one in [3]. Another example is presented in [6] where all homomorphisms of a groupoid into another one is constructed on the basis of the fact that the category of all groupoids is isomorphic to a suitable category of mono-unary algebras. Isomorphisms of a category of relational structures and a category of algebras were studied also in [4] and [5].

The present paper offers another possibility of using isomorphisms between categories. If $\mathbf{K}$ is a category and X, Y its objects, then all pairs (p, q), where p is a morphism of X into Y and q a morphism of Y into X, constitute a ternary semigroup with an operation defined in a natural way. If $\mathbf{K}'$ is a category isomorphic to $\mathbf{K}$ and X', Y' are objects corresponding to X and Y, respectively, then the ternary semigroup of morphisms formed by means of X' and Y' is proved to be isomorphic to the ternary semigroup of morphisms formed by means of X and Y. This result is applied in various situations in our examples.

We now present the details of our considerations. The instruments of the theory of categories used here can be easily found in [1] or in [7]. First, we recall some definitions of [5].

1991 *Mathematics Subject Classification*: 20N15, 08A02, 08A62, 18B10.
Key words and phrases: ternary semigroup, mono-n-ary structure, mono-n-ary algebra, category, homomorphism, strong homomorphism, isomorphism.
Received May 27, 1994.

2. Totally additive and atom-preserving mappings

For any set A we denote by $\boldsymbol{P}(A)$ its power set, i.e., $\boldsymbol{P}(A) = \{X;\ X \subseteq A\}$.

Let $A,\ A'$ be sets, H a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$. The mapping H is said to be *totally additive* if

$$H(X) = \bigcup\{H(\{x\});\ x \in X\}$$

holds for any set $X \in \boldsymbol{P}(A)$. The mapping H is referred to as *atom-preserving* if for any $x \in A$ there exists $x' \in A'$ such that $H(\{x\}) = \{x'\}$.

Let r be a relation from A to A', i.e., $r \subseteq A \times A'$. Then for any $X \in \boldsymbol{P}(A)$ we put

$$\mathbf{P}[r](X) = \{x' \in A';\ \text{there exists } x \in X \text{ with } (x, x') \in r\}.$$

Clearly $\mathbf{P}[r]$ is a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$.

Let H be a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$. Then we set

$$\mathbf{Q}[H] = \{(x, x') \in A \times A';\ x' \in H(\{x\})\}.$$

Then $\mathbf{Q}[H]$ is a relation from A to A'.

3. Categories $\mathbf{REL}n + 1$ and $\mathbf{ALG}n$

In what follows n is a positive integer.

If A is a set, we put $A^n = A \times \cdots \times A$ where A appears n times. A set $r \subseteq A^n$ is said to be an *n-ary relation on A* and the ordered pair (A, r) is called a *mono-n-ary structure*.

If (A, r), (A', r') are mono-$n+1$-ary structures and h is a mapping of A into A', then h is said to be a *strong homomorphism* of the mono-$n+1$-ary structure (A, r) into (A', r') whenever the following holds: For any $x_1, \ldots, x_n$ in A and any x'_{n+1} in A' the condition $(h(x_1), \ldots, h(x_n), x'_{n+1}) \in r'$ is satisfied if and only if there exists $x_{n+1} \in A$ such that $h(x_{n+1}) = x'_{n+1}$ and $(x_1, \ldots, x_n, x_{n+1}) \in r$.

We now recall the definition of the category $\mathbf{REL}n+1$. Objects of this category are mono-$n+1$-ary structures of the form (A, r). By a morphism of the object (A, r) into the object (A', r') we mean a strong homomorphism of the mono-$n+1$-ary structure (A, r) into (A', r'). It is easy to see (cf. [5]) that $\mathbf{REL}n + 1$ is a category.

Let A be a set and N an n-ary operation on $\boldsymbol{P}(A)$. Then the ordered pair $(\boldsymbol{P}(A), N)$ will be referred to as a *mono-n-ary algebra*. The operation N is said to be *totally additive* if

$$N(X_1, \ldots, X_n) = \bigcup\{N(\{x_1\}, \ldots, \{x_n\});\ (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n\}$$

holds for any $X_1, \ldots, X_n$ in $\boldsymbol{P}(A)$.

We now recall the definition of the category $\mathbf{ALG}n$. Objects of this category are mono-n-ary algebras of the form $(\boldsymbol{P}(A), N)$ where A is a set and N is a totally additive n-ary operation on $\boldsymbol{P}(A)$. By a morphism of the object $(\boldsymbol{P}(A), N)$ into the object $(\boldsymbol{P}(A'), N')$ in $\mathbf{ALG}n$ we mean a totally additive atom-preserving homomorphism of the mono-n-ary algebra $(\boldsymbol{P}(A), N)$ into $(\boldsymbol{P}(A'), N')$. It is easy to see (cf. [5]) that $\mathbf{ALG}n$ is a category.

4. Isomorphisms of categories $\mathbf{REL}n+1$ and $\mathbf{ALG}n$

Let (A, r) be a mono-$n+1$-ary structure. For arbitrary sets $X_1, \ldots, X_n$ in $\boldsymbol{P}(A)$ we put

$$\mathbf{R}[r](X_1, \ldots, X_n) = \{x_{n+1} \in A;\ \text{ there exist } x_1 \in X_1, \ldots, x_n \in X_n \text{ such that } (x_1, \ldots, x_n, x_{n+1}) \in r\}.$$

Clearly, $\mathbf{R}[r]$ is an n-ary operation on the set $\boldsymbol{P}(A)$. Hence $\mathbf{R}$ is an operator assigning an n-ary operation on $\boldsymbol{P}(A)$ to any $n+1$-ary relation on A.

If A is a set and N an n-ary operation on $\boldsymbol{P}(A)$, we put

$$\mathbf{S}[N] = \{(x_1, \ldots, x_n, x_{n+1}) \in A^{n+1};\ x_{n+1} \in N(\{x_1\}, \ldots, \{x_n\})\}.$$

Hence $\mathbf{S}$ is an operator assigning a $n+1$-ary relation on A to any n-ary operation on $\boldsymbol{P}(A)$.

We define two functors. F will be a functor from the category $\mathbf{REL}n+1$ to $\mathbf{ALG}n$ and G will be a functor from the category $\mathbf{ALG}n$ to $\mathbf{REL}n+1$. These functors will be defined by presenting the object mappings Fo, Go and the morphism mappings Fm, Gm.

If (A, r) is an object in the category $\mathbf{REL}n+1$ and h a morphism in this category, we put

$$Fo(A, r) = (\boldsymbol{P}(A), \mathbf{R}[r]),\ Fm(h) = \mathbf{P}[h].$$

If $(\boldsymbol{P}(A), N)$ is an object in the category $\mathbf{ALG}n$ and H is a morphism in this category, we set

$$Go(\boldsymbol{P}(A), N) = (A, \mathbf{S}[N]),\ Gm(H) = \mathbf{Q}[H].$$

Theorem 1. *Let n be a positive integer. Then F is a functor of the category $\mathbf{REL}n+1$ into $\mathbf{ALG}n$ and G is a functor of the category $\mathbf{ALG}n$ into $\mathbf{REL}n+1$ such that $F \circ G$ and $G \circ F$ are identity functors.*

Corollary. *Let n be a positive integer. Then the functor F is an isomorphism of the category $\mathbf{REL}n+1$ onto $\mathbf{ALG}n$ and the functor G is an isomorphism of the category $\mathbf{ALG}n$ onto $\mathbf{REL}n+1$.*

For the proofs see [5].

5. Ternary semigroups

A *ternary semigroup* (cf. [2], [8]) is an algebraic structure (A, f) such that A is a nonempty set and $f : A^3 \to A$ is a ternary operation satisfying the associative law:

$$f(f(x_1, x_2, x_3), x_4, x_5) = f(x_1, f(x_2, x_3, x_4), x_5) = f(x_1, x_2, f(x_3, x_4, x_5))$$

for all $x_1, \ldots, x_5$ in A.

Let (A, f), (A', f') be ternary semigroups. A mapping $h : A \to A'$ is called a *homomorphism* of (A, f) into (A', f') if

$$h(f(x_1, x_2, x_3)) = f'(h(x_1), h(x_2), h(x_3))$$

holds for any $x_1,\ x_2,\ x_3$ in A. A bijective homomorphism is said to be an *isomorphism.*

If X and Y are nonempty sets, we denote the set of all mappings of X into Y by $T(X, Y)$. Furthermore, we put $T[X, Y] = T(X, Y) \times T(Y, X)$. Define the ternary operation $f : (T[X, Y])^3 \to T[X, Y]$ by the rule:

$$f((p_1, q_1), (p_2, q_2), (p_3, q_3)) = (p_1 \circ q_2 \circ p_3, q_1 \circ p_2 \circ q_3)$$

for any $(p_i, q_i) \in T[X, Y]$ where $i = 1,\ 2,\ 3$. The algebraic structure $(T[X, Y], f)$ is a ternary semigroup and is called the *ternary semigroup of mappings of sets X and Y*. This ternary semigroup is said to be *disjoint* if $X \cap Y = \emptyset$.

A slightly modified argument applied in the proof of Theorem 3 in [2] yields the following theorem.

Theorem 2. *Every ternary semigroup is embeddable into a disjoint ternary semigroup $(T[X, Y], f)$ of mappings of some sets X and Y.*

Let $\mathbf{K}$ be a category, X and Y its objects. We denote by $Hom_{\mathbf{K}}(X, Y)$ the set of all morphisms of X into Y in $\mathbf{K}$. Suppose $Hom_{\mathbf{K}}(X, Y) \neq \emptyset \neq Hom_{\mathbf{K}}(Y, X)$. Then we put $Hom_{\mathbf{K}}[X, Y] = Hom_{\mathbf{K}}(X, Y) \times Hom_{\mathbf{K}}(Y, X)$. Define the mapping $f_{\mathbf{K}}[X, Y] : (Hom_{\mathbf{K}}[X, Y])^3 \to Hom_{\mathbf{K}}[X, Y]$ putting

$$(*) \quad f_{\mathbf{K}}[X, Y]((p_1, q_1), (p_2, q_2), (p_3, q_3)) = (p_1 \circ q_2 \circ p_3, q_1 \circ p_2 \circ q_3)$$

for all (p_i, q_i) in $Hom_{\mathbf{K}}[X, Y]$ where $i = 1,\ 2,\ 3$. Then $(Hom_{\mathbf{K}}[X, Y], f_{\mathbf{K}}[X, Y])$ is a ternary semigroup which is called the *ternary semigroup of morphisms of objects X and Y in the category $\mathbf{K}$*.

Let I be a functor from a category $\mathbf{K}$ to a category $\mathbf{K}'$. By Io, Im we denote the object mapping and the morphism mapping of the functor I, respectively.

Theorem 3. *Let $\mathbf{K}$ and $\mathbf{K}'$ be categories, I an isomorphism of $\mathbf{K}$ onto $\mathbf{K}'$. Assume that X and Y are objects in $\mathbf{K}$ such that $Hom_{\mathbf{K}}(X, Y) \neq \emptyset \neq Hom_{\mathbf{K}}(Y, X)$.*

Then the ternary semigroups of morphisms $(Hom_{\mathbf{K}}[X, Y], f_{\mathbf{K}}[X, Y])$ and $(Hom_{\mathbf{K}'}[Io(X), Io(Y)], f_{\mathbf{K}'}[Io(X), Io(Y)])$ are isomorphic. The corresponding isomorphism assigns to any pair $(p, q) \in Hom_{\mathbf{K}}[X, Y]$ the pair $(Im(p), Im(q)) \in Hom_{\mathbf{K}'}[Io(X), Io(Y)]$.

Proof. It is clear that the restrictions of the mapping Im are bijections of $Hom_{\mathbf{K}}(X, Y)$ onto $Hom_{\mathbf{K}'}(Io(X), Io(Y))$ and of $Hom_{\mathbf{K}}(Y, X)$ onto $Hom_{\mathbf{K}'}(Io(Y), Io(X))$. Let us put $H(p, q) = (Im(p), Im(q))$ for any $(p, q) \in Hom_{\mathbf{K}}[X, Y]$. Of course $H(p, q) \in Hom_{\mathbf{K}'}[Io(X), Io(Y)]$ for any $(p, q) \in Hom_{\mathbf{K}}[X, Y]$ and H is a bijection. Since I is a functor, we obtain for any elements $(p_1, q_1),\ (p_2, q_2),\ (p_3, q_3)$ in $Hom_{\mathbf{K}}[X, Y]$ the following relations:
$H(f_{\mathbf{K}}[X, Y]((p_1, q_1), (p_2, q_2), (p_3, q_3))) = H(p_1 \circ q_2 \circ p_3, q_1 \circ p_2 \circ q_3) = (Im(p_1 \circ q_2 \circ p_3), Im(q_1 \circ p_2 \circ q_3)) = (Im(p_1) \circ Im(q_2) \circ Im(p_3), Im(q_1) \circ Im(p_2) \circ Im(q_3)) =$

$f_{\mathbf{K}'}[Io(X), Io(Y)]((Im(p_1), Im(q_1)), (Im(p_2), Im(q_2)), (Im(p_3), Im(q_3))) =$
$f_{\mathbf{K}'}[Io(X), Io(Y)](H(p_1, q_1), H(p_2, q_2), H(p_3, q_3))$. Thus H is a homomorphism and, consequently, an isomorphism. □

6. Applications to categories $\mathbf{REL}n+1$ and $\mathbf{ALG}n$

We now apply Theorem 3 to a particular situation described in Corollary. We obtain

Theorem 4. *Let n be a positive integer. If (A, r) and (A', r') are objects in the category $\mathbf{REL}n+1$ such that*

$$Hom_{\mathbf{REL}n+1}[(A, r), (A', r')] \neq \emptyset \neq Hom_{\mathbf{REL}n+1}[(A', r'), (A, r)]\,,$$

then the ternary semigroups

$$(Hom_{\mathbf{REL}n+1}[(A, r), (A', r')], f_{\mathbf{REL}n+1}[(A, r), (A', r')])$$

and

$$(Hom_{\mathbf{ALG}n}[(\boldsymbol{P}(A), \mathbf{R}[r]), (\boldsymbol{P}(A'), \mathbf{R}[r'])], f_{\mathbf{ALG}n}[(\boldsymbol{P}(A), \mathbf{R}[r]), (\boldsymbol{P}(A'), \mathbf{R}[r'])])$$

are isomorphic. The corresponding isomorphism assigns to any pair $(p, q) \in Hom_{\mathbf{REL}n+1}[(A, r), (A', r')]$ the pair $(\mathbf{P}[p], \mathbf{P}[q]) \in Hom_{\mathbf{ALG}n}[(\boldsymbol{P}(A), \mathbf{R}[r]), (\boldsymbol{P}(A'), \mathbf{R}[r'])]$.

Example 1. The category $\mathbf{REL}2$ appears under the name $\mathbf{STR}$ and the category $\mathbf{ALG}1$ under the name $\mathbf{PMA}$ in [3].

Let A, A' be sets, $r = \{(x, y) \in A \times A;\ x \neq y\}$, $r' = \{(x', y') \in A' \times A';\ x' \neq y'\}$. Then (A, r), (A', r') are objects in $\mathbf{REL}2$.

A mapping $h : A \to A'$ is a strong homomorphism of (A, r) into (A', r') if and only if it is a bijection.

Indeed, if h is a bijection and $x \in A$, $y' \in A'$, then $(h(x), y') \in r'$ means $h(x) \neq y'$ which is equivalent to the existence of $y \in A$ such that $h(y) = y'$ and $x \neq y$, i.e., $(x, y) \in r$. Thus, h is a strong homomorphism.

On the other hand, if h is a strong homomorphism and x, y in A are such that $x \neq y$, then $(x, y) \in r$ which implies that $(h(x), h(y)) \in r'$, i.e., $h(x) \neq h(y)$. Thus, h is injective. Furthermore, if $y' \in A'$, $x \in A$ are arbitrary, then either $h(x) = y'$ or $(h(x), y') \in r'$ which implies the existence of y such that $h(y) = y'$ and $x \neq y$. Hence h is a bijection.

Furthermore, $\mathbf{R}[r](\emptyset) = \emptyset$, $\mathbf{R}[r](\{x\}) = \{y \in A;\ x \neq y\} = A - \{x\}$ for any $x \in A$, and $\mathbf{R}[r](X) = A$ for any $X \in \boldsymbol{P}(A)$ with at least two elements. Any morphism of $(\boldsymbol{P}(A), \mathbf{R}[r])$ into $(\boldsymbol{P}(A'), \mathbf{R}[r'])$ is of the form $\mathbf{P}[h]$ where h is a bijection of A onto A'.

Thus, the elements of $Hom_{\mathbf{REL}2}[(A, r), (A', r')]$ are of the form (p, q) where p is a bijection of A onto A' and q is a bijection of A' onto A. Simultaneously, $(\mathbf{P}[p], \mathbf{P}[q])$ is the general form of an element in $Hom_{\mathbf{ALG}1}[(\boldsymbol{P}(A), \mathbf{R}[r]), (\boldsymbol{P}(A'), \mathbf{R}[r'])]$. If we define the operations $f_{\mathbf{REL}2}[(A,r),(A',r')]$ and $f_{\mathbf{A}LG1}[(\boldsymbol{P}(A),\mathbf{R}[r]),(\boldsymbol{P}(A'),\mathbf{R}[r'])]$ by $(*)$, we obtain two isomorphic ternary semigroups.

Example 2. The category **REL**3 appears under the name **TER** and the category **ALG**2 under the name **PGR** in [4].

Let $A,\ A'$ be sets. Put $r=\{(x,y,z)\in A^3;\ \text{ either } x=z \text{ or } y=z\}$, $r'=\{(x',y',z')\in(A')^3;\ \text{ either } x'=z' \text{ or } y'=z'\}$. Then $(A,r),\ (A',r')$ are objects in the category **REL**3.

It is easy to see that any mapping h of A into A' is a strong homomorphism of (A,r) into (A',r'). Indeed, if $x\in A,\ y\in A,\ z'\in A'$, then $(h(x),h(y),z')\in r'$ holds if either $h(x)=z'$ or $h(y)=z'$. Since $(x,y,x)\in r,\ (x,y,y)\in r$ hold, h is a strong homomorphism.

Furthermore $\mathbf{R}[r](X,Y)=\{z\in A;\ x\in X,\ y\in Y,\ (x,y,z)\in r\}=X\cup Y$. Any morphism of $(\boldsymbol{P}(A),\mathbf{R}[r])$ into $(\boldsymbol{P}(A'),\mathbf{R}[r'])$ is of the form $\mathbf{P}[h]$ where h is a mapping of A into A'.

Thus, the elements of $Hom_{\mathbf{REL}3}[(A,r),(A',r')]$ are of the form (p,q) where p is a mapping of A into A' and q is a mapping of A' into A. Simultaneously, the elements in $Hom_{\mathbf{ALG}2}[(\boldsymbol{P}(A),\mathbf{R}[r]),(\boldsymbol{P}(A'),\mathbf{R}[r'])]$ coincide with the pairs of the form $(\mathbf{P}[p],\mathbf{P}[q])$. If defining the operations $f_{\mathbf{REL}3}[(A,r),(A',r')]$ and $f_{\mathbf{ALG}2}[(\boldsymbol{P}(A),\mathbf{R}[r]),(\boldsymbol{P}(A'),\mathbf{R}[r'])]$ by $(*)$, we obtain two isomorphic ternary semigroups.

Example 3. Let $A,\ A'$ be sets. Put $r=\{(x,x,x);x\in A\},\ r'=\{(x',x',x');x'\in A'\}$. Then $(A,r),\ (A',r')$ are objects in the category **REL**3.

We prove that the set $Hom_{\mathbf{REL}3}((A,r),(A',r'))$ coincides with the set of all injective mappings of A into A'.

Indeed, if $h:A\to A'$ is injective and $x\in A,\ y\in A,\ z'\in A'$, then $(h(x),h(y),z')\in r'$ means $h(x)=z'=h(y)$ which is equivalent to $x=y$ and $h(x)=z'$ and, therefore, to $(x,y,x)\in r,\ h(x)=z'$. Thus, h is a strong homomorphism.

On the other hand, if h is a strong homomorphism and $x\in A,\ y\in A$ are such that $h(x)=h(y)$, then $(h(x),h(y),h(x))\in r'$ which implies the existence of $z\in A$ with $(x,y,z)\in r$ and $h(z)=h(x)$, i.e., $x=y=z$. Thus, h is injective.

Furthermore, $\mathbf{R}[r](X,Y)=\{z\in A;\ (x,y,z)\in r,\ x\in X,\ y\in Y\}=X\cap Y$. Any morphism of $(\boldsymbol{P}(A),\mathbf{R}[r])$ into $(\boldsymbol{P}(A'),\mathbf{R}[r'])$ is of the form $\mathbf{P}[h]$ where h is an injective mapping of A into A'.

Thus, the elements of $Hom_{\mathbf{REL}3}[(A,r),(A',r')]$ are of the form (p,q) where p is an injective mapping of A into A' and q is an injective mapping of A' into A. Simultaneously, $(\mathbf{P}[p],\mathbf{P}[q])$ is the general form of an element in $Hom_{\mathbf{ALG}2}[(\boldsymbol{P}(A),\mathbf{R}[r]),(\boldsymbol{P}(A'),\mathbf{R}[r'])]$. Furthermore, if defining the operations $f_{\mathbf{REL}3}[(A,r),(A',r')]$ and $f_{\mathbf{ALG}2}[(\boldsymbol{P}(A),\mathbf{R}[r]),(\boldsymbol{P}(A'),\mathbf{R}[r'])]$ by $(*)$, we obtain two isomorphic ternary semigroups.

7. Concluding remarks

Though the assertion of Theorem 3 is not surprising and its proof is simple, this theorem enables to state that some ternary semigroups are isomorphic which would be laborious without the results presented here.

The problem solved in this paper is an example of a richer class of problems that are solved using isofunctors. Such problems may be found also in [3], [4], [5], [6], [7].

References

[1] MacLane, S., *Categories for the working mathematician,* Springer, New York - Heidelberg - Berlin 1971.

[2] Monk, D., Sioson, F. M., *m-Semigroups, semigroups, and function representation,* Fund. Math. **59** (1966), 233-241.

[3] Novotný, M., *Construction of all strong homomorphisms of binary structures,* Czech. Math. J. **41 (116)** (1991), 300-311.

[4] Novotný, M., *Ternary structures and groupoids,* Czech. Math. J. **41 (116)** (1991), 90-98.

[5] Novotný, M., *On some correspondences between relational structures and algebras,* Czech. Math. J. **43 (118)** (1993), 643-647.

[6] Novotný, M., *Construction of all homomorphisms of groupoids,* presented to Czech. Math. J.

[7] Pultr, A., Trnková, V., *Combinatorial, algebraic and topological representations of groups, semigroups and categories,* Academia, Prague 1980.

[8] Sioson, F. M., *Ideal theory in ternary semigroups,* Math. Japon. **10** (1965), 63-84.

Antoni Chronowski
Wyższa Szkoła Pedagogiczna
Instytut Matematyki
ul. Podchorążych 2
30-884 Kraków, POLAND

Miroslav Novotný
Faculty of Computer Science
Masaryk University
Burešova 20
602 00 Brno, CZECH REPUBLIC