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PROPERTY (A) OF THE *N-TH* ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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Abstract. The equation to be considered is

$$L_n y(t) + p(t)y(\tau(t)) = 0.$$

The aim of this paper is to derive sufficient conditions for property (A) of this equation.

In the paper a result of Džurina [2] concerning asymptotic properties of the third order linear differential equations with delay is extended to an *n-th* order delay differential equation.

We consider the differential equation

$$L_n y(t) + p(t)y(\tau(t)) = 0, \tag{1}$$

where $n \geqslant 3$,

$$L_n y(t) = \frac{1}{r_n(t)}\left(\frac{1}{r_{n-1}(t)} \cdots \left(\frac{y(t)}{r_0(t)}\right)' \cdots\right)',$$

$r_i(t)$, $i = 0, 1, \cdots, n$ are positive and continuous functions on some ray $[t_0, \infty)$, $\tau(t) < t$ is increasing function on $[t_0, \infty)$.

The expressions

$$L_0 y(t) = \frac{y(t)}{r_0(t)}, \quad L_i y(t) = \frac{1}{r_i(t)}\Big(L_{i-1}y(t)\Big)', \quad i = 1, 2, \cdots, n$$

called *quasi-derivatives* will be very helpful in the sequel. We will suppose throughout the paper that

$$\int_{t_0}^{\infty} r_i(s)\, ds = \infty \qquad i = 1, 2, \cdots, n-1.$$

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We restrict our considerations to nontrivial solutions of (1), which exist on $[t_0, \infty)$. Such a solution is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

Let $i_k \in \{1, \cdots, n-1\}$, $1 \le k \le n-1$ and $t, s \in [t_0, \infty)$. We define

$$I_0 = 1,$$
$$I_k(t, s; r_{i_k}, \cdots, r_{i_1}) = \int_s^t r_{i_k}(x) I_{k-1}(x, s; r_{i_{k-1}}, \cdots, r_{i_1})\, dx.$$

It is easy to verify that for $1 \le k \le n-1$

$$I_k(t, s; r_{i_k}, \cdots, r_{i_1}) = (-1)^k I_k(s, t; r_{i_1}, \cdots, r_{i_k}),$$
$$I_k(t, s; r_{i_k}, \cdots, r_{i_1}) = \int_s^t r_{i_1}(x) I_{k-1}(t, x; r_{i_k}, \cdots, r_{i_2})\, dx.$$

The following generalization of a lemma of Kiguradze [4] can be found in [7, Lemma 1 and Lemma 2].

Lemma 1. *Let $y(t)$ be a nonoscillatory solution of (1), then there exist an integer ℓ, $\ell \in \{0, 1, \cdots, n-1\}$ with $n + \ell$ odd and $t_1 \geqslant t_0$, such that for all $t \geqslant t_1$*

$$\begin{aligned} y(t) L_i y(t) &> 0, \quad 0 \leqslant i \leqslant \ell, \\ (-1)^{i-\ell} y(t) L_i y(t) &> 0, \quad \ell \leqslant i \leqslant n-1 \end{aligned} \tag{3}$$

and moreover if $y(t)$ is positive then

$$L_0 y(t) \geqslant L_\ell y(t) I_\ell(t, t_1; r_1, \cdots, r_\ell). \tag{4}$$

The following lemma is necessary in the proof of the main result of this paper.

Lemma 2. *Let $y(t)$ be a positive solution of (1). If $y(t)$ is of degree ℓ, $1 \leqslant \ell \leqslant n-1$, then*

$$L_\ell u(t) \geqslant \int_t^\infty I_{n-\ell-1}(s, t; r_{n-1}, \cdots, r_{\ell+1}) r_n(s) p(s) y(\tau(s))\, ds. \tag{5}$$

For a proof see e.g. [5, Theorem 1].

Remark. Relation (5) can be also easily obtained by repeated integration of (1) from t to ∞.

Following Foster and Grimmer [3] we say that $y(t)$ satisfying (3) is a function of degree ℓ. The set of all nonoscillatory solutions of degree ℓ of (1) is denoted by $\mathcal{N}_\ell$. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (1), then by Lemma 1

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \mathcal{N}_{n-1} \qquad \text{if } n \text{ is odd},$$
$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \mathcal{N}_{n-1} \qquad \text{if } n \text{ is even}.$$

We are interested in the situation when

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_0 \qquad \text{if } n \text{ is odd,} \\ \mathcal{N} &= \emptyset \qquad \text{if } n \text{ is even.} \end{aligned}$$

When this situation occurs, we say that (1) enjoys property (A). Property (A) has been studied by many authors see e.g.in [1], [7] and [8]. The main purpose of this paper is to adapt Džurina's method and technique known for third order delay equations [2] to establish criteria for property (A) of *n-th* order delay equations.

Theorem 1. *Let $g(t)$ be a continuous function satisfying*

$$g(t) > t, \quad \tau(g(t)) \leqslant t. \tag{6}$$

Define for all $\ell \in \{1, 2, \cdots, n-1\}$ and $s \geqslant t$, t large enough the functions

$$q_\ell(s,t) = I_{n-\ell-1}(s,t;r_{n-1},\cdots,r_{\ell+1})r_n(s)p(s)r_0(\tau(s))I_\ell(\tau(s),t_0;r_1,\cdots,r_2). \tag{7}$$

Assume that

$$\liminf_{t\to\infty} \int_t^{g(t)} q_\ell(s,t)\,ds > 1 \tag{8}$$

for all $\ell \in \{1, 2, \cdots, n-1\}$ such that $n + \ell$ is odd. Then equation (1) has property (A).

Proof. Suppose that $y(t)$ is a nonoscillatory and positive solution of (1) in a neighbourhood of infinity. With respect to Lemma 1, there exist a t_1 and an integer $\ell \in \{0, 1, \cdots, n-1\}$ with $n+\ell$ odd, such that (3) holds. To obtain a contradiction assume that $\ell \geqslant 1$. Then on the basis of Lemma 1

$$L_0 y(t) \geqslant L_\ell y(t) I_\ell(t, t_1; r_1, \cdots, r_\ell)$$

and by Lemma 2

$$L_\ell u(t) \geqslant \int_t^\infty I_{n-\ell-1}(s,t;r_{n-1},\cdots,r_{\ell+1})r_n(s)p(s)y(\tau(s))\,ds.$$

It follows from (3) that $L_\ell y(t)$ is a decreasing function. Combining the last two inequalities we see that

$$\begin{aligned} L_\ell u(t) &\geqslant \int_t^\infty I_{n-\ell-1}(s,t;r_{n-1},\cdots,r_{\ell+1})r_n(s)p(s)r_0(\tau(s)) \\ &\quad \times L_\ell y(\tau(s)) I_\ell(\tau(s),t_1;r_1,\cdots,r_\ell)\,ds \\ &\geqslant \int_t^{g(t)} I_{n-\ell-1}(s,t;r_{n-1},\cdots,r_{\ell+1})r_n(s)p(s)r_0(\tau(s)) \\ &\quad \times L_\ell y(\tau(s)) I_\ell(\tau(s),t_1;r_1,\cdots,r_\ell)\,ds \\ &\geqslant L_\ell u(\tau[g(t)]) \int_t^{g(t)} I_{n-\ell-1}(s,t;r_{n-1},\cdots,r_{\ell+1})r_n(s)p(s)r_0(\tau(s)) \\ &\quad \times I_\ell(\tau(s),t_1;r_1,\cdots,r_\ell)\,ds. \end{aligned}$$

Since $\tau(g(t)) \leqslant t$ and $L_\ell u(t)$ is decreasing, the previous inequalities yield

$$\begin{aligned}1 \geqslant \int_t^{g(t)} & I_{n-\ell-1}(s,t;r_{n-1},\cdots,r_{\ell+1})r_n(s)p(s)r_0(\tau(s)) \\ & \times I_\ell(\tau(s),t_1;r_1,\cdots,r_\ell)\,ds,\end{aligned}$$

which contradicts with (8). The proof is complete.

Theorem 1 extends Theorem 1 in [2] to $n-th$ order differential equations.

If we put $g(t)=\tau^{-1}(t)$, where $\tau^{-1}(t)$ is the inverse function to $\tau(t)$ we immediately have:

Corollary 1. *Let $q_\ell(s,t)$ be defined as in (7). Assume that for all $\ell \in \{1,2,\cdots,n-1\}$ such that $n+\ell$ is odd*

$$\liminf_{t\to\infty} \int_t^{\tau^{-1}(t)} q_\ell(s,t)\,ds > 1.$$

Then equation (1) has property (A).

Our results are new also for the particular case of (1), namely for the differential equation

$$y^{(n)}(t) + p(t)y(\tau(t)) = 0. \tag{9}$$

To illustrate this fact we compare our results with those of Naito [6] and Džurina [1]. At first note that (8) reduces for (9) to

$$\liminf_{t\to\infty} \int_t^{\tau^{-1}(t)} \frac{(s-t)^{n-\ell-1}}{(n-\ell-1)!}\frac{(\tau(s)-t_0)^\ell}{\ell!}p(s)\,ds > 1.$$

Theorem A. *Let*

$$\liminf_{t\to\infty} \tau(t)\int_t^\infty (\tau(s)-\tau(t))^{n-2}p(s)\,ds > \frac{(n-1)!}{4}.$$

Then equation (9) has property (A).

Theorem B. *Let*

$$\liminf_{t\to\infty} \tau^{n-1}(t)\int_t^\infty p(s)\,ds > \frac{M_1}{n-1},$$

where M_1 is the maximum of all local maxima of the polynomial

$$P_n(k) = -k(k-1)\cdots(k-n-1).$$

Then equation (9) has property (A).

Theorem A can be found in [6, Theorem 5 and 6] and Theorem B can be found in [1, Theorem 11].

Example 1. Let us consider the fourth order delay equation

$$y^{(IV)}(t) + \frac{a}{t^4} y(\lambda t) = 0 \tag{10}$$

with $a > 0$, $t \geqslant 1$, $0 < \lambda < 1$ and $a\lambda^3 < 1$. It is easy to verify that Theorems A and B fail for (10). On the other hand by Corollary 1 equation (10) has property (A) i.e. (10) is oscillatory provided

$$a\lambda^3 \ln \lambda > 6.$$

The above example shows that Theorem 1 and Corollary 1 are not included in the known criteria for property (A).

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