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PARALLELISABILITY CONDITIONS FOR DIFFERENTIABLE THREE–WEBS

Alena Vanžurová

Abstract. Our aim is to find conditions under which a 3-web on a smooth $2n$-dimensional manifold is locally equivalent with a web formed by three systems of parallel n-planes in $\mathbb{R}^{2n}$. We will present here a new approach to this "classical" problem using projectors onto the distributions of tangent subspaces to the leaves of foliations forming the web.

The parallelisability conditions for multicodimensional 3-webs were at first formulated by S.S. Chern, [3]. Later on M. A. Akivis, [1], interpreted these conditions as vanishing of both torsion and curvature tensors of a certain connection intimately related to the web, so called canonical Chern connection of a 3-web (M. Kikkawa, [8]).

We will find parallelisability conditions formulated in terms of projectors of a web, and will verify that they are equivalent with those derived by Akivis. At the same time we will show that the problem of parallelisability of a 3-web is equivalent with integrability of a (P, B)-structure (a couple of polynomial structures defining a 3-web).

All objects under consideration will be supposed of the class C^∞ (smooth).

1. Projectors of a 3-web

A 3-web on a $2n$-dimensional manifold is given by a triple of foliations (in general position) of codimension n which are usually defined by totally integrable systems of Pfaffian equations.

For our purposes let us choose the following definition.

Definition 1. Under a differentiable 3-web $\mathcal{W}$ on a manifold M_{2n} of dimension $2n$ we will understand here a triple $\mathcal{W} = (D_1, D_2, D_3)$ of (smooth) n-dimensional integrable distributions which are pairwise complementary.

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Three foliations of integral submanifolds of these distributions form a 3-web in a classical view-point. Given three-webs $\mathcal{W} = (D_1, D_2, D_3)$ on M_{2n}, and $\mathcal{W}' = (D'_1, D'_2, D'_3)$ on M'_{2n}, a diffeomorphism $f : M \to M'$ will be called a *web-morphism* or *web-equivalence* if it satisfies

$$Tf(D_\alpha) = D'_\alpha \quad \text{for } \alpha = 1, 2, 3. \tag{1}$$

Three-webs $\mathcal{W}$ and $\mathcal{W}'$ are *locally equivalent* if there exists a local diffeomorphism $f : M \to M'$ of the underlying manifolds which satisfies (1). About each point $x \in M$, a local web-morphism (equivalence) f induces a diffeomorphism of some open neighborhood $U \ni x$ onto a nbh of $f(x)$. Investigations in web geometry are often concerned with local invariants, i. e. properties invariant under local equivalences satisfying (1).

Given a 3-web $\mathcal{W} = (D_1, D_2, D_3)$, each couple D_α, D_β of complementary n-distributions forms an almost product structure denoted by $[D_\alpha, D_\beta]$ here which is integrable since both distributions forming it are integrable, [18]. The tangent bundle can be written as a Whitney sum:

$$TM = D_\alpha \oplus D_\beta, \qquad \alpha \neq \beta, \quad \alpha, \beta \in \{1, 2, 3\}. \tag{2}$$

This decompositions determine web projectors P^β_α associated with $\mathcal{W}$, that is $(1,1)$ tensor fields on M satisfying

$$\begin{aligned} (P^\beta_\alpha)^2 = P^\beta_\alpha, \quad P^\beta_\alpha + P^\alpha_\beta = I, \quad \ker P^\alpha_\beta = \operatorname{im} P^\beta_\alpha = D_\alpha, \\ P^\alpha_\beta P^\gamma_\beta = P^\gamma_\beta \qquad \text{for } \alpha \neq \beta \neq \gamma \neq \alpha, \\ P^\alpha_\beta P^\gamma_\alpha = 0 \qquad \text{for } \gamma \neq \alpha \neq \beta, \end{aligned}$$

and $[P^\beta_\alpha, P^\beta_\alpha] = 0$. The last condition means integrability of both D_α and D_β.

Now a web-morphism can be characterized by the property that its tangent mapping Tf commutes with web-projectors.

Besides projectors, it is useful to introduce web-associated isomorphisms of the tangent bundle, [11], [15], by

$$B_\gamma = P^\alpha_\gamma - P^\gamma_\beta = P^\beta_\gamma - P^\gamma_\alpha.$$

2. Regular webs

A local chart[1] $(U; x^1, \ldots, x^{2n})$ about $x \in M$ will be called *adapted* with respect to the almost-product structure $[D_1, D_2]$ if the coordinate vector fields $\partial_i = \partial/\partial x^i$, $i = 1, ..., n$ span D_1 on U, and $\partial_{n+i} = \partial/\partial x^{n+i}$ span D_2 on U. It can be verified that with respect to a holonomic frame $(\partial_i \mid \partial_{n+i})$, the matrix expressions of the projectors P^1_3, P^3_1 and the involutory automorphism B_3 are

$$P^1_3 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{P} & \boldsymbol{I} \end{pmatrix}, \qquad P^3_1 = \begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ -\boldsymbol{P} & \mathbf{0} \end{pmatrix}, \qquad B_3 = \begin{pmatrix} \mathbf{0} & \boldsymbol{Q} \\ \boldsymbol{P} & \mathbf{0} \end{pmatrix}.$$

Here $\boldsymbol{P} = (p^k_i)$, $p^k_i = p^k_i(x^1, \ldots, x^{2n})$ is a regular (n,n)-matrix at any point, $\boldsymbol{I}$ and $\mathbf{0}$ denote zero and unit (n,n)-matrices respectively, $\boldsymbol{Q} = \boldsymbol{P}^{-1} = (q^j_k)$.

[1] U denotes an open domain of M containing x.

Definition 2. A frame $(X_i \mid X_{n+i})$ (which may be non-holonomic in general) will be called *web-adapted* if X_i span D_1, X_{n+i} span D_2, and $X_i + X_{n+i}$ span D_3, $i = 1, \ldots, n$.

Definition 3. A 3-web $\mathcal{W}$ on M will be called here *regular* if there exist local $[D_1, D_2]$-adapted coordinates such that the corresponding holonomic frame $(\partial_i \mid \partial_{n+i})$ is web-adapted, that is, ∂_i span D_1, ∂_{n+i} span D_2, and $\partial_i + \partial_{n+i}$ span D_3, $i = 1, \ldots, n$.

Remark. It can be verified that $\mathcal{W}$ is regular if and only if the almost product structures $[D_1, D_2]$, $[D_2, D_3]$, and $[D_1, D_3]$ are simultaneously integrable.

It is obvious that a 3-web is regular iff it is locally equivalent with a "parallel web" formed by three systems of parallel n-planes in $\mathbb{R}^{2n}$. Regular webs are frequently called parallelisable.

Given a web we will find necessary and sufficient conditions for its regularity using components of the projector P_1^3.

Since $[D_1, D_2]$ is integrable we can find via Frobenius Theorem a transformation f (at least locally) of our (local) adapted coordinates $(x^i; x^{n+i})$ under which both D_1 and D_2 remain unchanged, in other words, $(x'^i; x'^{n+i})$ are again adapted:

$$x'^{\,i} = f^i(x^1, \ldots, x^n), \qquad x'^{\,n+i} = f^{n+i}(x^{n+1}, \ldots, x^{2n}), \qquad i = 1, \ldots, n \tag{3}$$

that is,

$$\frac{\partial f^i}{\partial x^{n+j}} = 0, \qquad \frac{\partial f^{n+i}}{\partial x^j} = 0. \tag{4}$$

Let us denote

$$\boldsymbol{F} = \left(f^i_j\right) \text{ where } f^i_j = \frac{\partial f^i}{\partial x^j}, \quad \tilde{\boldsymbol{F}} = \left(\tilde{f}^i_j\right) \text{ where } \tilde{f}^i_j = \frac{\partial f^{n+i}}{\partial x^{n+j}},$$

and similarly for the inverse transformation g, $x^i = g^i(x'^{\,j})$, $x^{n+i} = g^{n+i}(x'^{\,n+j})$:

$$\boldsymbol{G} = \left(g^i_j\right) \text{ where } g^i_j = \frac{\partial g^i}{\partial x'^{\,j}}, \quad \tilde{\boldsymbol{G}} = \left(\tilde{g}^i_j\right) \text{ where } \tilde{g}^i_j = \frac{\partial g^{n+i}}{\partial x'^{\,n+j}}.$$

Lemma 1. *Let $f = (f^1, \ldots, f^{2n})$ be a local coordinate transformation (in an open nbd U of some point $x \in M$) given by (3). Then the n-tuple of vector fields*

$$\frac{\partial}{\partial x'^{\,i}} + \frac{\partial}{\partial x'^{\,n+i}}, \qquad i = 1, \ldots, n$$

forms a basis of D_3 on U if and only if on U, the components of f and the components $p^k_i(x^1, \ldots, x^{2n})$ of the projector P_1^3 satisfy the following system of differential equations:

$$\sum_i \sum_k \left(\frac{\partial f^s}{\partial x_k} \cdot p^k_i\right) - \frac{\partial f^{n+s}}{\partial x^{n+i}} = 0 \qquad \text{for } i, s = 1, \ldots, n, \tag{5}$$

or equivalently,

$$p_i^k = \frac{\partial f^{n+s}}{\partial x^{n+i}} \cdot \frac{\partial g^k}{\partial x'^{\,s}}. \tag{6}$$

Proof. Using the inverse transformation $g = (g^1, \dots, g^{2n})$ we can write

$$P_1^3\left(\frac{\partial}{\partial x'^{\,j}}\right) = \frac{\partial g^k}{\partial x'^{\,j}} \cdot \partial_k, \qquad P_1^3\left(\frac{\partial}{\partial x'^{\,n+j}}\right) = -\frac{\partial g^{n+s}}{\partial x'^{\,n+j}} \cdot p_s^k \cdot \partial_k.$$

Now the tangent vector

$$P_1^3\left(\frac{\partial}{\partial x'^{\,j}} + \frac{\partial}{\partial x'^{\,n+j}}\right) = \left(\frac{\partial g^k}{\partial x'^{\,j}} - \frac{\partial g^{n+s}}{\partial x'^{\,n+j}}\, p_s^k\right) \partial_k$$

is equal to a zero vector for any $j \in \{1, \dots, n\}$ if and only if all coefficients vanish:

$$\frac{\partial g^{n+s}}{\partial x'^{\,n+j}}\, p_s^k - \frac{\partial g^k}{\partial x'^{\,j}} = 0, \qquad k = 1, \dots, n.$$

That is,

$$\tilde{\boldsymbol{G}} \cdot \boldsymbol{P} - \boldsymbol{G} = \boldsymbol{0}.$$

Multiplication by the regular matrix $\tilde{\boldsymbol{F}}$ from the left yields $\boldsymbol{P} - \tilde{\boldsymbol{F}}\boldsymbol{G} = \boldsymbol{0}$, or

$$\boldsymbol{P} = \tilde{\boldsymbol{F}}\boldsymbol{G}$$

which is (6), and further

$$\boldsymbol{P} \cdot \boldsymbol{F} = \tilde{\boldsymbol{F}}$$

which can be written as (5).

We see now that our system of partial differential equations (4), (5) is solvable iff the block $\boldsymbol{P}$ in the matrix representation of the projector P_1^3 can be written as a product of two vector valued matrices each depending only on one family of coordinates,

$$\boldsymbol{P}(x^i; x^{n+i}) = \tilde{\boldsymbol{F}}(x^{n+i}) \cdot \boldsymbol{G}(x'^{\,i}) = \tilde{\boldsymbol{F}}(x^{n+i}) \cdot \boldsymbol{F}^{-1}(x^i). \tag{7}$$

At the same time, the factors can be regarded as Jacobi matrices of some transformations from $\mathbb{R}^n$ into $\mathbb{R}^n$.

The problem when such a decomposition does exist can be solved with help of the results obtained in [12], [5]. J. Šimša in [12] answered a more general question: when a smooth matrix function $\boldsymbol{H}$ in $p+q$ variables, non-singular at each point, is factorizable into the form

$$\boldsymbol{H}(z^1, \dots z^p; y^1, \dots, y^q) = \boldsymbol{A}(z^1, \dots, z^p) \cdot \boldsymbol{B}(y^1, \dots, y^q), \tag{8}$$

or into the product of matrices depending on a single variable $\boldsymbol{H}(z^1, \dots, z^k) = \boldsymbol{A}(z^1) \cdot \dots \cdot \boldsymbol{A}(z^k)$. In the case which interests us the result is:

Lemma 2. (Šimša condition) *Let* $Z = Z_1 \times \cdots \times Z_p \subset \mathbb{R}^p$, $Y = Y_1 \times \cdots \times Y_q \subset \mathbb{R}^q$ *be open intervals (where* Z_j, $j = 1, \dots p$ *and* Y_s, $s = 1, \dots, q$ *are open intervals in* $\mathbb{R}$*). Let* $\boldsymbol{H} : Z \times Y \to GL(n, \mathbb{R})$ *(or* $\boldsymbol{H} : Z \times Y \to GL(n, \mathbb{C})$*, respectively) be a mapping which has the partial derivatives* $\boldsymbol{H}_{z^j} = \dfrac{\partial \boldsymbol{H}}{\partial z^j}$, $\boldsymbol{H}_{y^s} = \dfrac{\partial \boldsymbol{H}}{\partial y^s}$ *and* $\boldsymbol{H}_{y^j z^s} = \dfrac{\partial^2 \boldsymbol{H}}{\partial z^j\, \partial y^s}$, $1 \leqq j \leqq p$, $1 \leqq s \leqq q$ *(in some order of differentiation) on the open interval* $Z \times Y$*. Then the mapping* $\boldsymbol{H}$ *can be written in the form* (8) *if and only if it suffices the following system of partial differentiable equations:*

$$\boldsymbol{H}_{z^j y^s} = \boldsymbol{H}_{z^j} \cdot \boldsymbol{H}^{-1} \cdot \boldsymbol{H}_{y^s} \qquad \text{on } Z \times Y, \quad 1 \leqq j \leqq p, \quad 1 \leqq s \leqq q \tag{9}$$

where the lower indexis mean partial derivations with respect to the corresponding subscribed variables. Moreover, the factors $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are exactly pairs of the form*

$$\begin{aligned} \boldsymbol{A}(z^1, \dots, z^p) &= \boldsymbol{H}(z^1, \dots z^p; u^1, \dots, u^q) \cdot \boldsymbol{C} \\ \boldsymbol{B}(y^1, \dots, y^q) &= \boldsymbol{D} \cdot \boldsymbol{H}(v^1, \dots, v^p; y^1, \dots, y^q) \end{aligned} \tag{10}$$

where the elements $u^s \in Y_s$ *and* $v^j \in Z_j$ *are chosen arbitrarily, and the* (n, n)*-matrices* $\boldsymbol{C}$ *and* $\boldsymbol{D}$ *over* $\mathbb{R}$ *(or over* $\mathbb{C}$ *respectively) satisfy*

$$\boldsymbol{C} \cdot \boldsymbol{D} = \boldsymbol{H}^{-1}(v^1, \dots, v^p; u^1, \dots, u^q).$$

Using the formula $\boldsymbol{H}^{-1}\boldsymbol{H}_{z^j} + \boldsymbol{H}^{-1}_{z^j}\boldsymbol{H} = \boldsymbol{0}$ which arises by derivation of the equality $\boldsymbol{H}^{-1} \cdot \boldsymbol{H} = \boldsymbol{I}$ we can write (9) in the form

$$\boldsymbol{H}_{z^j y^s} = \boldsymbol{H}_{z^j} \cdot \boldsymbol{H}^{-1}_{y^s} \cdot \boldsymbol{H}. \tag{11}$$

For our purposes we will also need the following:

Lemma 3. ([19]) *Let* $a_1(z), \dots, a_p(z)$ *be differentiable functions of a vector variable* $z = (z^1, \dots, z^p)$ *having continuous partial derivations on some open interval* $Z \subset \mathbb{R}^p$*. The necessary and sufficient condition for the existence of a differentiable mapping* $\varphi : \mathbb{R}^p \to \mathbb{R}^p$ *such that*

$$\frac{\partial \varphi}{\partial z^i} = a_i(z), \qquad i = 1, \dots, p$$

is

$$\frac{\partial a_i}{\partial z^j} = \frac{\partial a_j}{\partial z^i}. \tag{12}$$

Corollary. *A matrix-valued function* $\boldsymbol{A} = \left(a_j^k(z)\right)$, $z \in \mathbb{R}^p$ *is in a "Jacobi matrix-like" form* $a_j^k = \dfrac{\partial \varphi^k}{\partial z^j}$ *for some mapping* $\varphi = (\varphi^1, \dots, \varphi^p) : \mathbb{R}^p \to \mathbb{R}^p$ *if and only if*

$$\frac{\partial a_i^k}{\partial z^j} = \frac{\partial a_j^k}{\partial z^i}. \tag{13}$$

Remark. Lemma 3 is sometimes formulated as follows: Let X be a differentiable (at least C^1) vector field on a domain in E^p with components $X = (a_1, \dots, a_p)$. Then there exists a potencial f of X (i. e. $X = \operatorname{grad} f$ for some function f) iff (12) is satisfied. This theorem is a dual version of so called Poincaré Theorem.

Lemma 4. *The system* (4), (5) *of partial differentiable equations is solvable on* U *if and only if the block matrix* $\boldsymbol{P}$ *of the projector* P_1^3 *with respect to adapted coordinates* $(x^i; x^{n+i})$ *satisfies*

$$\boldsymbol{P}_{x^{n+j}x^s} \cdot \boldsymbol{Q} + \boldsymbol{P}_{x^{n+j}} \cdot \boldsymbol{Q}_{x^s} = \boldsymbol{0}, \tag{14}$$

$$(\boldsymbol{P}_{x^{n+i}})_j^k = (\boldsymbol{P}_{x^{n+j}})_i^k, \qquad (\boldsymbol{Q}_{x^i})_j^k = (\boldsymbol{Q}_{x^j})_i^k \tag{15}$$

where $\boldsymbol{Q} = \boldsymbol{P}^{-1}$.

Proof. If we put $p = q = n$, $\boldsymbol{H} = \boldsymbol{P}$, $z^j = x^{n+j}$, $y^j = x^j$, we obtain by Lemma 2 and its corollary (11): The necessary and sufficient condition for the existence of a decomposition $\boldsymbol{P}(x^i; x^{n+i}) = \boldsymbol{A}(x^{n+i}) \cdot \boldsymbol{B}(x^i)$ is

$$\boldsymbol{P}_{x^{n+j}x^s} + \boldsymbol{P}_{x^{n+j}} \cdot \boldsymbol{Q}_{x^s} \cdot \boldsymbol{P} = \boldsymbol{0}, \tag{16}$$

and the factors of any decomposition of the above form are exactly matrices

$$\boldsymbol{A}(x^{n+1}, \dots, x^{2n}) = \boldsymbol{P}(u^1, \dots, u^n, x^{n+1}, \dots, x^{2n}) \cdot \boldsymbol{C},$$
$$\boldsymbol{B}(x^1, \dots, x^n) = \boldsymbol{D} \cdot \boldsymbol{P}(x^1, \dots, x^n, u^{n+1}, \dots, u^{2n})$$

with $\boldsymbol{C} \cdot \boldsymbol{D} = \boldsymbol{Q}(u^1, \dots, u^{2n})$ for some choice of a constant point $u = (u^1, \dots, u^{2n})$ in a coordinate neighborhood U. Multiplication of (16) by $\boldsymbol{Q}$ gives (14). The entries of the matrix factors of the factorization can be written as

$$(\boldsymbol{A})_j^i = (\tilde{\boldsymbol{F}})_j^i = \frac{\partial f^{n+i}}{\partial x^{n+j}}$$

and

$$(\boldsymbol{B})_j^i = (\boldsymbol{G})_j^i = (\boldsymbol{F}^{-1})_j^i, \qquad (\boldsymbol{B}^{-1})_j^i = (\boldsymbol{F})_j^i = \frac{\partial f^i}{\partial x^j}$$

if and only if

$$\frac{\partial(\boldsymbol{A})_i^k}{\partial x^{n+j}} = \frac{\partial(\boldsymbol{A})_j^k}{\partial x^{n+i}}, \qquad \frac{\partial(\boldsymbol{B}^{-1})_i^k}{\partial x^j} = \frac{\partial(\boldsymbol{B}^{-1})_j^k}{\partial x^i},$$

that is

$$\sum_s ((\boldsymbol{P}_{x^{n+j}})_i^s - (\boldsymbol{P}_{x^{n+i}})_j^s)(\boldsymbol{C})_s^k = 0, \quad \sum_s ((\boldsymbol{Q}_{x^{n+j}})_i^s - (\boldsymbol{Q}_{x^{n+i}})_j^s)(\boldsymbol{D}^{-1})_s^k = 0. \tag{17}$$

Since $\boldsymbol{C}$ and $\boldsymbol{D}^{-1}$ are non-singular the conditions (17) are equivalent with (15).

As a consequence it follows:

Theorem 1. *A 3-web is regular (equivalently, all $[D_\alpha, D_\beta]$ are simultaneously integrable) if and only if the conditions* (14) *and* (15) *are fullfilled.*

3. Chern connection. Expressions in local coordinates.

Consider a 3-web $\mathcal{W} = (D_1, D_2, D_3)$ on M_{2n}, and for shortness, let us denote

$$P = P_1^3, \qquad B = B_3.$$

Then $\mathcal{W}$ can be regarded as a couple (P, B) of polynomial structures on M, [16], [13], [11], satisfying

$$P^2 - P = 0, \qquad B^2 - I = 0 \tag{18}$$

with characteristic polynomials

$$p(\xi) = \xi(\xi - 1), \qquad b(\xi) = (\xi - 1)(\xi + 1),$$

having in addition the following properties:

$$\begin{gathered} D_1 = \ker P, \quad D_2 = \ker(I - P), \quad \text{and} \quad D_3 = \ker(B - I) \quad \text{are integrable,} \\ BP = (I - P)B. \end{gathered} \tag{19}$$

On the other hand, any couple (P, B) of smooth $(1,1)$-tensor fields on M_{2n} satisfying (18) and (19) gives rise to a 3-web $(\ker P, \ker(I - P), \ker(B - I))$, [11]. The first condition in (19) can be written as[2]

$$[P, P] = 0, \quad [B, B](X, Y) = 0 \ \text{ for } X, Y \in \ker(B - I).$$

Definition 4. A (P, B)-structure will be defined as a couple of smooth $(1,1)$-tensor fields on M_{2n} satisfying (18), (19). We say that a (P, B)-structure is *integrable* if there exist local coordinates in a nbd of each point such that the matrix expressions of P and B with respect to a coordinate frame are

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{0} \end{pmatrix}.$$

Obviously, a (P, B)-structure is integrable if and only if the corresponding 3-web is regular.

On the base web-manifold M, a canonical Chern connection of the web $\mathcal{W}$ is given by the formula

$$\begin{aligned} \nabla_X Y = {} & P[(I - P)X, PY] + (I - P)[PX, (I - P)Y] \\ & + PB[PX, BPY] + (I - P)B[(I - P)X, B(I - P)Y]. \end{aligned}$$

[2] The Nijenhuis tensor of vector 1-forms P and B is given by [P,B](X,Y)=[PX,BY]+[BX,PY] +PB[X,Y]+BP[X,Y]-P[X,BY]-B[X,PY]- P[BX,Y]-B[PX,Y].

With respect to ∇, the tensor fields P and B are covariantly constant, and the torsion tensor T of this connection satisfies $T(PX,(1-P)Y)=0$ for any vector fields X, Y on M.

In our adapted coordinates, the formula for an evaluation of the Lie bracket of vector fields $X = X^l\partial_l + X^{n+l}\partial_{n+l}$, and Y is

$$\begin{aligned}[X,Y] &= \left(\frac{\partial Y^k}{\partial x^l}X^l + \frac{\partial Y^k}{\partial x^{n+l}}X^{n+l} - \frac{\partial X^k}{\partial x^l}Y^l - \frac{\partial X^k}{\partial x^{n+l}}Y^{n+l}\right)\partial_k \\ &+\left(\frac{\partial Y^{n+k}}{\partial x^l}X^l + \frac{Y^{n+k}}{\partial x^{n+l}}X^{n+l} - \frac{\partial X^{n+k}}{\partial x^l}Y^l - \frac{\partial X^{n+k}}{\partial x^{n+l}}Y^{n+l}\right)\partial_{n+k}.\end{aligned}$$

We evaluate the Chern connection ∇ with respect to a $[D_1,D_2]$-adapted holonomic frame $(\partial_i \mid \partial_{n+i})$:

$$\begin{aligned}\nabla_{\partial_i}\partial_j &= \frac{\partial q^k_j}{\partial x_i}\, p^s_k\, \partial_s, \qquad \nabla_{\partial_{n+i}}\partial_{n+j} = \frac{\partial p^k_j}{\partial x_{n+i}}\, q^s_k\, \partial_{n+s}, \\ \nabla_{\partial_i}\partial_{n+j} &= \nabla_{\partial_{n+i}}\partial_j = 0.\end{aligned} \tag{20}$$

The only non-zero Christoffel symbols are

$$\Gamma^s_{ij} = \frac{\partial q^k_j}{\partial x^i}\, p^s_k, \qquad \Gamma^{n+s}_{n+i,n+j} = \frac{p^k_j}{\partial x^{n+i}}\, q^s_k.$$

In evaluations of the torsion tensor $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ and the curvature tensor $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ the only non-zero terms are

$$\begin{aligned}T(\partial_i,\partial_j) &= \left(\frac{\partial q^k_j}{\partial x^{n+i}} - \frac{\partial q^k_i}{\partial x^{n+j}}\right) p^s_k\, \partial_s, \\ T(\partial_{n+i},\partial_{n+j}) &= \left(\frac{\partial p^k_j}{\partial x^{n+i}} - \frac{\partial p^k_i}{\partial x^{n+j}}\right) q^s_k\, \partial_{n+s}, \\ R(\partial_i,\partial_{n+j})\,\partial_l &= \left(\frac{\partial^2 q^k_l}{\partial x^i\partial x^{n+j}}\cdot p^s_k + \frac{\partial q^k_l}{\partial x^i}\cdot\frac{\partial p^s_k}{\partial x^{n+j}}\right)\partial_s = \frac{\partial}{\partial x^{n+j}}\left(\frac{\partial q^k_l}{\partial x^i}p^s_k\right)\partial_s, \\ R(\partial_i,\partial_{n+j})\,\partial_{n+l} &= \left(\frac{\partial^2 p^k_l}{\partial x^{n+j}\partial x^i}\cdot q^s_k + \frac{\partial p^k_l}{\partial x^{n+j}}\cdot\frac{\partial q^s_k}{\partial x^i}\right)\partial_{n+s} = \frac{\partial}{\partial x^i}\left(\frac{\partial p^k_l}{\partial x^{n+j}}q^s_k\right)\partial_{n+s}.\end{aligned}$$

So the only non-zero components are

$$\begin{aligned}&T^s_{i,j} = \left(\frac{\partial q^k_j}{\partial x^i} - \frac{\partial q^k_i}{\partial x^j}\right) p^s_k, \qquad T^{n+s}_{n+i,n+j} = \left(\frac{\partial p^k_j}{\partial x^{n+i}} - \frac{\partial p^k_i}{\partial x^{n+j}}\right) q^s_k, \\ &R^{n+s}_{i,n+j,n+l} = (\boldsymbol{P}_{x^{n+j}x^i}\cdot\boldsymbol{Q} + \boldsymbol{P}_{x^{n+j}}\cdot\boldsymbol{Q}_{x^i})^s_l = ((\boldsymbol{P}_{x^{n+j}}\boldsymbol{Q})_{x^i})^s_l, \\ &R^s_{i,n+j,l} = -(\boldsymbol{Q}_{x^i x^{n+j}}\cdot\boldsymbol{P} + \boldsymbol{Q}_{x^i}\boldsymbol{P}_{x^{n+j}})^s_l = (-\boldsymbol{P}^{-1}\cdot(\boldsymbol{P}_{x^{n+j}}\cdot\boldsymbol{Q})_{x^i}\cdot\boldsymbol{P})^s_l \\ &\quad = (-\boldsymbol{P}^{-1}\cdot(\boldsymbol{P}_{x^{n+j}x^i}\cdot\boldsymbol{Q} + \boldsymbol{P}_{x^{n+j}}\cdot\boldsymbol{Q}_{x^i})\cdot\boldsymbol{P})^s_l.\end{aligned}$$

The conditions (15) can be written as

$$
\begin{aligned}
&\frac{\partial p_i^k}{\partial x^{n+j}} = \frac{\partial p_j^k}{\partial x^{n+i}} \quad (1.\ \text{integrability condition}),\\
&\frac{\partial q_i^k}{\partial x^j} = \frac{\partial q_j^k}{\partial x^i} \quad (2.\ \text{integrability condition}).
\end{aligned}
\tag{21}
$$

We see that the condition (15) is equivalent with $T=0$, and (14) is equivalent with $R=0$. Further, it can be verified that the condition (15) can be expressed by vanishing of the Nijenhuis bracket $[P,B]$, [17]:

$$
\begin{aligned}
[P,B](\partial_i,\partial_j) &= \left(\frac{\partial q_j^k}{\partial x^i} - \frac{\partial q_i^k}{\partial x^j}\right)\partial_{n+k},\\
[P,B](\partial_i,\partial_{n+j}) &= [P,B](\partial_{n+i},\partial_j) = 0,\\
[P,B](\partial_{n+i},\partial_{n+j}) &= \left(\frac{\partial p_i^k}{\partial x^{n+j}} - \frac{\partial p_j^k}{\partial x^{n+i}}\right)\partial_k.
\end{aligned}
$$

So we proved[3]

$$
\begin{aligned}
[P,B](X,Y) &= BT(X,Y) \qquad \text{for} \quad X,Y\in D_1,\\
[P,B](X,Y) &= -BT(X,Y) \quad \text{for} \quad X,Y\in D_2.
\end{aligned}
$$

Since B is an automorphism of T_xM at any point x we conclude

$$T=0 \Longleftrightarrow [P,B]=0.$$

Similarly, it can be verified the following:

$$T=0 \Longleftrightarrow [P_1^2,P_1^3]=0, \quad [P_2^1,P_2^3]=0.$$

Theorem 2. *A (P,B)-structure is integrable (and the corresponding 3-web is regular if and only if $[P,B]=0$, and the conditions* (14) *hold.*

We succeeded to substitute the Chern connection of a 3-web, which is more or less an additional object, by projectors, that is by tensor fields which are quite naturally related to the web, or can be even regarded as 1-forms directly defining web-distributions. This approach may be useful also for investigation of other types of webs.

It remains an open question if it will be possible to substitute the curvature tensor of the Chern connection by a suitable (1,3)-tensor field (or fields) constructed from projectors in some direct and natural way, some "concomitants" of projectors defined e. g. by help of Frölicher-Nijenhuis bracket.

[3] In [17], the assertion is proved in an invariant way.

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Alena Vanžurová
Department of Algebra and Geometry
Fac. Sci., Palacký University
Tomkova 40
779 00 Olomouc, CZECH REPUBLIC