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THE GENERALIZED BOUNDARY VALUE PROBLEM IS A FREDHOLM MAPPING OF INDEX ZERO

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Abstract. In the paper it is proved that each generalized boundary value problem for the n-th order linear differential equation generates a Fredholm mapping of index zero.

This contribution is a short note on the paper of V.Šeda [2].

The main goal of this paper is to prove that each generalized boundary value problem for the n-th order linear differential equation

$$x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \cdots + a_n(t)x(t) = 0, \tag{1}$$

$$l_i(x) = 0 \qquad i = 1, \dots, n, \tag{2}$$

where $a_i(t) \in C([a,b])$ and $l_i : C^n([a,b]) \to R$ are continuous linear functionals, generates a Fredholm mapping of index zero.

Functionals $l_i, \quad i = 1, \dots, n$ are assumed to be linearly independent. Cf. [2, Lemma 10].

The result is based on Nikol'skij's theorem.

Theorem. *(Nikol'skij)* [3, p.233]. *A linear bounded operator $A : X \to Y$ is Fredholm of index zero if and only if*

$$A = C + T$$

where X and Y are Banach spaces, C is a linear homeomorphism of X onto Y and $T : X \to Y$ is a linear completely continuous operator.

We denote $X = \{x \in C^n([a,b]),\ l_i(x) = 0 \quad i = 1, \dots, n\}$ and $Y = C([a,b])$.

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Theorem. *The operator* $A : X \to Y$

$$Ax(t) = x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \cdots + a_n(t)x(t)$$

is a Fredholm mapping of index zero.

Proof. We shall find a suitable homeomorphism C and a completely continuous operator T such that $A = C + T$.

As the boundary conditions (2) are linearly independent no functional l_i is identically equal to zero and for every $i = 1, \ldots, n$ we can represent the factor space

$$C^n([a,b])/_{\{l_i(x)=0\}}$$

by the one dimensional subspace $\{c\phi_i(t),\ c \in R\}$ of $C^n([a,b])$, where $\phi_i(t) \in C^n([a,b])$ is a suitable (not uniquely determined) function.

Now we choose functions $\{\phi_i\}_{i=1}^n$ by the induction.

1. In the first step we choose ϕ_1 positive. If $l_1(1) \neq 0$, then we set $\phi_1 = 1$. If $l_1(1) = 0$, we choose arbitrary $\bar{\phi}_1$ such that $l_1(\bar{\phi}_1) \neq 0$. Being from $C^n([a,b])$, $\bar{\phi}_1$ has a minimum on $[a,b]$ and there is a constant $k \geq 0$ such that

$$\phi_1(t) = \bar{\phi}_1(t) + k > 0 \qquad \text{for } t \in [a,b].$$

Obviously

$$l_1(\phi_1) = l_1(\bar{\phi}_1) \neq 0.$$

Functionals l_1 and $-l_1$ represent the same boundary condition. We choose between them such, and denote it again l_1, that $l_1(\phi_1) > 0$.

2. Assume that we have found functions $\phi_1, \ldots, \phi_{k-1}$ associated with functionals $l_1, \ldots, l_{k-1}$ such that the Wronskian

$$\begin{aligned} W(\phi_1, \ldots, \phi_i) > 0 \qquad & \text{for } t \in [a,b] \\ l_i(\phi_i) > 0, \qquad l_j(\phi_i) = 0 & \end{aligned} \tag{3}$$

holds for every $i = 1, \ldots, k-1$ and every $j < i$.

We find the function ϕ_k associted with the functional l_k, such that (3) holds for $i = 1, \ldots, k$ and $j < i$.

At first we find a function $\bar{\phi}_k$ satisfying the condition

$$W(\phi_1, \ldots, \phi_{k-1}, \bar{\phi}_k) = 1.$$

As the function $\bar{\phi}_k$ is a solution of the above (k-1)-th order ODE it is

$$\bar{\phi}_k(t) = y(t) + c_1\phi_1(t) + \cdots + c_{k-1}\phi_{k-1}(t).$$

Now beginning by c_1, we choose the coefficients $\{c_i\}_{i=1}^{k-1}$ such that $l_j(\bar{\phi}_k) = 0$ for $j = 1, \ldots, k-1$. The choice is possible while $l_j(\phi_i) = 0$ for $j < i$.

If $l_k(\bar{\phi}_k) \neq 0$, we set $\phi_k = \bar{\phi}_k$.

If $l_k(\bar{\phi}_k) = 0$, we choose the function $\tilde{\phi}_k$ arbitrary but such that $l_j(\tilde{\phi}_k) = 0$ for $j = 1, \ldots, k-1$ and $l_k(\tilde{\phi}_k) \neq 0$, and we calculate the Wronskian

$$W(\phi_1, \ldots, \phi_{k-1}, \tilde{\phi}_k) = d(t).$$

The function $d(t)$ is continuous. We set

$$\phi_k = \tilde{\phi}_k + c\bar{\phi}_k$$

where $c \in R$, $c > |min\ d(t)|$ is a constant.

Now it is easy to prove that

$$W(\phi_1, \ldots, \phi_k) > 0$$

and $l_j(\phi_k) = 0$ for $j = 1, \ldots, k-1$ and $l_k(\phi_k) \neq 0$.

At last we again choose between functionals l_k, $-l_k$ such, denoting it again l_k, that $l_k(\phi_k) > 0$.

Thus we have found n linearly independent functions $\{\phi_i\}_{i=1}^n$, $\phi_i \in C^n([a,b])$ satisfying (3) for each $i = 1, \ldots, n$ and each $j < i$. These functions generate the basis of the set of solutions of the homogeneous linear differential equation

$$W(\phi_1, \ldots, \phi_n, x) = 0, \tag{4}$$

or dividing by the first coefficient, the equation

$$x^{(n)}(t) + c_1(t)x^{(n-1)}(t) + \cdots + c_n(t)x(t) = 0. \tag{5}$$

That means, the boundary value problem (5), (2) has only the trivial solution.

Then the operator $C : X \to Y$ given by

$$Cx(t) = x^{(n)}(t) + c_1(t)x^{(n-1)}(t) + \cdots + c_n(t)x(t),$$

i.e. the operator generated by the problem (5), (2) is continuous, invertible and onto Y. The Banach inverse function theorem means that C is also a homeomorphism.

Then

$$A = C + T$$

where $T : X \to Y$

$$Tx(t) = (a_1(t) - c_1(t))\, x^{(n-1)}(t) + \cdots + (a_n(t) - c_n(t))\, x(t).$$

Because $X \subseteq C^n([a,b])$ and the highest derivative is of the order $n-1$, the Arsela-Ascoli theorem follows that T is a completely continuous operator.

We remark that the basis $\{\phi_i\}_{i=1}^n$ of the set of solutions of the equation (5) has property (3) and especially $W(\phi_1, \dots, \phi_i) > 0$ for each $t \in [a,b]$, and each $i = 1, \dots, n$.

The property (3) implies that the equation (5) is disconjugate on $[a,b]$. (See [1]).

Corollary. *To each system of n independent generalized boundary coditions* (2) *there is a disconjugate on* $[a,b]$ *ordinary differential equation of n-th order such that the boundary value problem has only the trivial solution.*

Lets consider the generalized boundary value problem

$$x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \dots + a_n(t)x(t) + f(t, x, \dots, x^{(m)}) = q(t), \tag{6}$$

$$l_i(x) = 0 \qquad i = 1, \dots, n, \tag{2}$$

where $f \in C([a,b] \times R^{m+1})$, $m < n$, $q \in C([a,b])$ and $l_i : C^{n-1}([a,b]) \to R$ $i = 1, \dots, n$ is a linear continuous functional [2, p.11-12].

The right side of the equation (6) defines an operator $C + (T + B) : X \to Y$, where $B : X \to Y$ is the completely continuous Nemickij's operator given by the function f.

Obviously the restriction of l_i on $C^n([a,b])$ is a continuous linear functional.

As the consequence of our theorem we obtain that the operator $C + (T + B)$ is a completely continuous perturbation of a linear homeomorphism. (Confer [2, Lemma 10, 12, Theorem 5, 6] where an aditional assumption is supposed.)

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