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ON THE DARBOUX TRANSFORMATION II

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Abstract. Automorphisms of the family of all Sturm-Liouville equations $y'' = qy$ are investigated. The classical Darboux transformation arises as a particular case of a general result.

1. Introduction

To make this part independent of [7], we recall the classical Darboux result.

Let nonvanishing functions $y = y(x)$, $z = z(x)$ be solutions of the Sturm-Liouville equations

$$y'' = p(x)y, \quad z'' = q(x)z \tag{1}$$

where the potentials $p(x), q(x)$ differ by a constant: $p - q = \lambda \in \mathbf{R}$. Then the function $\tilde{y} = y' - yz'/z$ satisfies the Sturm-Liouville equation

$$\tilde{y}'' = \tilde{p}(x)\tilde{y} \quad (\tilde{p} = p - 2(z'/z)'). \tag{2}$$

We moreover recall the following generalization derived in [7] by elementary method.

Retaining the notation and all assumptions as above, *let us consider new variables*

$$U = y'/y + z'/z, \quad V = y'/y - z'/z \tag{3}$$

and choose quite arbitrarily the functions $h = h(\lambda)$, $g = g(\lambda, x, V) \neq 0$. Denoting moreover

$$\tilde{U} = \frac{1}{g}\left(h - \frac{\partial g}{\partial x} - V'\frac{\partial g}{\partial V}\right), \quad \tilde{V} = g, \tag{4}$$

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then the functions

$$\tilde{y} = e^{\frac{1}{2}\int(\tilde{U}+\tilde{V})dx}, \quad \tilde{z} = e^{\frac{1}{2}\int(\tilde{U}-\tilde{V})dx} \tag{5}$$

satisfy certain Sturm-Liouville equations

$$\tilde{y}'' = \tilde{p}(x)\tilde{y}, \quad \tilde{z}'' = \tilde{q}(x)\tilde{z}$$

such that the difference of new potentials $\tilde{p}, \tilde{q}$ *is equal to the constant* $\tilde{p}-\tilde{q} = h(\lambda) \in$ $\mathbf{R}$ *depending on the choice of the previous difference* $p-q=\lambda$. The generalization resists on the fact that the identity $y''/y - z''/z = p - q = \lambda \in \mathbf{R}$ simplifies as

$$V' + UV = \lambda, \quad \lambda' = 0. \tag{6}$$

in terms of variables (3).

For the particular choice $h = A\lambda$, $g = BV$ $(A, B \in \mathbf{R})$, the formulae (5) can be rewritten as

$$\begin{aligned} \tilde{y} &= V^{\frac{1}{2}(\frac{A}{B}-1)} y^{\frac{1}{2}(\frac{A}{B}+B)} z^{\frac{1}{2}(\frac{A}{B}-B)}, \\ \tilde{z} &= V^{\frac{1}{2}(\frac{A}{B}-1)} y^{\frac{1}{2}(\frac{A}{B}-B)} z^{\frac{1}{2}(\frac{A}{B}+B)} \end{aligned} \tag{7}$$

and are quadrature-free. The more intricate choice $h = A\lambda$, $g = B\lambda/V$ gives

$$\begin{aligned} \tilde{y} &= V^{\frac{1}{2}(1+B)} y^{\frac{1}{2}(\frac{A}{B}+B)} z^{-\frac{1}{2}(\frac{A}{B}-B)}, \\ \tilde{z} &= V^{\frac{1}{2}(1-B)} y^{\frac{1}{2}(\frac{A}{B}-B)} z^{-\frac{1}{2}(\frac{A}{B}+B)} \end{aligned} \tag{8}$$

and this may be regarded a generalization of the classical result. Our present aim is to determine *all* quadrature-free formulae. The method of proof inspired from [4,5] will not be elementary and may be of a certain independent interest.

2. Fundamental concepts

Our reasonings will be carried out in the space $\mathbf{R}^\infty$ of all real sequences $P = (p^1, p^2, \ldots)$, $p^i \in \mathbf{R}$. Denoting by $t^1, t^2, \ldots$ the coordinates in $\mathbf{R}^\infty$ defined by $t^i(P) \equiv p^i$, we shall deal with the family $\mathcal{F}$ of C^∞-smooth *functions* of the kind $f = f(t^1, \ldots t^m)$, each depending on a finite number $m = m(f)$ of coordinates, *vector fields* $Z = z^1\partial/\partial t^1 + z^2\partial/\partial t^2 + \ldots (z^i \in \mathcal{F})$ given by infinite series with quite arbitrary terms, and *differential forms* $\varphi = a^1dt^1 + \cdots + a^ndt^n$ $(n = n(\varphi),\ a^i \in \mathcal{F})$. In order to simplify the exposition, the domain of definition of functions, vector fields, and differential forms will be not specified. (If necessary, one can suppose that all functions under consideration are defined on a certain *box* given by certain inequalities $a^i < p^i < b^i$.) Accepting this common convention, we shall nevertheless speak of various $\mathcal{F}$-modules $\mathcal{V}$. Recall that a sequence $\xi^1, \xi^2, \dots \in \mathcal{V}$ is called a *basis* of such $\mathcal{F}$-module $\mathcal{V}$ if every $\xi \in \mathcal{V}$ can be uniquely expressed by a finite sum $\xi = z^1\xi^1 + \cdots + z^m\xi^m$ $(m = m(\xi),\ z^i \in \mathcal{F})$. A submodule $\mathcal{U} \subset \mathcal{V}$ is called *generic* if there exists a basis of $\mathcal{U}$ which may be completed to give a basis of $\mathcal{V}$. Denoting by $\ell(\mathcal{V})$ the cardinality of a basis (i.e., either $\ell(\mathcal{V})$ is a natural number or $\ell(\mathcal{V}) = \infty$), clearly $\ell(\mathcal{U}) \leq \ell(\mathcal{V})$ if $\mathcal{U} \subset \mathcal{V}$ is a generic submodule and $\ell(\mathcal{U}) < \ell(\mathcal{V})$ if moreover $\ell(\mathcal{U}) < \infty$ and $\mathcal{U} \neq \mathcal{V}$. For example, the totality Φ of all differential forms φ is an $\mathcal{F}$-module with the basis $dt^1, dt^2, \ldots$ and differential forms vanishing at a fixed point *do not* constitute a generic submodule of Φ. The totality $\mathcal{T}$ of all vector fields Z is an $\mathcal{F}$-module but it does not have any basis in the above sense.

The vector field $Z = z^1\partial/\partial t^1 + z^2\partial/\partial t^2 + \ldots$ can be *substituted* into the differential form $\varphi = a^1dt^1 + \cdots + a^ndt^n$ which provides the function denoted $\varphi(Z) = a^1z^1 + \cdots + a^nz^n \in \mathcal{F}$. For the particular case $\varphi = df = \partial f/\partial t^1.dt^1 + \cdots + \partial f/\partial t^m.dt^m$, we abbreviate $Zf = df(Z) = z^1\partial f/\partial t^1 + \cdots + z^m\partial f/\partial t^m$ which is common "directional derivative" of function f generalized on the infinite-dimensional space $\mathbf{R}^\infty$. In particular $Zt^i \equiv z^i$ for every coordinate t^i. Quite analogous is the *Lie derivative* $\mathcal{L}_Z\varphi \in \Phi$ which may be regarded for a "directional derivative" of a form φ in the direction of the vector field Z. Using the well-known rules

$$\mathcal{L}_Z(\varphi + \psi) = \mathcal{L}_Z\varphi + \mathcal{L}_Z\psi, \mathcal{L}_Z(f\varphi) = Zf.\varphi + f\mathcal{L}_Z\varphi, \mathcal{L}_Zdf = dZf,$$

it may be easily calculated in terms of coordinates:

$$\begin{aligned} \mathcal{L}_Z(a^1dt^1 + \cdots + a^ndt^n) &= \sum_i Za^idt^i + \sum_i a^idZt^i = \\ &= \sum_i \left(\sum_j z^j \frac{\partial a^i}{\partial t^j} \right) dt^i + \sum_i a^idz^i = \\ &= \sum_i \sum_j \left(z^j \frac{\partial a^i}{\partial t^j} + a^j \frac{\partial z^j}{\partial t^i} \right) dt^i. \end{aligned} \tag{9}$$

In particular, if $Z = \partial/\partial t^1$ then the operator $\mathcal{L}_Z$ turns into the "derivation with respect to the parameter" given by

$$\mathcal{L}_{\partial/\partial t^1}(a^1dt^1 + \cdots + a^ndt^n) = \Sigma \frac{\partial a^i}{\partial t^1} dt^i,$$

which is a very particular case of (9).

Admissible mappings $\mathbf{F}$ of the space $\mathbf{R}^\infty$ which assign a point $\mathbf{F}P = Q = (q^1, q^2, \ldots)$ to a point $P = (p^1, p^2, \ldots)$ are given by a certain infinite array of equations

$$q^i \equiv f^i(p^1, \ldots, p^{m(i)}) \quad (i = 1, 2, \ldots) \tag{10}$$

where $f^i \in \mathcal{F}$. Then, for every $f \in \mathcal{F}$, we introduce the *transformed function* $\mathbf{F}^*f = f \circ \mathbf{F}$, equivalently, $\mathbf{F}^*f(P) = f(Q)$. Moreover, using the well-known rules

$$\mathbf{F}^*(\varphi + \psi) = \mathbf{F}^*\varphi + \mathbf{F}^*\psi,\ \mathbf{F}^*(f\varphi) = \mathbf{F}^*f.\mathbf{F}^*\varphi,\ \mathbf{F}^*df = d\mathbf{F}^*f,$$

the *transformed differential form*

$$\mathbf{F}^*\varphi = \mathbf{F}^*\Sigma a^i dt^i = \Sigma \mathbf{F}^* a^i . \mathbf{F}^* dt^i = \Sigma \mathbf{F}^* a^i . d\mathbf{F}^* t^i$$

can be calculated in terms of coordinates. Quite explicitly, using the formulae (10), one can then obtain

$$\mathbf{F}^* f = f(f^1, \dots, f^m), \tag{11}$$

in particular $\mathbf{F}^* t^i \equiv f^i$. Moreover

$$\mathbf{F}^*\varphi = \Sigma a^i(f^1, \dots, f^{n(i)}) df^i = \Sigma a^i(f^1, \dots, f^{n(i)}) \Sigma \frac{\partial f^i}{\partial t^j} dt^j \tag{12}$$

(where $a^i \equiv a^i(t^1, \dots, t^{n(i)})$), in particular $\mathbf{F}^* dt^i \equiv \Sigma \partial f^i / \partial t^j . dt^j$. Transformation of vector fields causes some troubles. Assuming that the inverse $\mathbf{F}^{-1} = \mathbf{G}$ exists, hence (10) can be inverted by certain formulae

$$p^i \equiv g^i(q^1, \dots, q^{n(i)}) \qquad (i = 1, 2, \dots)$$

where $g^i \in \mathcal{F}$, we may introduce the *transformed vector field* $\mathbf{F}_* Z$ by the property

$$(\mathbf{F}_* Z) f = \mathbf{G}^* Z(\mathbf{F}^* f) \qquad (f \in \mathcal{F} \text{ is varying}). \tag{13}$$

In particular we have

$$\begin{aligned} (\mathbf{F}_* Z) t^j &\equiv \mathbf{G}^* Z f^j = \mathbf{G}^* \Sigma z^i \frac{\partial f^j}{\partial t^i} = \\ &= \Sigma z^i (g^1, \dots, g^k) \frac{\partial f^j}{\partial t^i} (g^1, \dots, g^m) \end{aligned} \tag{14}$$

for every coordinate t^j, hence finally

$$\mathbf{F}_* Z = \Sigma \bar{z}^j \frac{\partial}{\partial t^j} \tag{15}$$

where $\bar{z}^j = (\mathbf{F}_* Z) t^j$ is given as above.

3. Underdetermined differential equations

We shall deal with systems of ordinary differential equations which involve more unknown functions $y^1 = y^1(x), \dots, y^m = y^m(x)$ than is the total number of differential equations under consideration. Since every such a higher order system can be represented by an equivalent first order system, we may assume, that our equations are

$$\begin{gathered} dy^1/dx = f^1(x, y^1, \dots, y^m, dy^{c+1}/dx, \dots, dy^m/dx), \\ \dots \\ dy^c/dx = f^c(x, y^1, \dots, y^m, dy^{c+1}/dx, \dots, dy^m/dx), \end{gathered} \tag{16}$$

without any essential loss of generality. Here $1 \leq c < m$ in order to avoid trivialities. Assuming the functions f^i to be C^∞-smooth, such a system admits a lot of solutions: the functions $y^{c+1} = y^{c+1}(x), \dots, y^m = y^m(x)$ can be arbitrarily chosen and then the remaining $y^1 = y(x), \dots, y^c = y^c(x)$ are uniquely determined by choosing the initial values $y^1(a) = a^1, \dots, y^c(a) = a^c$ at a certain point $x = a$. We shall be interested in automorphisms of certain particular systems of the kind (16) preserving the independent variable x. In full generality, such automorphisms will transform every solution

$$y^1 = y^1(x), \dots, y^m = y^m(x) \tag{17}$$

of the system (16) into the new solution

$$y^i \equiv h^i(x, y^1(x), \dots, y^m(x), \dots, d^a y^1(x)/dx^a, \dots, d^a y^m(x)/dx^a) \ (i = 1, \dots, m),$$

where $h^1, \dots, h^m$ are certain very special and fixed functions and the constant a is not apriori limited. It follows that all derivatives of the solution (17) must be taken into account, i.e., we are compelled to use the space $\mathbf{R}^\infty$ with coordinates

$$x, y_0^1, \dots, y_0^m, y_1^{c+1}, \dots, y_1^m, y_2^{c+1}, \dots, y_2^m, \dots, \tag{18}$$

where y_s^j stands for the derivative $d^s y^j/dx^s$. Note that the variables $y_1^1, \dots, y_1^c, y_2^1, \dots, y_2^c, \dots$ must not be included into the coordinates. This is the consequence of formulae (16) and their derivatives, for instance

$$d^2 y^k/dx^2 = \frac{\partial f^k}{\partial x} + \sum_{j=1}^{m} \frac{\partial f^k}{\partial y_0^j} f^j + \sum_{i=c+1}^{m} \frac{\partial f^k}{\partial y_1^i} y_2^i \quad (k = 1, \dots, c)$$

and analogously for higher order derivatives $d^s y^k/dx^s$ $(s = 3, 4, \dots, k = 1, \dots, c)$. But in order to obtain a clear theory, it is better to reformulate the problem in terms of differential forms.

According to the above reasonings, we find ourselves in the space $\mathbf{R}^\infty$ with coordinates (18). We introduce the differential forms

$$\begin{aligned} &\omega^k = dy_0^k - f^k dx \ (k = 1, \dots, c; f^k \equiv f^k(x, y_0^1, \dots, y_0^m, y_1^{c+1}, \dots, y_1^m)) \\ &\vartheta_s^j \equiv dy_s^{c+j} - y_{s+1}^{c+j} dx \qquad (j = 1, \dots, n; s = 0, 1, \dots) \end{aligned} \tag{19}$$

which may be regarded for a good substitute of the original system (16) and moreover express the sense of the variables y_s^j: a mapping $\mathbf{F}$ of an interval $a < x < b$ into $\mathbf{R}^\infty$ given by

$$\mathbf{F}(x) = (x, y_0^1(x), \dots, y_0^m(x), y_1^{c+1}(x), \dots, y_1^m(x), y_2^{c+1}(x), \dots, y_2^m(x), \dots) \in \mathbf{R}^\infty$$

in terms of the coordinates (18) clearly satisfies

$$\mathbf{F}^*\omega^k = \mathbf{F}^*\vartheta^j_s \equiv 0 \tag{20}$$

if and only if $y^1(x) = y^1_0(x), \dots, y^m(x) = y^m_0(x)$ is a solution of the system (16) and moreover $y^j_s(x) \equiv d^s y^j(x)/dx^s$ are higher order derivatives.

But the most important achievement of this reformulation is that it can be expressed without any use of coordinates. This is achievement as follows. Denoting by

$$\Phi = \{dx, dy^1_0, \dots, dy^m_0, dy^{c+1}_1, \dots, dy^m_1, dy^{c+1}_2, \dots, dy^m_2, \dots\}$$

the common module of all differential forms on our space with coordinates (18), let $\Omega \subset \Phi$ be the submodule

$$\Omega = \{\omega^1, \dots, \omega^c, \vartheta^1_0, \dots \vartheta^n_0, \vartheta^1_1, \dots, \vartheta^n_1, \vartheta^1_2, \dots, \vartheta^n_2, \dots\},$$

i.e., the submodule generated by all forms (19), which constitute even a basis of Ω. One can observe that the property (20) can be expressed only in terms of the submodule $\Omega \subset \Phi$: it is equivalent to $\mathbf{F}^*\omega \equiv 0$ ($\omega \in \Omega$).

The submodule $\Omega \subset \Phi$ is of a certain special kind. First of all, it is of codimension 1 since $dx \notin \Omega$ but the form dx together with all forms from Ω generate Φ. It follows that the submodule $\Omega^\perp \subset \mathcal{T}$ of all vector fields $Z \in \mathcal{T}$ satisfying $\omega(Z) = 0$ ($\omega \in \Omega$) consists of all multiples of a single vector field, e.g., of the vector field

$$X = \partial/\partial x + \sum_{j=1}^{m} f^j \partial/\partial y^j_0 + \sum_{i=c+1}^{m} \sum_{s=0}^{\infty} y^i_{s+1} \partial/\partial y^i_s. \tag{21}$$

One can calculate the Lie derivatives

$$\mathcal{L}_X\omega^k \equiv \sum_j \frac{\partial f^k}{\partial y^1_0}\omega^j + \sum_j \sum_s \frac{\partial f^k}{\partial y^j_s}\vartheta^j_s, \ \mathcal{L}_X\vartheta^j_s \equiv \vartheta^j_{s+1}. \tag{22}$$

These equations are not preserved if X is replaced by an arbitrary vector field $Z \in \Omega^\perp$. Let us however consider the filtration

$$\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega = \cup\Omega_s \tag{23}$$

consisting of submodules

$$\Omega_s \equiv \{\omega^1, \dots, \omega^c, \vartheta^1_0, \dots, \vartheta^n_0, \dots, \vartheta^1_{s-1}, \dots, \vartheta^n_{s-1}\} \subset \Omega.$$

Then the equations imply $\mathcal{L}_X\Omega_s \subset \Omega_{s+1}$ and using the general rule $\mathcal{L}_{fX}\varphi \equiv \varphi(X)df + f\mathcal{L}_X\varphi$ (valid for all $X \in \mathcal{T}$, $\varphi \in \Phi$), it follows that

$$\mathcal{L}_Z\Omega_s \subset \Omega_{s+1}(\text{all } s),\ \Omega_s + \mathcal{L}_Z\Omega_s = \Omega_{s+1}\ (s \text{ large}), \tag{24}$$

for every nonvanishing vector field $Z \in \Omega^\perp$. These properties of the submodule $\Omega \subset \Phi$ are quite enough in the following sense: *in the module Φ of all forms on $\mathbf{R}^\infty$, let $\Omega \subset \Phi$ be a submodule of codimension 1 such that there exists a filtration* (23) *satisfying* (24) *for an arbitrary* (equivalently: *appropriate*) *nonvanishing vector field $Z \in \Omega^\perp \subset \mathcal{T}$. Then Ω may be regarded for the submodule corresponding to a certain system* (16) *in the construction as stated above*. Expressively saying, these submodules $\Omega \subset \Phi$ called *diffieties* represent the systems (16) in a coordinate-free manner. In reality, we shall not need this result so that more comments are unnecessary.

4. Standard filtrations

Recall that diffiety is a submodule $\Omega \subset \Phi$ of the module Φ of all differential forms such that $\ell(\Phi/\Omega) = 1$ and that there exist filtration (23) satisfying (24) with $0 \neq Z \in \Omega^{\perp}$. Later on, we shall be interested in *automorphisms* of the diffiety Ω, i.e., in invertible transformations $\mathbf{F}$ such that $\mathbf{F}^*\Omega = \Omega$. This is an extremely difficult task and it will be facilitated by using a certain special filtration, cf. [4,5].

For an arbitrary submodule $\Gamma \subset \Omega$ of a diffiety Ω, let $\operatorname{Ker}\Gamma \subset \Gamma$ be the submodule of all $\gamma \in \Gamma$ such that $\mathcal{L}_X\gamma \in \Gamma$ for a certain nonvanishing vector field $X \in \Omega^{\perp}$. (One can observe that the choice of the field X is not important. This follows from the rule $\mathcal{L}_{fX}\omega \equiv f\mathcal{L}_X\omega$ valid for any $f \in \mathcal{F}$, $X \in \Omega^{\perp}$, $\omega \in \Omega$.) In equivalent terms, $\gamma \in \Gamma$ belongs to $\operatorname{Ker}\Gamma$ if the composition

$$\gamma \in \Gamma \to \mathcal{L}_X\gamma \in \Omega \to \text{ the class } [\mathcal{L}_X\gamma] \text{ in } \Omega/\Gamma$$

is vanishing. Since this composition $\gamma \to [\mathcal{L}_X\gamma]$ is $\mathcal{F}$-linear (which is a consequence of the equations

$$\alpha + \beta \to [\mathcal{L}_X(\alpha+\beta)] = [\mathcal{L}_X\alpha] + [\mathcal{L}_X\beta],$$
$$f\gamma \to [\mathcal{L}_X(f\gamma)] = [Xf\gamma + f\mathcal{L}_X\gamma] = f[\mathcal{L}_X\gamma]$$

where $\alpha, \beta, \gamma \in \Gamma$), it follows that $\operatorname{Ker}\Gamma$ is indeed a module. Moreover $\Gamma/\operatorname{Ker}\Gamma$ is injectively transformed into Ω/Γ and even more: if $\operatorname{Im}\Gamma \subset \Omega$ is the submodule of Ω generated by all forms of the kind $\mathcal{L}_X\gamma$ ($\gamma \in \Gamma$) then $\Gamma/\operatorname{Ker}\Gamma$ is bijectively transformed onto $\operatorname{Im}\Gamma/\Gamma$ (by means of $\mathcal{L}_X$), hence

$$\ell(\Gamma/\operatorname{Ker}\Gamma) = \ell(\operatorname{Im}\Gamma/\Gamma). \tag{25}$$

This self-evident equation will be soon needed. Then a filtration (23) satisfying (24) will be called a *standard* one if moreover

$$\operatorname{Ker}\Omega_{s+1} \equiv \Omega_s \ (s \geq 0),\ \operatorname{Ker}^2\Omega_0 = \operatorname{Ker}\Omega_0 \neq \Omega_0 \tag{26}$$

is satisfied. There may exist many standard filtrations of a given diffiety excepting a certain very interesting particular case which will be introduced.

In virtue of (24_2), there exist a finite number of forms $\omega^1, \ldots, \omega^\mu \in \Omega$ such that every $\omega \in \Omega$ can be represented by a finite linear combination of certain forms of the kind $\mathcal{L}_Z^k\omega^j$ ($k = 0, 1, \ldots; j = 1, \ldots, \mu$). In view of future needs, we shall be however interested only in the case $\mu = 1$ in a certain "approximative sense". This vague term can be precisely expressed as follows: we shall be interested only in such diffieties Ω that every form $\omega \in \Omega$ satisfies a congruence

$$\omega \cong \Sigma a^k\mathcal{L}_Z^k\omega^1 \pmod{\gamma^1, \ldots, \gamma^c} \tag{27}$$

with (a certain finite sum and) appropriate fixed forms $\gamma^1, \ldots, \gamma^c \in \Omega$. Alternatively, (27) is expressed by

$$\omega = \Sigma a^k\mathcal{L}_Z^k\omega^1 + \Sigma c^j\gamma^j \quad (a^k, c^j \in \mathcal{F}). \tag{28}$$

To avoid some exceptional singular points, all the forms $\gamma^1,\dots,\gamma^c,\omega^1,\mathcal{L}_X\omega^1,\mathcal{L}_X^2\omega^1$, ... will be assumed linearly independent and may be therefore used for a basis of Ω. As yet they are not uniquely determined (for instance, one can choose ω^1 for γ^{c+1} and then $\mathcal{L}_X\omega^1$ for the new ω^1, as well) but this unconvenience can be suppressed. We shall prove a theorem which may be regarded for a convenient substitute for the uniqueness of the basis of Ω.

5. Theorem

Assuming (27), *there exists a unique standard filtration of the diffiety* $\Omega\subset\Phi$.

Proof. Since $\gamma^j\in\Omega$, we have also $\mathcal{L}_X\gamma^j\in\Omega$ and hence

$$\mathcal{L}_X\gamma^j\in\{\gamma^1,\dots,\gamma^c,\omega^1,\dots,\mathcal{L}_X^S\omega^1\}\quad (j=1,\dots,c; S \text{ large enough})$$

in virtue of (27) applied on the forms $\omega=\gamma^j$. So we may introduce the (in general non-standard) filtration (23) with terms

$$\Omega_s\equiv\{\gamma^1,\dots,\gamma^c,\omega^1,\mathcal{L}_X\omega^1,\dots,\mathcal{L}_X^{S+s}\omega^1\}\quad (s=0,1,\dots); \tag{29}$$

note that the requirements (24) are clearly satisfied. One can moreover see that (24_2) is true even for all $s\ge 0$. This filtration will be successively improved to the sought result.

One can directly find that $\operatorname{Ker}\Omega_{s+1}\equiv\Omega_s$ for every $s\ge 0$. This is the property (26_1), however, the property (26_2) is not yet guaranteed.

Continuing the proof, let us successively define $\Omega_{-1}=\operatorname{Ker}\Omega_0$, $\Omega_{-2}=\operatorname{Ker}\Omega_{-1}$, and so on. So we obtain a descending sequence $\Omega_0\supset\Omega_{-1}\supset\Omega_{-2}\supset\dots$ which is therefore stationary after a certain term:

$$\Omega_{-R}\ne\Omega_{-R-1}\quad\text{but}\quad \Omega_{-R-1}=\Omega_{-R-2}=\dots\ . \tag{30}$$

Let us introduce the modules

$$\bar\Omega_0=\Omega_{-R},\bar\Omega_1=\Omega_{-R+1},\dots,\bar\Omega_R=\Omega_0,\bar\Omega_{R+1}=\Omega_1,\dots\ .$$

Clearly $\bar\Omega_0\subset\bar\Omega_1\subset\dots\subset\Omega=\cup\bar\Omega_s$ and we shall verify that the requirements (24), (26) are satisfied (for the modules $\bar\Omega_s$ at the place of Ω_s), i.e., that we have a standard filtration.

First of all, since $\bar\Omega_s\equiv\operatorname{Ker}\bar\Omega_{s+1}$, the module $\bar\Omega_s$ consists of all $\omega\in\bar\Omega_{s+1}$ with $\mathcal{L}_X\omega\in\bar\Omega_{s+1}$ hence $\mathcal{L}_X\bar\Omega_s\subset\bar\Omega_{s+1}$ and (24_1) is true. Moreover, we have

$$1\le\ell(\bar\Omega_{s+1}/\bar\Omega_s)\equiv\ell(\bar\Omega_{s+1}/\operatorname{Ker}\bar\Omega_{s+1})=\ell(\operatorname{Im}\bar\Omega_{s+1}/\bar\Omega_{s+1})\le$$
$$\le\ell(\bar\Omega_{s+2}/\bar\Omega_{s+1})$$

by using the definition of $\operatorname{Ker}\bar\Omega_{s+1}$, (26), ($24_1$). However $\bar\Omega_s\equiv\Omega_{s-R}$ for s large enough ($s\ge R$) hence

$$\ell(\bar\Omega_{s+2}/\bar\Omega_{s+1})\equiv\ell(\Omega_{s-R+2}/\Omega_{s-R+1})=1$$

(see (29)) and it follows $\ell(\bar\Omega_{s+1}/\bar\Omega_s) \equiv 1$ for all $s \geq 0$. Moreover clearly

$$1 \leq \ell(\bar\Omega_0/\mathrm{Ker}\,\bar\Omega_0) = \ell(\mathrm{Im}\ \bar\Omega_0/\bar\Omega_0) \leq \ell(\bar\Omega_1/\bar\Omega_0) = 1$$

whence $\ell(\bar\Omega_0/\mathrm{Ker}\,\bar\Omega_0) = 1$. Altogether taken, it follows

$$\ell(\bar\Omega_0) = \ell(\mathrm{Ker}\,\bar\Omega_0) + 1,\ \ \ell(\bar\Omega_{s+1}) = \ell(\bar\Omega_s) + 1 \quad (s \geq 0).$$

So if we take an *initial form* defined by the properties $\bar\omega \in \bar\Omega_0$ but $\bar\omega \notin \mathrm{Ker}\,\bar\Omega_0$, then $\mathcal{L}_X\bar\omega \in \bar\Omega_1$ but $\mathcal{L}_X\bar\omega \notin \bar\Omega_0 = \mathrm{Ker}\,\Omega_1$, consequently $\mathcal{L}_X^2\bar\omega \in \bar\Omega_2$ but $\mathcal{L}_X^2\bar\omega \notin \bar\Omega_1$, and so on. Denoting by $\bar\gamma^1, \ldots, \bar\gamma^c$ a basis of $\mathrm{Ker}\,\bar\Omega_0$ it follows that

$$\bar\Omega_s \equiv \{\bar\gamma^1, \ldots, \bar\gamma^c, \bar\omega, \mathcal{L}_X\bar\omega, \ldots, \mathcal{L}_X^s\bar\omega\} \tag{31}$$

for all $s \geq 0$. Recall moreover the property

$$\mathrm{Ker}\,^2\bar\Omega_0 = \mathrm{Ker}\,^2\Omega_{-R} = \Omega_{-R-2} = \Omega_{-R-1} = \mathrm{Ker}\,\bar\Omega_0$$

(see (30)) which means that $\mathcal{L}_X\mathrm{Ker}\,\bar\Omega_0 \subset \mathrm{Ker}\,\bar\Omega_0$ hence

$$\mathcal{L}_X\bar\gamma^k \in \{\bar\gamma^1, \ldots, \bar\gamma^c\} \quad (k = 1, \ldots, c). \tag{32}$$

Using (31), (32), the remaining properties (24_2), (26) can be verified at once.

Concerning the uniqueness let (23) be a standard filtration and we have to prove that $\Omega_s \equiv \bar\Omega_s$, where $\bar\Omega_s$ are the modules (31). For this aim, recall that $\mathcal{L}_X$ transforms $\mathrm{Ker}\,\Omega_0$ into itself:

$$\mathrm{Ker}\,\Omega_0 = \mathrm{Ker}\,^2\Omega_0 \to \mathrm{Ker}\,\Omega_0, \tag{33}$$

and moreover yields the *injective* linear mappings

$$\Omega_0/\mathrm{Ker}\,\Omega_0 \to \mathrm{Im}\ \Omega_0/\Omega_0 \subset \Omega_1/\Omega_0, \tag{34}$$

$$\Omega_{s+1}/\mathrm{Ker}\,\Omega_{s+1} = \Omega_{s+1}/\Omega_s \to \mathrm{Im}\ \Omega_{s+1}/\Omega_{s+1} \subset \Omega_{s+2}/\Omega_{s+1}. \tag{35}$$

It follows easily that $\mathrm{Ker}\,\Omega_0$ consists just of such forms $\omega \in \Omega$ that all terms of the series $\omega, \mathcal{L}_X\omega, \mathcal{L}_X^2\omega, \ldots$ are contained in a finite-dimensional module (namely in $\mathrm{Ker}\,\Omega_0$). But recalling (31) and (32), the same property have just the forms lying in the module $\{\bar\gamma^1, \ldots, \bar\gamma^c\} = \mathrm{Ker}\,\bar\Omega_0$, hence $\mathrm{Ker}\,\Omega_0 = \mathrm{Ker}\,\bar\Omega_0$. Moreover, in virtue of the injectivity of the mappings (34), (35), necessarily $\bar\omega \notin \mathrm{Ker}\,\Omega_0$. On the other hand, the module $\mathrm{Ker}\,\Omega_0 = \mathrm{Ker}\,\bar\Omega_0$ together with all forms $\bar\omega, \mathcal{L}_X\bar\omega, \mathcal{L}_X^2\bar\omega, \ldots$ generate the whole diffiety Ω (see (31)), hence in virtue of (34) and (35), necessarily $\bar\omega \in \Omega_0$ and even

$$\Omega_s \equiv \{\bar\gamma^1, \ldots, \bar\gamma^c, \bar\omega, \ldots, \mathcal{L}_X^s\bar\omega\} \tag{36}$$

(using the injectivity (34) and (35)) by induction on s. This is the desired result.

6. The resolving diffiety

After all necessary preliminaries, we come to the main task. Recalling the original equations (1), one can see that three functions y, z, p of independent variable x and their derivatives (in principle, of all orders), and a variable parameter λ (ensuring the difference $p - q = \lambda \in \mathbf{R}$) are in the play. Abbreviating the derivatives by lower indices (explicitly $f_s \equiv d^s f/dx^s$ where f will be any of y, z, p), the equations (1) permit to explicitly determine all derivatives y_s, z_s $(s \geq 2)$ in terms of p_s and $q_s = (p-\lambda)_s$ (where $(p-\lambda)_0 = p - \lambda$ but $(p-\lambda)_s \equiv p_s$ if $s \geq 1$) so we find ourselves in the infinite-dimensional space with coordinates

$$x, \lambda, y_0, y_1, z_0, z_1, p_0 (= p), p_1, p_2, \dots \tag{37}$$

and the structure of the problem is determined by the equations

$$d\lambda = dy_0 - y_1 dx = dz_0 - z_1 dx = dy_1 - py_0 dx = dz_1 - (p-\lambda) z_0 dx = 0, \tag{38}$$

$$dp_s - p_{s+1} dx \equiv 0 \quad (s = 0, 1, \dots). \tag{39}$$

Here (38) are substitute for the requirements $\lambda \in \mathbf{R}$, and (1), moreover (38) and (39) ensure the true sense of the derivatives (we refer to (19), (20) for more details). Individually the equations (38), (39) do not make a good sense: in the spirit of Section 3 above we have to introduce the diffiety Ω generated by all differential forms appearing in (38) and (39).

More explicitly, to abbreviate the notation, we introduce the vector field X expressed in terms of coordinates (37) by

$$X = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + py_0 \frac{\partial}{\partial y_1} + z_1 \frac{\partial}{\partial z_0} + (p-\lambda) z_0 \frac{\partial}{\partial z_1} + \sum_{i=0}^{\infty} p_{s+1} \frac{\partial}{\partial p_s}, \tag{40}$$

the recurrences

$$y_{s+1} \equiv X y_s,\ z_{s+1} \equiv X z_s,\ p_{s+1} \equiv X p_s \in \mathcal{F} \quad (s = 0, 1, \dots) \tag{41}$$

for certain special functions (including the coordinates (37)) and the differential forms

$$\eta_s \equiv dy_s - y_{s+1} dx,\ \xi_s \equiv dz_s - z_{s+1} dx,\ \pi_s \equiv dp_s - p_{s+1} dx. \tag{42}$$

One can then verify the formulae $\mathcal{L}_X d\lambda = 0$,

$$\mathcal{L}_X \eta_s \equiv \eta_{s+1},\ \mathcal{L}_X \xi_s \equiv \xi_{s+1},\ \mathcal{L}_X \pi_s \equiv \pi_{s+1}. \tag{43}$$

At the same time clearly

$$\begin{gathered} y_2 = py_0,\ y_3 = Xy_2 = p_1 y_0 + py_1, \dots, \\ z_2 = (p-\lambda) z_0,\ z_3 = Xz_2 = p_1 z_0 + (p-\lambda) z_1, \dots \end{gathered} \tag{44}$$

and therefore (in particular)

$$\eta_2 = p\eta_0 + y_0\pi,\ \xi_2 = (p-\lambda)\xi_0 + z_0(\pi - d\lambda) \tag{45}$$

by direct calculations. In terms of the new notation clearly

$$\Omega = \{d\lambda, \eta_0, \eta_1, \xi_0, \xi_1, \pi_0, \pi_1, \dots\}. \tag{46}$$

We have indeed a diffiety: the forms from Ω together with the single form $dx \notin \Omega$ generate the module Φ of all differential forms, and there is filtration (23) defined (for instance) by

$$\Omega_s \equiv \{d\lambda, \eta_0, \eta_1, \xi_0, \xi_1, \pi_0, \dots, \pi_s\}, \tag{47}$$

which satisfies the requirement (24), see (43) and (45). Our next aim will be to determine the (unique) standard filtration of Ω. The result will be employed to the study of certain automorphisms of Ω which may be regarded for a generalization of the Darboux transformation.

7. Calculation of standard filtration

First of all, $\mathrm{Ker}\,\Omega_{s+1} \equiv \Omega_s$ $(s=0,1,\dots)$ and (denoting $\Omega_{-1} = \{d\lambda, \eta_0, \eta_1, \xi_0, \xi_1\}$) clearly $\mathrm{Ker}\,\Omega_0 = \Omega_{-1}$ in virtue of (43), (45). Second, in order to determine $\mathrm{Ker}\,\Omega_{-1}$, recall again the formulae

$$d\lambda \to 0,\ \eta_0 \to \eta_1, \xi_0 \to \xi_1, \eta_1 \to p\eta_0 + y_0\pi,\ \xi_1 \to (p-\lambda)\xi_0 + z_0(\pi - d\lambda)$$

(the arrows stand for $\mathcal{L}_X$) which imply

$$z_0\eta_1 - y_0\xi_1 \to z_1\eta_1 - y_1\xi_1 + z_0p\eta_0 - y_0(p-\lambda)\xi_0,$$

whence we may introduce

$$\Omega_{-2} = \mathrm{Ker}\,\Omega_{-1} = \{d\lambda, \eta_0, \xi_0, z_0\eta_1 - y_0\xi_1\}. \tag{48}$$

Third, in order to determine $\mathrm{Ker}\,\Omega_{-2}$, we may use the above formulae (especially (45)) which imply

$$z_0\eta_1 - y_0\xi_1 \to z_1\eta_1 - y_1\xi_1 + z_0p\eta_0 - y_0((p-\lambda)\xi_0 - z_0 d\lambda),$$

$$z_1\eta_0 - y_1\xi_0 \to z_1\eta_1 - y_1\xi_1 + (p-\lambda)z_0\eta_0 - py_0\xi_0,$$

$$z_0\eta_0 - y_0\xi_0 \to z_0\eta_1 - y_0\xi_1 + z_1\eta_0 - y_1\xi_0.$$

It follows that

$$\Omega_{-3} = \mathrm{Ker}\,\Omega_{-2} = \{d\lambda, z_0\eta_0 - y_0\xi_0, z_0\eta_1 - y_0\xi_1 - (z_1\eta_0 - y_1\xi_0)\}. \tag{49}$$

Denoting the last forms in the braces α and β, respectively, one can see that there is the alternative formula

$$\beta = d(z_0y_1 - y_0z_1) - X(z_0y_1 - y_0z_1)dx.$$

Fourth, in order to determine $\operatorname{Ker}\Omega_{-3}$, we use the obvious equations

$$\alpha = z_0\eta_0 - y_0\xi_0,\ \beta - z_0\eta_1 + y_0\xi_1 = y_1\xi_0 - z_1\eta_0$$

to conveniently express η_0 and ξ_0:

$$\eta_0 = (y_0(\beta - z_0\eta_1 + y_0\xi_1) + y_1\alpha)/(z_0y_1 - y_0z_1),$$

$$\xi_0 = (z_0(\beta - z_0\eta_1 + y_0\xi_1) + z_1\alpha)/(z_0y_1 - y_0z_1).$$

Then we obtain (use also (45))

$$\alpha \to z_0\eta_1 - y_0\xi_1 + z_1\eta_0 - y_1\xi_0 = 2(z_0\eta_1 - y_0\xi_1) - \beta,$$

$$\beta \to \lambda(y_0\xi_0 + z_0\eta_0) + y_0z_0d\lambda =$$

$$= \lambda(2y_0z_0(\beta - z_0\eta_1 + y_0\xi_1) + (y_0z_1 + z_0y_1)\alpha)/(z_0y_1 - y_0z_1) + y_0z_0d\lambda,$$

more briefly

$$\alpha \to 2(z_0\eta_1 - y_0\xi_1),$$

$$\beta \to -2\lambda y_0z_0(z_0\eta_1 - y_0\xi_1)/(z_0y_1 - y_0z_1)$$

modulo $d\lambda, \alpha, \beta$. Consequently

$$\Omega_{-4} = \operatorname{Ker}\Omega_{-3} = \{d\lambda, \lambda y_0z_0\alpha + (z_0y_1 - y_0z_1)\beta\}. \tag{50}$$

Then, obviously, $\Omega_{-5} = \operatorname{Ker}\Omega_{-4} = \{d\lambda\}$ and $\Omega_{-6} = \operatorname{Ker}\Omega_{-5} = \{d\lambda\} = \operatorname{Ker}{}^2\Omega_{-4}$. We are done: *the sequence*

$$\bar\Omega_0 = \Omega_{-4} \subset \bar\Omega_1 = \Omega_{-3} \subset \cdots = \Omega = \cup\bar\Omega_s \tag{51}$$

provides the sought standard filtration of Ω.

Comparing this result with the general one (31), we have $c = 1$, $\bar\gamma^1 = d\lambda$ and the initial form $\bar\omega = \lambda y_0z_0\alpha + (z_0y_1 - y_0z_1)\beta$. Note that the form $d\lambda$ may be replaced by any nonvanishing multiple $ad\lambda$ and analogously, any form of the kind $b\bar\omega + cd\lambda$ ($b \neq 0$) may be regarded for an initial form: after this change, the filtration (51) does not change.

8. Determination of automorphisms

Every automorphism $\mathbf{F}$ of our diffiety Ω obviously preserves all terms $\bar{\Omega}_s$ of the standard filtration (51), and also the submodule $\operatorname{Ker}\bar{\Omega}_0 = \{d\lambda\} \subset \bar{\Omega}_0$ (as follows at once from the uniqueness of these objects and the obvious fact that automorphisms of diffieties turn standard filtrations into standard filtrations). We shall explicitly employ only the invariance of $\operatorname{Ker}\bar{\Omega}_0$ and the first term $\bar{\Omega}_0$. This is expressed by

$$\mathbf{F}^* d\lambda = a d\lambda, \ \mathbf{F}^* \bar{\omega} = b\bar{\omega} + c d\lambda \tag{52}$$

with certain coefficients a, b, c (where $a \neq 0$, $b \neq 0$). On the other hand, we shall systematically employ that the automorphisms $\mathbf{F}$ preserve the submodule $\Omega^{\perp} \subset \mathcal{T}$. This is expressed by $\mathbf{F}_* X = gX$ $(g \neq 0)$ where X is the vector field (40) generating $\Omega^{\perp}$. Note that the unknown factor g satisfies the equation

$$g = gXx = (\mathbf{F}_* X)x = \mathbf{G}^* X \mathbf{F}^* x, \tag{53}$$

which is a particular case of (13).

In order to calculate such automorphisms in full generality, the following method will be examined: first, (52_1) means that $\mathbf{F}^*\lambda$ is a function of merely λ (explicitly $\mathbf{F}^*\lambda = h(\lambda)$), second, (52_2) taken only modulo $d\lambda$ means that $\mathbf{F}$ may be regarded for a contact transformation (the form $\bar{\omega}$ is multiplied by a factor b), which permits to specify the transformations $\mathbf{F}^*u, \mathbf{F}^*v, \mathbf{F}^*w$ for certain function u, v, w closely related to $\bar{\omega}$ (see below), third, the rule (13) makes it possible to investigate even the functions

$$\mathbf{F}^*u_s, \mathbf{F}^*v_s, \mathbf{F}^*w_s, \quad (u_s \equiv X^s u, \ v_s \equiv X^s v, \ w_s \equiv X^s w), \tag{54}$$

fourth, we may expect that the crucial variables x, y_0, z_0 can be expressed in terms of functions u_s, v_s, w_s $(s = 0, 1, \ldots)$ and then the sought $\mathbf{F}^*x, \mathbf{F}^*y_0, \mathbf{F}^*z_0$ can be easily found (since they are expressed in terms of functions (54) by the same formulae), finally, the knowledge of the last triple of the transformed functions may be regarded for the sought result.

In reality we are however not interested in quite general automorphisms $\mathbf{F}$ which are rather far from the classical Darboux transformations: we have to look only for special automorphisms $\mathbf{F}$ which preserve the independent variable x, that is, which satisfy $\mathbf{F}^*x = x$. In this special case clearly $g = \mathbf{G}^*Xx = \mathbf{G}^*1 = 1$ by using (53), hence $\mathbf{F}_*X = X$ and the rule (13) simplifies as $\mathbf{F}^*Xf = X\mathbf{F}^*f$. It follows that the functions (54) can be easily found in terms of $\mathbf{F}^*u, \mathbf{F}^*v, \mathbf{F}^*w$:

$$\mathbf{F}^*u_s \equiv X^s\mathbf{F}^*u, \mathbf{F}^*v_s \equiv X^s\mathbf{F}^*v, \mathbf{F}^*w_s \equiv X^s\mathbf{F}^*w.$$

We shall moreover see that the presumed invariance of x will imply certain very strong additional requirement for these functions.

Passing to explicit calculations, recall that we suppose $\mathbf{F}^*x = x$ (hence $\mathbf{F}^*Xf \equiv X\mathbf{F}^*f$ for every function f) and $\mathbf{F}^*\lambda = h(\lambda)$ with a certain function h. We turn to (52_2) which will be considered only modulo $d\lambda$ and dx. Since clearly

$$\bar{\omega} \cong \lambda y_0 z_0(z_0 dy_0 - y_0 dz_0) + (z_0 y_1 - y_0 z_1)d(z_0 y_1 - y_0 z_1) \tag{55}$$

modulo dx, we find it useful to introduce the functions $u = y_0 z_0$, $v = y_0/z_0$, $w = y_0 z_1 - z_0 y_1$. Then

$$\bar{\omega} \cong w^2\{\lambda(\frac{u}{w})^2 d\ln v + d\ln w\}$$

(so that the form $\{\dots\}$ may be used instead of $\bar{\omega}$) and (52_2) is expressed by

$$h\mathbf{F}^*(\frac{u}{w})^2 d\ln \mathbf{F}^*v + d\ln \mathbf{F}^*w \cong b(\lambda(\frac{u}{w})^2 d\ln v + d\ln w). \tag{56}$$

If x, λ are kept fixed for a moment, this congruence means that two subcases should be distinguished: either both $\ln \mathbf{F}^*v$ and $\ln \mathbf{F}^*w$ are functions of $\ln v$ and $\ln w$, or there is only one relation between these quadruple of functions (so that either $\ln \mathbf{F}^*v$ or $\ln \mathbf{F}^*w$ may be represented as a composed function of the remaining triple). We shall separately deal with these subcases.

9. The "point transformation" subcase

In accordance with the above remark, we begin with the assumption

$$\ln \mathbf{F}^*v = k(\ln v, \ln w, \lambda, x), \ln \mathbf{F}^*w = l(\ln v, \ln w, \lambda, x). \tag{57}$$

Substitution into (56) yields the identities

$$hk_1\mathbf{F}^*(\frac{u}{w})^2 + l_1 = b\lambda(\frac{u}{w})^2, hk_2\mathbf{F}^*(\frac{u}{w})^2 + l_2 = b$$

(where the lower indices denote partial derivatives with the respect to the arguments at the noted places, e.g., $k_1 = \partial k/\partial \ln v$). It follows easily

$$\left(\frac{\mathbf{F}^*w}{\mathbf{F}^*u}\right)^2 = \mathbf{F}^*(\frac{w}{u})^2 = h\frac{\lambda k_2(u/w)^2 - k_1}{l_1 - \lambda l_2(u/w)^2} \tag{58}$$

by elimination of the unimportant function b. In principle the formulae (57), (58) determine $\mathbf{F}^*u, \mathbf{F}^*v, \mathbf{F}^*w$ and then clearly

$$\mathbf{F}^*y_0 = (\mathbf{F}^*u\mathbf{F}^*v)^{1/2},\ \mathbf{F}^*z_0 = (\mathbf{F}^*u/\mathbf{F}^*v)^{1/2} \tag{59}$$

(as follows from the obvious formulae $y_0 = \sqrt{uv}$, $z_0 = \sqrt{u/v}$). However in reality we are as yet dealing only with necessary conditions: all the automorphisms $\mathbf{F}$ must be of the mentioned kind but the converse need not be true.

In order to obtain necessary and sufficient requirements for the functions h, k, l, we shall proceed as follows. One can find that the functions

$$x, \lambda, u_0 = u,\ v_0 = v, u_1, v_1, u_2, u_3, \dots \tag{60}$$

can be used for coordinates instead of the original ones (37), as well. This follows from the obvious formulae

$$y_0 = \sqrt{uv},\ z_0 = \sqrt{\frac{u}{v}},\ p = \frac{y_2}{y_0} = \frac{X^2\sqrt{uv}}{\sqrt{uv}},$$

the recurrences $y_s \equiv X^s y_0$, $z_s \equiv X^s z_0$, $p_s \equiv X^s p$ and the identity

$$X^2 \ln v + X \ln u . X \ln v = \lambda \tag{61}$$

(identical with (6_1)) which permits to express v_2 in terms of u_0, u_1, u_2, v_0, v_1. In general, formulae for v_s $(s \geq 3)$ are obtained if X is repeatedly applied to (61). It follows that all w_s $(s \geq 0)$ can be expressed in terms of new coordinates. Using $Xv = X(y_0/z_0) = y_1/z_0 - y_0 z_1/(z_0)^2$, clearly

$$w = uX \ln v = uv_1/v \tag{62}$$

and then $w_s \equiv X^s\ (uX \ln v)$. These are all interrelations between the functions u_s, v_s, w_s. The corresponding interrelations must be satisfied for the transforms $\mathbf{F}^* u_s, \mathbf{F}^* v_s, \mathbf{F}^* w_s$, which provides the additional requirements for the functions h, k, l. It is however quite sufficient to ensure only the relations corresponding to (61), (62), that is, the relations

$$X^2 \ln \mathbf{F}^* v + X \ln \mathbf{F}^* u . X \ln \mathbf{F}^* v = h, \mathbf{F}^* w = \mathbf{F}^* u . X \ln \mathbf{F}^* v, \tag{63}$$

since the higher order ones (involving v_s for $s > 2$ and w_s for $s > 0$) appear if the operator X is repeatedly applied to (63) and therefore do not bring any novelty.

We shall first deal with (63_2). Using (58), (57_1) and the identity

$$Xk = k_1 v_1/v + k_2 w_1/w + k_4\ (k_4 = \partial k/\partial x)$$

where $v_1/v = w/u$ (see (62)) and $w_1/w = \lambda u/w$ (direct verification), we obtain the requirement

$$h \frac{\lambda k_2 (u/w)^2 - k_1}{l_1 - \lambda l_2 (u/w)^2} = (k_1 \frac{w}{u} + \lambda k_2 \frac{u}{w} + k_4)^2. \tag{64}$$

Quite analogously, using (63_2) and its consequence, the derived identity

$$X^2 \ln \mathbf{F}^* v = X \left(\frac{\mathbf{F}^* w}{\mathbf{F}^* u} \right) = \frac{X \mathbf{F}^* w}{\mathbf{F}^* u} - \mathbf{F}^* w \frac{X \mathbf{F}^* u}{(\mathbf{F}^* u)^2},$$

the requirement (63_1) can be found equivalent to the equation $X\mathbf{F}^*w = h\mathbf{F}^*u$, that is, to the equation

$$X \ln \mathbf{F}^* w = h\mathbf{F}^*(u/w). \tag{65}$$

Using (58), this can be explicitly rewritten as

$$\frac{\lambda k_2(u/w)^2 - k_1}{l_1 - \lambda l_2(u/w)^2} = (l_1\frac{w}{u} + \lambda l_2\frac{u}{w} + l_4)^{-2}. \tag{66}$$

In particular we have

$$(k_1\frac{w}{u} + \lambda k_2\frac{u}{w} + k_4)(l_1\frac{w}{u} + \lambda l_2\frac{u}{w} + l_4) = h,$$

whence easily

$$k_1l_1 = k_2l_2 = k_1l_4 + l_1k_4 = k_2l_4 + l_2k_4 = 0,\ l_4k_4 + \lambda(k_1l_2 + l_1k_2) = h. \tag{67}$$

At this place note that the Jacobian $k_2l_1 - l_2k_1$ of the functions k, l with respect to the variables $\ln v, \ln w$ should not vanish (otherwise the functions (60) cannot be used for coordinates after applying the automorphism $\mathbf{F}$ which is a contradiction). Moreover (64), (66) are possible only if $l_4 = k_4 = 0$ (otherwise the left hand sides would not be even functions of the variable u), hence (67) reduces to $k_1l_1 = k_2l_2 = 0$, $k_1l_2 + l_1k_2 = h/\lambda$. Altogether taken, one can find that this may happen if one of the following possibilities holds true:

$$\text{either}\quad k_1 \neq 0,\ l_1 = k_2 = 0,\ k_1l_2 = h/\lambda, \tag{68}$$

$$\text{or}\quad k_2 \neq 0,\ l_2 = k_1 = 0,\ k_2l_1 = h/\lambda. \tag{69}$$

We shall discuss (68), (69) separately.

In the first subcase (68), we have $h = h(\lambda)$, $k = k(\ln v, \lambda)$, $l = l(\ln w, \lambda)$ and the identity (64) simplifies into relation (68_3), that is,

$$h(\lambda)/\lambda = l'(\ln w, \lambda)k'(\ln v, \lambda)$$

where the primes denote the partial derivatives with respect to the first argument. (The identity (66) may be omitted; it is a consequence of (64) and (67).) It follows $l = a(\lambda)\ln w$, $k = b(\lambda)\ln v$ with the relation $h(\lambda)/\lambda = a(\lambda)b(\lambda)$. So we have

$$\mathbf{F}^*v = v^{b(\lambda)}, \mathbf{F}^*w = w^{a(\lambda)}, \mathbf{F}^*u = \frac{1}{b}uw^{a(\lambda)-1},$$

where the last formula follows from (58) by direct calculation (using moreover $\mathbf{F}^*w = w^{a(\lambda)}$ and $a(\lambda)b(\lambda) = h(\lambda)/\lambda$). Returning to the variables y_s, z_s, we obtain the final result

$$\begin{aligned} \mathbf{F}^*y &= (uw^{a-1}/b.v^b)^{1/2} = (yzw^{a-1}(y/z)^b/b)^{1/2} = (y^{1+b}z^{1-b}w^{a-1}/b)^{1/2}, \\ \mathbf{F}^*z &= (uw^{a-1}/b.v^{-b})^{1/2} = (y^{1-b}z^{1+b}w^{a-1}/b)^{1/2}. \end{aligned} \tag{70}$$

This may be regarded for a mere modification of the identity, which is obtained if $a = b = 1$. Interesting transformation appears even if $a = 1$ and $b = b(\lambda)$ arbitrary, then the potentials of the transformed functions differ by the value $h = a(\lambda)b(\lambda)\lambda = b(\lambda)\lambda$.

In the second subcase (69), we have $h = h(\lambda)$, $k = k(\ln w, \lambda)$, $l = l(\ln v, \lambda)$ and the identity (64) simplifies into the relation (69_3), that is,

$$h(\lambda)/\lambda = l'(\ln v, \lambda)k'(\ln w, \lambda).$$

(Then (66) is automatically satisfied and may be omitted.) It follows $l = a(\lambda)\ln v$, $k = b(\lambda)\ln w$ with the relation $h(\lambda)/\lambda = a(\lambda)b(\lambda)$. Using (58), we obtain

$$\mathbf{F}^*v = w^{b(\lambda)}, \mathbf{F}^*w = v^{a(\lambda)}, \mathbf{F}^*u = v^{a(\lambda)}w/u\lambda b(\lambda)$$

and consequently

$$\begin{aligned} \mathbf{F}^*y &= (v^a w/u\lambda b.w^b)^{1/2} = (y^{a-1}w^{1+b}/\lambda b z^{1+a})^{1/2}, \\ \mathbf{F}^*z &= (v^a w/u\lambda b.w^{-b})^{1/2} = (y^{a-1}w^{1-b}/\lambda b z^{1+a})^{1/2}. \end{aligned} \tag{71}$$

The classical Darboux transformation appears if $a = b = 1$.

We shall prove in the concluding Section 10 that the (seemingly more general) "contact transformation" subcase does not give any x-preserving automorphisms, hence (70) *and* (71) *represent the most general automorphisms of our diffiety.* Comparing this with the quadrature-free formulae (7) and (8) in Introduction, it is necessary to substitute $V = w/u = w/yz$, $\frac{A}{B} = a =$ const., $A\lambda = h(\lambda) = \lambda ab$ (hence $A = ab$, $B = A/a = b$) and it follows that (7) and (8) are particular cases of (70) and (71) if $a = a(\lambda)$, $b = b(\lambda)$ are merely constants.

So we may conclude: *the formulae*(7)*and*(8)*with*$A = A(\lambda)$, $B = B(\lambda) \neq 0$*nonconstant cannot be already generalized. They provide the most general quadrature-free and* $x-$ *preserving automorphisms of couples of Sturm-Liouville problems* (1) *with potentials* p, q *differing by a constant.*

10. The "contact transformation" subcase

Recalling (56), let us suppose the existence of a relation of the kind

$$\ln \mathbf{F}^*w = k(\ln \mathbf{F}^*v, \ln w, \ln v, \lambda, x). \tag{72}$$

Substitution into (56) gives immediately

$$h\mathbf{F}^*(\frac{u}{w})^2 + k_1 = 0,\ \ k_2 = b,\ \ k_3 = b\lambda(\frac{u}{v})^2, \tag{73}$$

whence

$$k_3 = k_2\lambda(\frac{u}{v})^2 \tag{74}$$

by elimination of b. If we deal with a contact transformation, the functions $\mathbf{F}^*w$, $\mathbf{F}^*u$, $\mathbf{F}^*v$ can be calculated in terms of variables

$$\ln w, \ln v, \lambda, x \tag{75}$$

by virtue of equations (72), (73_1), (74). However the identities $X \ln v = -w/u$ and $X \ln w.w = \lambda u/w$ are true, hence the transformed (63_2) and (65) must be satisfied, too. They ensure that the functions $\mathbf{F}^*Xv$, $\mathbf{F}^*Xw$ can be expressed in terms of variables (75). Finally, applying X on the equation (74) one can calculate Xu in terms of $\mathbf{F}^*Xv$ and the variables (75), hence $Xu = u_1$ would be expressible in terms of (75). But this is a contradiction since the functions (60) may be chosen for coordinates.

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