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PARAMETRIZED RELAXATION FOR EVOLUTION INCLUSIONS OF THE SUBDIFFERENTIAL TYPE

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Abstract. In this paper we consider parametric nonlinear evolution inclusions driven by time-dependent subdifferentials. First we prove some continuous dependence results for the solution set (of both the convex and nonconvex problems) and for the set of solution-selector pairs (of the convex problem). Then we derive a continuous version of the "Filippov-Gronwall" inequality and using it, we prove the parametric relaxation theorem. An example of a parabolic distributed parameter system is also worked out in detail.

1. Introduction

It is well known that if the orientor field (set-valued vector field) of a differential inclusion is Lipschitz continuous in the state variable, then the solution set of the differential inclusion is dense in that of the convexified problem (i.e. the differential inclusion obtained by replacing the orientor field by its closed, convex hull). We refer to the book of Aubin-Cellina [2] (theorem 2, p. 124) for differential inclusions in $\mathbb{R}^N$ and to Papageorgiou [21], Zhu [32] for differential inclusions in Banach spaces, for further details on this issue. Such a density result is known in the literature as "relaxation theorem" and plays an important role in control theory, in connection with the study of the relaxed system and in the derivation of "bang-bang principles". Recently, the relaxation theorem was extended by Frankowska [12] (theorem 2.5 and corollary 2.6) to semilinear evolution inclusions and by Papageorgiou [23], [24] to nonlinear, nonautonomous evolution inclusions of the subdifferential type (in fact in [24] a stronger result was obtained; namely that the set of "extremal solutions"-i.e. solutions moving through the extreme points of the orientor field - is dense in the solution set of the convexified problem). In a recent paper, Fryszkowski-Rzezuchowski [13], considered parametrized differential inclusions, and proved a continuous analog of the relaxation result. Their proof was based on a parametric version of the well known Filippov approximation result, which was obtained by Colombo et al. [6]. Recall (see Aubin-Cellina [2], theorem

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1, p. 120 and Filippov [10]), that according to Filippov's result, given the multivalued Cauchy problem $\dot{x}(t) \in F(t, x(t))$ *a.e.* , $x(0) = x_0$ in which the orientor field $F(t, x)$ is $h - Lipschitz$ continuous in x, with closed but not necessarily convex values and $y(\cdot)$ an absolutely continuous function on $T = [0, b]$ such that $y(0) = x_0$ and $t \to d(\dot{y}(t), F(t, y(t))$ is integrable, then there exists a solution $x(\cdot)$ of the differential inclusion such that $\int_0^t \|\dot{y}(s) - \dot{x}(s)\| \, ds \leq \int_0^t p(s) \, exp \left[\int_s^t k(\tau) \, d\tau \right] ds$, with $k(t) \in L^1_+$ being the Lipschitz constant for $F(t, \cdot)$. This estimate is a very useful tool in the study of differential inclusions and among other things, shows that the solution set $S(x_0)$ of the multivalued Cauchy problem, as a multifunction of the initial condition x_0, is $h - Lipschitz$ continuous. The original result of Filippov [10], was extended to Caratheodory-type orientor fields by Himmelberg-Van Vleck [15] and to semilinear evolution inclusions by Frankowska [12] and Papageorgiou [26]. When dealing with parametrized differential inclusions, due to the lack of uniqueness of a solution to obtain a continuous version of the above "Filippov-Gronwall inequality", we need to slightly relax the estimate by allowing for an arbitrarily small error $\epsilon > 0$ (see Colombo et al. [6]).

The purpose of this paper is to extend the parametric relaxation result of Fryszkowski-Rzezuchowski [13], to nonlinear evolution inclusions of the subdifferential type. Such inclusions are important in the study of infinite dimensional systems, because they model differential inclusions with multivalued terms; see for example Flytzanis-Papageorgiou [11], Papageorgiou [22] and Tiba [27]. Our approach follows that of Fryszkowski-Rzezuchowski [13], and so we also prove a continuous version of the "Filippov-Gronwall inequality", extending this way to a class of nonlinear parametric evolution inclusions the corresponding result in Frankowska [12] and Papageorgiou [26].

2. Preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper, we will be using the following notations:

$$P_{f(c)}(X) = \{A \subseteq X : \quad \text{nonempty, closed (convex)}\}$$
$$\text{and} \quad P_{(w)k(c)}(X) = \{A \subseteq X : \quad \text{nonempty, (weakly-) compact, (convex)}\}$$

A multifunction (set-valued function) $F : \Omega \to P_f(X)$ is said to be "measurable" if and only if for all $x \in X$, the $\mathbb{R}_+ - valued$ function $\omega \to d(x, F(\omega)) = inf\{\|x - z\| : z \in F(\omega)\}$ is measurable. We will say that $F(\cdot)$ is "graph measurable" if and only if $GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field. Recall that measurability implies graph measurability, but the converse is not in general true. It is true if Σ is a Souslin family and this is the case if there is a finite measure $\mu(\cdot)$ defined on (Ω, Σ) with respect to which Σ is complete. For details we refer to the survey paper of Wagner [28]. Next let $\mu(\cdot)$ be a finite measure defined on (Ω, Σ). By $S^p_F(1 \leq p \leq \infty)$ we will denote the set of measurable selectors of $F(\cdot)$, that belong in the Lebesgue-Bochner space $L^p(X)$; i.e. $S_F^p = \{f \in L^p(X) : f(\omega) \in F(\omega)\mu - a.e.\}$. In general, this set may be empty. A straightforward application of Aumann's selection theorem (see Wagner [28], theorem 5.10) shows that for a graph measurable multifunction, S_F^p is nonempty if and only if $\omega \to \inf\{||x|| : x \in F(\omega)\} \in L_+^p$. A set $K \subseteq L^p(X)$ is said to be decomposable if for all $(f, g, A) \in K \times K \times \Sigma$, we have that $\chi_A f + \chi_{A^c} g \in K$. Clearly the set S_F^p is decomposable in the space $L^p(X)$.

On $P_f(X)$ we can define a generalized metric, known in the literature as Hausdorff metric, by setting for $A, B \in P_f(X)$

$$h(A, B) = \max\left[\sup_{a\in A} d(a, B), \sup_{i\in B} d(b, A)\right]$$

where $d(a, B) = \inf\{||a-b|| : b \in B\}$ and $d(b, A) = \inf\{||b-a|| : a \in A\}$. It is well-known (see for example Klein-Thompson [16]), that the metric space $(P_f(X), h)$ is complete. A multifunction $F : X \to P_f(X)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from X into the metric space $(P_f(X), h)$.

If Y, Z are Hausdorff topological spaces and $G : Y \to 2^Z \setminus \{\emptyset\}$, we say that $G(\cdot)$ is lower semicontinuous ($l.s.c.$), if for every $C \subseteq Z$ closed $G^+(C) = \{y \in Y : G(y) \subseteq C\}$ is closed (or equivalently for every $U \subseteq Z$ open, $G^-(U) = \{y \in Y : G(y) \cap U \neq \emptyset\}$ is open). If Y, Z are metric spaces, then lower semicontinuity is equivalent to saying that if $y_n \to y$ in Y, then $G(y) \subseteq \underline{\lim}\, G(y_n) = \{z \in Z : \lim d(z, G(y_n)) = 0\} = \{z \in Z : z = \lim z_n, z_n \in G(y_n) n \geq 1\}$. Also in this case lower semicontinuity of $G(\cdot)$ is equivalent to the upper semicontinuity of the distance function $y \to d(z, G(y))$ for every $z \in Z$ (see DeBlasi-Myjak [8]).

Let $\varphi : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We will say that $\varphi(\cdot)$ is proper, if it is not identically $+\infty$. Assume that $\varphi(\cdot)$ is proper, convex and lower semicontinuous ($l.s.c.$) (usually this family of $\bar{\mathbb{R}}$-valued functions is denoted by $\Gamma_0(X)$). By dom φ, we will denote the effective domain of $\varphi(\cdot)$; i.e. $\operatorname{dom}\varphi = \{x \in X : \varphi(x) < +\infty\}$. The subdifferential of $\varphi(\cdot)$ at x, is the set $\partial\varphi(x) = \{x^* \in X^* : (x^*, y - x) \leq \varphi(y) - \varphi(x)$ for all $y \in \operatorname{dom}\varphi\}$ (here $(\cdot, \cdot)$ stands for the duality brackets for the pair (X, X^*)). If $\varphi(\cdot)$ is Gateaux differentiable, then $\partial\varphi(x) = \{\varphi'(x)\}$. We will say that $\varphi \in \Gamma_0(X)$ is of compact type, if for every $\lambda \in \mathbb{R}_+$, the level set $\{x \in X : ||x||^2 + \varphi(x) \leq \lambda\}$ is compact.

Our mathematical setting is the following: $T = [0, b]$, H is a separable Hilbert space (the state space) and Λ a complete metric space (the parameter space). We will be considering the following two multivalued Cauchy problems:

$$\left\{ \begin{array}{c} \dot{x}(t) \in \partial\varphi(t, x(t)) + F(t, x(t), \lambda) \;\; a.e. \\ x(0) = v(\lambda) \end{array} \right\} \tag{1}$$

and its convexified counterpart

$$\left\{ \begin{array}{c} -\dot{x}(t) \in \partial\varphi(t, x(t)) + \overline{\operatorname{conv}} F(t, x(t), \lambda) \quad a.e. \\ x(0) = v(\lambda) \end{array} \right\} \tag{2}$$

By a "strong solution" of (1) (resp. of (2)), we mean a function $x \in W^{1,2}(T,H)$ such that $x(0) = x_0$ and there exists $f_\lambda \in L^2(H)$ with $f_\lambda(t) \in F(t,x(t),\lambda)$ *a.e.* (resp. $f_\lambda(t) \in \overline{\text{conv}}\, F(t,x(t),\lambda)$ *a.e.*), such that $-\dot{x}(t) \in \partial\varphi(t,x(t)) + f_\lambda(t)$ *a.e.* Recall (see theorem 2.2 of Barbu [3]), that $W^{1,2}(T,H)$ can be identified with $AC^{1,2}(T,H)$ the space of all absolutely continuous functions from T into H with strong derivative in $L^2(H)$. So for every $y \in W^{1,2}(T,H)$ we can find a $y_1 \in AC^{1,2}(T,H)$ such that $y = y_1$ *a.e.* on $(0,b)$ (i.e. every equivalence class in $W^{1,2}(T,H)$ has an absolutely continuous representative, whose strong derivative $\dot{x}(\cdot) \in L^2(T,H)$). We know that if Y is a Banach space with the RNP (Radon-Nikodym property), then every absolutely continuous function $x : T \to Y$, is strongly differentiable *a.e.* and $\dot{x} \in L^1(Y)$. The class of RNP-spaces, includes reflexive Banach spaces and separable dual spaces. For details we refer to Diestel-Uhl [9].

Following Yotsutani [31], we will make the following hypothesis on $\varphi(t,x)$ which will be valid throughout this work.

$\underline{H(\varphi)}$: $\varphi : T \times H \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a function such that

(1) for every $t \in T$, $\varphi(t,\cdot)$ is proper, convex, *l.s.c.* (i.e. $\varphi(t,\cdot) \in \Gamma_0(H)$) and is also of compact type,

(2) for any $\alpha \in [0,1]$ and $\beta = 2$ if $\alpha \in [0,1/2]$ or $\beta = \frac{1}{1-\alpha}$ if $\alpha \in [1/2,1]$ and for any positive integer r, there exists a constant $K_r > 0$, an absolutely continuous function $g_r : T \to \mathbb{R}$ with $\dot{g}_r \in L^\beta(T)$ and a function of bounded variation $h_r : T \to \mathbb{R}$ such that if $t \in T$, $x \in \text{dom}\,\varphi(t,\cdot)$ with $\|x\| \le r$ and $s \in [t,b]$, then there exists $\widehat{x} \in \text{dom}\,\varphi(s,\cdot)$ satisfying
$$\|\widehat{x} - x\| \le |g_r(s) - g_r(t)|(\varphi(t,x) + K_r)^\alpha$$
and $$\varphi(s,\widehat{x}) \le \varphi(t,x) + |h_r(s) - h_r(t)|(\varphi(t,x) + K_r)\,.$$

Remarks. (a) This hypothesis is more general than the ones used by Watanabe [29] and Yamada [30].

(b) If $\varphi(t,\cdot) = \varphi(\cdot) \in \Gamma_0(H)$ (i.e. there is no t-dependence) and $\varphi(\cdot)$ is of compact type, then clearly $H(\varphi)$ is satisfied. Also assume that $K : T \to P_{kc}(H)$ is a multifunction such that $h(K(t'),K(t)) \le \int_t^{t'} v(s)\,ds$ for all $0 \le t \le t' \le b$ and with $v(\cdot) \in L^2_+$ and let $\varphi(t,x) = \delta_{K(t)}(x)$ where $\delta_{K(t)}(x)$ is the indicator function of $K(t)$; i.e. $\delta_{K(t)}(x) = 0$ if $x \in K(t)$, $+\infty$ otherwise. Then it is clear that hypothesis $H(\varphi)$ is satisfied with $\alpha = 0$, $K_r = 1$, $\dot{g}_r = v$ and $h_r = 0$. Recall that $\partial\varphi(t,x) = N_{K(t)}(x)$ for all $(t,x) \in GrK$, where $N_{K(t)}(x)$ is the normal cone to $K(t)$ at x. Evolution inclusions of the form $-\dot{x}(t) \in N_{K(t)}(x(t)) + F(t,x(t))$ *a.e.* arise in mechanics (see Moreau [17], where $F = 0$) and can be useful in the optimal control of variational inequalities. In fact, when $K(t) = K$ (i.e. no $t - dependence$), they are called "Differential Variational Inequalities" and they are equivalent to the projected differential inclusion $\dot{x}(t) \in \text{proj}(F(t,x(t)); T_K(x(t)))$ *a.e.* This was first proved by Cornet [7] (see also Aubin-Cellina [2], chapter 5, section 6). Projected differential inclusions arise in mathematical economics, in the study of planning procedures, as well as in other applied problems with state constraints.

In what follows by $S(\lambda) \subseteq C(T,H)$, we will denote the solution set of (1) and by $S_r(\lambda) \subseteq C(T,H)$ the solution set of (2). Also if $h \in L^2(T,H)$, by $q_\lambda(h)(\cdot) \in C(T,H)$ we will denote the unique solution of the Cauchy problem $-\dot{x}(t) \in \partial\varphi(t,x(t)) + h(t)$ *a.e.* $x(0) = v(\lambda) \in \overline{\operatorname{dom}\varphi(0,\cdot)}$ (its existence and uniqueness follows from the theorem of Yotsutani [31]). Let $P(\lambda) \subseteq C(T,H)\times L^2(T,H)$ be defined by $P(\lambda) = \{(x,h) : x = q_\lambda(h) \text{ and } h \in S^2_{F(\cdot,x(\cdot))}\}$. Similarly $P_r(\lambda) = \{(x,h) : x = q_\lambda(h) \text{ and } h \in S^2_{\overline{\operatorname{conv}}F(\cdot,x(\cdot))}\} \subseteq C(T,H) \times L^2(T,H)$.

3. Continuous dependence result

In this section we establish the continuity properties of the multifunction $\lambda \to P_r(\lambda)$ and from that result, we deduce the continuity properties of $\lambda \to S_r(\lambda)$ and $\lambda \to S(\lambda)$. For this we will need the following hypotheses:

$\underline{H(F)_1:}$ $F : T \times H \times \Lambda \to P_f(H)$ is a multifunction such that

(1) $t \to F(t,x,\lambda)$ is measurable,

(2) $h(F(t,x,\lambda), F(t,y,\lambda)) \leq k_B(t)\|x-y\|$ *a.e.* for all $\lambda \in B \subseteq \Lambda$ compact and with $k_B(\cdot) \in L^2_+$,

(3) $|F(t,x,\lambda)| = \sup\{\|v\| : v \in F(t,x,\lambda)\} \leq a_B(t) + c_B(t)\|x\|$ *a.e.* for all $\lambda \in B \subseteq \Lambda$ compact and with $a_B, c_B \in L^2_+$,

(4) $\lambda \to \overline{\operatorname{conv}}F(t,x,\lambda)$ is *l.s.c.*

$\underline{H_0:}$ $v : \Lambda \to \operatorname{dom} \varphi(0,\cdot)$ is continuous.

From Papageorgiou [23] (theorems 3.1 and 3.2) we know that with hypotheses $H(\varphi)$, $H(F)$ and H_0 valid, for all $\lambda \in \Lambda$, the sets $S(\lambda)$ and $S_r(\lambda)$ are nonempty and in fact $S_r(\lambda)$ is compact in $C(T,H)$ (see [23], theorem 4.1). Also $P_r(\lambda)$ is a compact subset of $C(T,H) \times L^2(T,H)_w$, where $L^2(T,H)_w$ denotes the Hilbert space $L^2(T,H)$ furnished with the weak topology. In fact, it is easy to see that $P_r(\lambda) \in P_f(C(T,H) \times L^2(T,H))$ for all $\lambda \in \Lambda$.

Theorem 3.1. *If hypotheses $H(\varphi)$, $H(F)_1$, and H_0 hold, then*

$$\lambda \to P_r(\lambda) \quad \text{is } l.s.c. \text{ from } \Lambda \text{ into } P_f(C(T,H) \times L^2(T,H)).$$

Proof. We need to show that if $\lambda_n \to \lambda$, then $P_r(\lambda) \subseteq \underline{\lim}\, P_r(\lambda_n)$. To this end, let $[x,f] \in P_r(\lambda)$. By definition we have

$$-\dot{x}(t) \in \partial\varphi(t,x(t)) + f(t) a.e., x(0) = v(\lambda)$$

with $f(t) \in \overline{\operatorname{conv}}\, F(t,x(t),\lambda)$ *a.e.* Because of hypothesis $H(F)_1$ (4) and theorem 4.1 of Papageorgiou [20], we know that $\lambda \to S^2_{\overline{\operatorname{conv}}\,F(\cdot,x(\cdot),\lambda)}$ is *l.s.c.* So we can find $f_n \in S^2_{\overline{\operatorname{conv}}\,F(\cdot,x(\cdot),\lambda_n)}$ such that $f_n \xrightarrow{s} f$ in $L^2(T,H)$. Let $y_n(\cdot)$ be the unique strong solution of the Cauchy problem

$$\left\{\begin{array}{c} -\dot{y}_n(t) \in \partial\varphi(t, y_n(t)) + f_n(t) \quad a.e. \\ y_n(0) = v(\lambda) \end{array}\right\}.$$

Exploiting the monotonicity of the subdifferential operator, we get that

$$(-\dot{y}_n(t) + \dot{x}(t), x(t) - y_n(t)) \leq (f_n(t) - f(t), x(t) - y_n(t)) \quad a.e.$$

$$\text{thus} \quad \frac{1}{2}\frac{d}{dt}\|y_n(t) - x(t)\|^2 \leq \|f_n(t) - f(t)\| \cdot \|y_n(t) - x(t)\| \quad a.e.$$

Integrating the above inequality from 0 to t, we get

$$\|y_n(t) - x(t)\|^2 \leq 2\int_0^t \|f_n(s) - f(s)\| \cdot \|y_n(s) - x(s)\|\, ds\,.$$

Applying lemma A.5, p. 157 of Brezis [5], we get

$$\|y_n(t) - x(t)\| \leq 2\|f_n - f\|_1 \quad \text{for all} \quad t \in T$$

$$\text{and so} \quad y_n \to x \quad \text{in} \quad C(T, H) \quad \text{as} \quad n \to \infty\,.$$

Next let $m(t, \lambda_n) = \mathrm{proj}(f(t); \overline{\mathrm{conv}}\, F(t, x(t), \lambda_n))$ and $u(t, z, \lambda_n) = \mathrm{proj}(m(t, \lambda_n); \overline{\mathrm{conv}}\, F(t, z, \lambda_n))$.

Here $\mathrm{proj}(\cdot; \overline{\mathrm{conv}}\, F(t, z, \lambda))$ denotes the metric projection on the set $F(t, z, \lambda)$ for all $(t, z, \lambda) \in T \times H \times \Lambda$. Note that because of hypotheses $H(F)_1$ (1) and (2) and theorem 3.3 of [19] we know that $(t, x) \to F(t, x, \lambda_n)$ is measurable, so $t \to F(t, x(t), \lambda_n)$ is measurable thus $t \to \overline{\mathrm{conv}}\ F(t, x(t), \lambda_n)$ measurable $\Rightarrow t \to d(f(t), \overline{\mathrm{conv}}\ F(t, x(t), \lambda_n)) = \|f(t) - m(t, \lambda_n)\|$ is Borel measurable $\Rightarrow t \to m(t, \lambda_n)$ is Borel measurable. Similarly we can establish that $t \to u(t, z, \lambda_n)$ is Borel measurable, while from hypothesis $H(F)_1$ (2) and theorem 3.33, p. 322 of Attouch [1], we have that $z \to u(t, z, \lambda_n)$ $n \geq 1$ is continuous.

Let $x_n(\cdot) \in W^{1,2}(T, H)$ be a solution of the evolution inclusion

$$\left\{\begin{array}{c} -\dot{x}_n(t) \in \partial\varphi(t, x_n(t)) + u(t, x_n(t), \lambda_n)\, a.e. \\ x_n(0) = v(\lambda_n) \end{array}\right\}$$

(see theorem 3.1 of Papageorgiou [23]). Exploiting once again the monotonicity of the subdifferential operator, we get

$$(-\dot{x}_n(t) + \dot{y}_n(t), y_n(t) - x_n(t)) \leq (u(t, x_n(t), \lambda_n) - f_n(t), y_n(t) - x_n(t))\, a.e.$$

$$\text{thus} \quad \frac{1}{2}\|x_n(t) - y_n(t)\|^2 \leq \frac{1}{2}\|v(\lambda_n) - v(\lambda)\|^2 + \int_0^t \|u(s, x_n(s), \lambda_n) - f_n(s)\| \cdot \|y_n(s) - x_n(s)\|\, ds$$

$$\text{hence} \quad \|x_n(t) - y_n(t)\| \leq \|v(\lambda_n) - v(\lambda)\| + \int_0^t \|u(s, x_n(s), \lambda_n) - f_n(s)\|\, ds$$

(see lemma A.5, p. 157 of Brezis [5]).

Note that

$$\begin{aligned}
&\int_0^t \|u(s,x_n(s),\lambda_n)-f_n(s)\|\,ds\\
&\le \int_0^t \|u(s,x_n(s),\lambda_n)-u(s,x(s),\lambda_n)\|ds+\int_0^t \|u(s,x(s),\lambda_n)-f_n(s)\|\,ds\\
&\le \int_0^t h(F(s,x_n(s),\lambda_n),F(s,x(s),\lambda_n))ds+\int_0^t \|u(s,x(s),\lambda_n)-f(s)\|ds\\
&+\int_0^t \|f(s)-f_n(s)\|\,ds\\
&\le \int_0^t k_B(s)\|x_n(s)-x(s)\|ds+\int_0^t \|u(s,x(s),\lambda_n)-f(s)\|ds+\int_0^t \|f(s)-f_n(s)\|\,ds\\
&\le \int_0^t k_B(s)\|x_n(s)-y_n(s)\|ds+\int_0^b k_B(s)\|y_n(s)-x(s)\|ds\\
&+\int_0^b \|u(s,x(s),\lambda_n)-f(s)\|\,ds+\int_0^b \|f(s)-f_n(s)\|\,ds, \quad \text{with} \quad B=\{\lambda_n,\lambda\}_{n\ge 1}\,.
\end{aligned}$$

We know that

$$\begin{aligned}
&\int_0^b k_B(s)\|y_n(s)-x(s)\|\,ds \to 0 \text{ as } n\to\infty\,,\\
&\int_0^b \|f(s)-f_n(s)\|\,ds \to 0 \text{ as } n\to\infty
\end{aligned}$$

and $\int\limits_0^b \|u(s,x(s),\lambda_n)-f(s)\|\,ds=\int\limits_0^b d(f(s),\overline{\text{conv}}\,F(s,x(s),\lambda_n))ds$. Because of hypothesis $H(F)_1$ (4), we know that $\lambda\to d(f(s),\overline{\text{conv}}\,F(s,x(s),\lambda))$ is *u.s.c.* So from Fatou's lemma we have

$$\begin{aligned}
\overline{\lim}\int_0^b \|u(s,x(s),\lambda_n)-f(s)\|\,ds &\le \int_0^b \overline{\lim}\, d(f(s)\,,\overline{\text{conv}}\,F(s,x(s),\lambda_n))\,ds\\
&\le \int_0^b d(f(s),\overline{\text{conv}}\,F(s,x(s),\lambda))\,ds=0\,.
\end{aligned}$$

So given $\epsilon>0$, we can find $n_0(\epsilon)\ge 1$ such that for $n\ge n_0(\epsilon)$, we have

$$\|x_n(t)-y_n(t)\|\le \epsilon+\int_0^t k_B(s)\|x_n(s)-y_n(s)\|\,ds,\quad t\in T$$

$$\text{thus } \|x_n(t)-y_n(t)\|\le \epsilon\exp\|k_B\|_1 \quad \text{for all} \quad t\in T \text{ and all } n\ge n_0(\epsilon)$$

$$\text{hence} \quad x_n\to x \text{ in } C(T,H)\,.$$

Also note that

$$\begin{aligned} &\int_0^b \|u(t, x_n(t), \lambda_n) - f(t)\|^2\, dt \\ \le 2 &\int_0^b \|u(t, x_n(t), \lambda_n) - u(t, x(t), \lambda_n)\|^2\, dt + 2\int_0^b \|u(t, x(t), \lambda_n) - f(t)\|^2\, dt \\ &\le 2\int_0^b k_B(t)\|x_n(t) - x(t)\|^2\, dt + 2\int_0^b \|u(t, x(t), \lambda_n) - f(t)\|^2\, dt \to 0 \\ &\qquad \text{as} \quad n \to \infty\,. \end{aligned}$$

Set $h_n(\cdot) = u(\cdot, x_n(\cdot), \lambda_n)$. Then $[x_n, h_n] \in P_r(\lambda_n)$ and we have just seen that $[x_n, h_n] \to [x, h]$ in $C(T, H) \times L^2(T, H)$. Therefore we have

$$\begin{aligned} &P_r(\lambda) \subseteq \underline{\lim}\, P_r(\lambda_n) \text{ in } C(T, H) \times L^2(T, H) \\ &\quad \text{and so} \quad \lambda \to P_r(\lambda) \quad \text{is } l.s.c. \end{aligned}$$

□

From the above proof we also get:

Theorem 3.2. *If hypotheses $H(\varphi)$, $H(F)_1$ and H_0 hold, then $\lambda \to S_r(\lambda)$ is l.s.c. from Λ into $P_k(C(T, H))$.*

Since for every $\lambda \in \Lambda$, $S_r(\lambda) = \overline{S(\lambda)}$, the closure taken in $C(T, H)$ (see [23], theorem 5.1), we also have:

Theorem 3.3. *If hypotheses $H(\varphi)$, $H(F)_1$ and H_0 hold, then $\lambda \to S(\lambda)$ is l.s.c. from Λ into $2^{C(T,H)} \setminus \{\emptyset\}$.*

Proof. We know that $S_r(\lambda) = \overline{S(\lambda)}$ and that $\lambda \to S_r(\lambda)$ is *l.s.c.* (see theorem 3.2). Then the lower semicontinuity of $\lambda \to S(\lambda)$ follows from proposition 7.3.3, p. 85 of Klein-Thompson [16]. □

Remark. Our results extend theorem 1 of Colombo et al. [6] and theorem 4.2 of Zhu [32], who considered differential inclusions in separable Banach spaces, but with no subdifferential operators present. Furthermore, here we have a continuous dependence result for the multifunction $\lambda \to P_r(\lambda)$ too. Note that Colombo et al. [6] assume that $(t, x, \lambda) \to F(t, x, \lambda)$ is measurable. However it is enough to assume that $t \to F(t, x, \lambda)$ is measurable. Then because of the h-continuity of $x \to F(t, x, \lambda)$ (see hypothesis $H(F)_1$ (2) and theorem 3.3 of Papageorgiou [22], we know that $(t, x) \to F(t, x, \lambda)$ is jointly measurable. Furthermore, we should point out that in both [6] and [32], it is assumed that $\lambda \to F(t, x, \lambda)$ is *l.s.c.*, while here we only require that $\lambda \to \overline{\text{conv}}\, F(t, x, \lambda)$ is *l.s.c.* From Klein-Thompson [16] (theorem 7.2.7, p. 82), we know that our hypothesis is less restrictive. Finally, additional continuous dependence results for evolution inclusions in which the parameter appears in the subdifferential operator too, can be found in [25].

4. Filippov-Gronwall inequality

In this section, we prove a continuous version of the "Filippov-Gronwall" estimate. With the help of this result, in section 5, we will prove the desired continuous version of the relaxation theorem. Our result extends theorem 2 of Colombo et al. [6], which is about differential inclusions in Banach spaces with no subdifferential operator present and theorem 1.2 of Frankowska [12] and theorem 4.1 of Papageorgiou [26], which deal with semilinear evolution inclusions and no parameter $\lambda \in \Lambda$ is present.

We will need the following hypothesis on the orientor field $F(t,x,\lambda)$:

$\underline{H(F)_2:}$ $F : T \times H \times \Lambda \to P_f(H)$ is a multifunction such that

(1) $t \to F(t,x,\lambda)$ is measurable,

(2) $h(F(t,x,\lambda), F(t,y,\lambda)) \leq k(t)\|x-y\|$ *a.e.* for all $(t,\lambda) \in T \times \Lambda$ and with $k(\cdot) \in L^1_+$,

(3) $|F(t,x,\lambda)| \leq a(t) + c(t)\|x\|$ *a.e.* for all $\lambda \in \Lambda$, with $a, c \in L^2_+$,

(4) $\lambda \to F(t,x,\lambda)$ is *l.s.c.*

Note that if $\lambda \to [y(\lambda), g(\lambda)]$ is a continuous map from Λ into $C(T,H) \times L^2(H)$, then there exists $p_g : \Lambda \to L^1_+$ a continuous map such that for all $\lambda \in \Lambda$

$$d(g(\lambda)(t), F(t, y(\lambda)(t), \lambda)) \leq p_g(\lambda)(t) \, a.e. \text{ on } T. \tag{3}$$

For example, we can take $p_g(\lambda)(t) = \|g(\lambda)(t)\| + a(t) + c(t)\|y(\lambda)(t)\|$.

In what follows, given $h \in L^2(H)$, by $q_\lambda(h)(\cdot) \in C(T,H)$, we will denote the unique strong solution of

$$\left\{ \begin{array}{c} -\dot{z}(t) \in \partial\varphi(t, z(t)) + h(t) \quad a.e. \text{ on } T \\ z(0) = v(\lambda) \end{array} \right\}$$

Theorem 4.1. *If hypotheses $H(\varphi)$, $H(F)_2$, H_0 hold, $\lambda \to [y(\lambda), g(\lambda)]$ is a continuous map from Λ into $C(T,H) \times L^2(T,H)$, with $y(\lambda) = q_\lambda(g(\lambda))$, $\epsilon > 0$ and $p_g : \Lambda \to L^1_+$ is a continuous map satisfying (3), then there exists $\lambda \to [x(\lambda), r(\lambda)]$ a continuous map from Λ into $C(T,H) \times L^2(T,H)$ with $[x(\lambda), r(\lambda)] \in P(\lambda)$ and for all $\lambda \in \Lambda$ we have*

$$\|x(\lambda)(t) - y(\lambda)(t)\| \leq b\epsilon e^{\theta(t)} + \int_0^t p_g(\lambda)(s) \exp[\theta(t) - \theta(s)]\, ds, \quad t \in T$$

$$\text{with} \quad \theta(t) = \int_0^t k(s)\, ds\,.$$

Proof. Let $\Gamma_0(\lambda)(t) = \{v \in F(t, y(\lambda)(t), \lambda) : \|v - g(\lambda)(t)\| < p_g(\lambda)(t) + \epsilon\}$. Clearly $\Gamma_0(\lambda)\ (t) \neq \emptyset$ *a.e.* and by modifying it on a Lebesgue null set, we may assume without any loss of generality that $\Gamma_0(\lambda)(t) \neq \emptyset$ for all $t \in T$. Because of hypotheses

$H(F)_2$ (1) and (2) and theorem 3.3. of [22], we know that $(t,x) \to F(t,x,\lambda)$ is measurable. So $\operatorname{Gr}\Gamma_0(\cdot) \in \mathcal{L}(T) \times B(H)$, with $\mathcal{L}(T)$ being the Lebesgue σ-field of T and $B(H)$ the Borel σ-field of H. Applying Aumann's selection theorem, we deduce that there exists $z : T \to H$ a Lebesgue measurable selector of $\Gamma_0(\lambda)(\cdot)$. Then define $R_0 : \Lambda \to 2^{L^1(T,H)}$ by

$$R_0(\lambda) = \left\{ z \in S^1_{F(\cdot,y(\lambda)(\cdot),\lambda)} : \|z(t) - g(\lambda)(t)\| < p_g(\lambda)(t) + \epsilon \, a.e. \right\} .$$

We have just seen that for all $\lambda \in \Lambda$, $R_0(\lambda) \neq \emptyset$. Furthermore, from proposition 4 of Bressan-Colombo [4], we know that $\lambda \to R_0(\lambda)$ is *l.s.c.* with decomposable values. Hence $\lambda \to \overline{R_0(\lambda)}$ is *l.s.c.* with decomposable values. Then apply theorem 3 of Bressan-Colombo [4], to get $r_0 : \Lambda \to L^1(T,H)$ a continuous map such that $r_0(\lambda) \in R_0(\lambda)$ for all $\lambda \in \Lambda$. Then we have $\|g(\lambda)(t) - r_0(\lambda)(t)\| \leq p_g(\lambda)(t) + \epsilon$ *a.e.* for all $\lambda \in \Lambda$. Let $x_1(\lambda)(\cdot) \in W^{1,2}(T,H)$ be the unique strong solution of

$$\left\{ \begin{array}{c} -\dot{x}(t) \in \partial\varphi(t,x(t)) + r_0(\lambda)(t) \, a.e. \\ x(0) = v(\lambda) \end{array} \right\}$$

We claim that by induction, we get two sequences $\{x_n(\lambda)(\cdot)\}_{n\geq 1} \subseteq W^{1,2}(T,H)$ and $\{r_n(\lambda)(\cdot)\}_{n\geq 0} \subseteq L^2(T,H)$ satisfying

(i) $x_n(\lambda) = q_\lambda(r_{n-1}(\lambda))$,

(ii) $\lambda \to x_n(\lambda)$ is continuous from Λ into $C(T,H)$ and $\lambda \to r_n(\lambda)$ is continuous from Λ into $L^2(T,H)$,

(iii) $r_n(\lambda)(t) \in F(t, x_n(t), \lambda)$ *a.e.* ,

(iv) $\|r_n(\lambda)(t) - r_{n-1}(\lambda)(t)\| \leq k(t)\beta_n(\lambda)(t)$ *a.e.*

where $\beta_n(\lambda)(t) = \int\limits_0^t p_g(\lambda)(s)\frac{(\theta(t)-\theta(s))^{n-1}}{(n-1)!}\,ds + b\left(\sum\limits_{k=0}^n \frac{\epsilon}{2^{k+1}}\right)\frac{\theta(t)^{n-1}}{(n-1)!}$, with $\theta(t) = \int\limits_0^t k(s)\,ds$.

Suppose we were able to produce $\{x_k(\lambda)(\cdot)\}_{k=1}^n$ and $\{r_k(\lambda)(\cdot)\}_{k=0}^n$ satisfying properties (i) $\to$ (iv) above. Let $x_{n+1}(\lambda) = q_\lambda(r_n(\lambda))$. As before, because of the monotonicity of the subdifferential operator, we get

$$\begin{aligned}
&\|x_{n+1}(\lambda)(t) - x_n(\lambda)(t)\| \leq \int_0^t \|r_n(\lambda)(s) - r_{n-1}(\lambda)(s)\| \leq \int_0^t k(s)\beta_n(\lambda)(s)\,ds \\
&\leq \int_0^t k(s)\int_0^s p_g(\lambda)(\tau)\frac{(\theta(s)-\theta(\tau))^{n-1}}{(n-1)!}\,d\tau\,ds + b\left(\sum_{k=0}^n \frac{\epsilon}{2^{k+1}}\right)\int_0^t k(s)\frac{\theta(s)^{n-1}}{(n-1)!}\,ds \\
&\leq \int_0^t p_g(\lambda)(s)\int_s^t k(\tau)\frac{(\theta(\tau)-\theta(s))^{n-1}}{(n-1)!}\,d\tau\,ds + b\left(\sum_{k=0}^n \frac{\epsilon}{2^{k+1}}\right)\frac{\theta(t)^n}{n!} \\
&= \int_0^t p_g(\lambda)(s)\int_s^t \frac{d}{d\tau}\left(\frac{(\theta(\tau)-\theta(s))^n}{n!}\right)d\tau\,ds + b\left(\sum_{k=0}^n \frac{\epsilon}{2^{k+1}}\right)\frac{\theta(t)^n}{n!} \\
&= \int_0^t p_g(\lambda)(s)\frac{(\theta(t)-\theta(s))^n}{n!}\,ds + b\left(\sum_{k=0}^n \frac{\epsilon}{2^{k+1}}\right)\frac{\theta(t)^n}{n!} < \beta_{n+1}(\lambda)(t)\,a.e.
\end{aligned}$$

Also from hypothesis $H(F)_2$ (2), we have

$$d(r_n(\lambda)(t), F(t, x_{n+1}(\lambda)(t), \lambda)) \leq k(t)\|x_n(\lambda)(t) - x_{n+1}(\lambda)(t)\| < k(t)\beta_{n+1}(\lambda)(t) \quad a.e.$$

Define $R_{n+1} : \Lambda \to 2^{L^1(H)}$ by

$$R_{n+1}(\lambda) = \left\{ z \in S^1_{F(\cdot, x_{n+1}(\lambda)(\cdot), \lambda)} : \|z(t) - r_n(\lambda)(t)\| < k(t)\beta_{n+1}(\lambda)(t) \quad a.e. \right\}$$

As we did with $R_0(\cdot)$, we can verify using Aumann's selection theorem that $R_{n+1}(\lambda) \neq \emptyset$ for all $\lambda \in \Lambda$, while proposition 4 of Bressan-Colombo [4], tells us that $\lambda \to R_{n+1}(\lambda)$ is $l.s.c.$ with decomposable values. Hence $\lambda \to \overline{R_{n+1}(\lambda)}$ is $l.s.c.$ with decomposable values. Therefore theorem 3 of Bressan-Colombo [4], gives us $r_{n+1} : \Lambda \to L^1(H)$ a continuous map such that $r_{n+1}(\lambda) \in \overline{R_{n+1}(\lambda)}$ for all $\lambda \in \Lambda$. So $r_{n+1}(\lambda)(t) \in F(t, x_{n+1}(\lambda)(t), \lambda)$ $a.e.$ and $\|r_{n+1}(\lambda)(t) - r_n(\lambda)(t)\| \leq k(t)\beta_{n+1}(\lambda)(t)$ $a.e.$ So by induction we have established the existence of two sequences $\{x_n(\lambda)(\cdot)\}_{n \geq 1}$ and $\{r_n(\lambda)(\cdot)\}_{n \geq 0}$ satisfying $(i) \to (iv)$ above. Then we have:

$$\begin{aligned}\int_0^b \|r_n(\lambda)(t) - r_{n-1}(\lambda)(t)\| dt &< \int_0^b k(t)\beta_n(\lambda)(t) dt < \beta_{n+1}(\lambda)(b) \\ &= \int_0^b \int_0^t p_g(\lambda)(s) \frac{(\theta(t) - \theta(s))^n}{n!} \, ds \, dt + b\left(\sum_{k=0}^{n+1} \frac{\epsilon}{2^{k+1}}\right) \frac{\theta(b)^n}{n!} \\ &\leq b\frac{\theta(b)^n}{n!} \|p_g(\lambda)\|_1 + b\epsilon \frac{\|\theta\|_\infty^n}{n!}.\end{aligned}$$

Note that since $\lambda \to p_g(\lambda)$ is continuous from Λ into $L^1_+ \Rightarrow \lambda \to \|p_g(\lambda)\|_1$ is continuous, hence locally bounded. Therefore from the above inequality, and since $\|x_{n+1}(\lambda) - x_n(\lambda)\|_\infty \leq \|r_n(\lambda) - r_{n-1}(\lambda)\|_1$, we get that $\{x_n(\lambda)(\cdot)\}_{n \geq 1} \subseteq C(T, H)$ and $\{r_n(\lambda)(\cdot)\}_{n \geq 0} \subseteq L^1(T, H)$ are Cauchy sequences, locally uniformly in $\lambda \in \Lambda$. So we get that

$$x_n(\lambda) \to x(\lambda) \text{ in } C(T, H)$$

and $r_n(\lambda) \to r(\lambda)$ in $L^1(T, H)$ as $n \to \infty$, locally uniformly in Λ. Therefore $\lambda \to x(\lambda)$ is continuous from Λ into $C(T, H)$ and $\lambda \to r(\lambda)$ is continuous from Λ into $L^1(T, H)$. Furthermore, because of hypothesis $H(F)_2$ (3), we actually have that $\lambda \to r(\lambda)$ is continuous from Λ into $L^2(T, H)$. In addition from hypothesis $H(F)_2$ (2) we get that $r(\lambda)(t) \in F(t, x(\lambda)(t), \lambda)$ $a.e.$

Let $w(\lambda)(\cdot) \in W^{1,2}(T, H)$ be defined by $w(\lambda) = q_\lambda(r(\lambda))$. As before, from the monotonicity of the subdifferential operator, we have

$$\begin{gathered}\|x_n(\lambda)(t) - w(\lambda)(t)\| \leq \int_0^b \|r_n(\lambda)(t) - r(\lambda)(t)\| \, dt \to 0 \quad \text{as } n \to \infty \\ \text{so} \quad x_n(\lambda) \to w(\lambda) \quad \text{in} \quad C(T, H) \quad \text{as} \quad n \to \infty \\ \text{thus} \quad w(\lambda) = x(\lambda)\end{gathered}$$

So we have that $\lambda \to [x(\lambda), r(\lambda)]$ is continuous from Λ into $C(T,H) \times L^2(T,H)$ and for all $\lambda \in \Lambda$ we have $[x(\lambda), r(\lambda)] \in P(\lambda)$.

Finally from the triangle inequality, we have

$$\|y(\lambda)(t) - x_n(\lambda)(t)\| \le \|y(\lambda)(t) - x_1(\lambda)(t)\| + \sum_{k=1}^{n-1} \|x_k(\lambda)(t) - x_{k+1}(\lambda)(t)\| .$$

Recall that

$$\begin{aligned}
\|x_k(\lambda)(t) - x_{k+1}(\lambda)(t)\| &\le \int_0^t \|r_k(\lambda)(s) - r_{k-1}(\lambda)(s)\| \, ds \\
&\le \int_0^t k(s)\beta_k(\lambda)(s)\, ds \\
\le \int_0^t k(s) \int_0^s p_g(\lambda)(s) \frac{(\theta(s)-\theta(\tau))^{k-1}}{(k-1)!}\, d\tau\, ds &+ b\epsilon \int_0^t k(s) \frac{\theta(s)^{k-1}}{(k-1)!}\, ds \\
\le \frac{1}{k!}\int_0^t p_g(\lambda)(\tau) \int_\tau^t \frac{d}{ds}(\theta(s)-\theta(\tau))^k\, ds\, d\tau &+ \frac{b\epsilon}{k!}\int_0^t \frac{d}{ds}\theta(s)^k\, ds \\
= \frac{1}{k!}\int_0^t p_g(\lambda)(s)(\theta(t)-\theta(s))^k\, ds &+ \frac{b\epsilon}{k!}\theta(t)^k .
\end{aligned}$$

Also $\|y(\lambda)(t) - x_1(\lambda)(t)\| \le \int_0^t \|g(\lambda)(s) - r_0(\lambda)(s)\| ds \le \int_0^t (p_g(\lambda)(s) + \epsilon)\, ds$.

Summing up with respect to $k \ge 0$ and passing to the limit as $n \to \infty$, we get

$$\|y(\lambda)(t) - x(\lambda)(t)\| \le b\epsilon \exp(\theta(t)) + \int_0^t p_g(\lambda)(s) \exp(\theta(t)-\theta(s))\, ds, \quad t \in T .$$

$\square$

Remark. It is easy to see from the above proof, that we also have

$$\|r(\lambda)(t) - g(\lambda)(t)\| \le b\epsilon \exp(\theta(t)) + \int_0^t p_g(\lambda)(s)\exp(\theta(t)-\theta(s))\, ds + p_g(\lambda)(t)\, a.e.$$

5. Parametric relaxation theorem

In this section we use the parametric "Filippov-Gronwall inequality" proved in section 4 (theorem 4.1), to establish a parametric version of the relaxation theorem. We will need the following stronger variant of hypothesis H_0 :

H_0' : $v : \Lambda \to dom\varphi(0,\cdot)$ is continuous and bounded, and $\sup_{\lambda\in\Lambda} \varphi(0, v(\lambda)) < \infty$.

First note that if $x(\cdot) \in S_r(\lambda)$, then if $z_\lambda = q_\lambda(0)$, we have

$$\|x(t) - z_\lambda(t)\| \le \int_0^t \|f(s)\|\, ds, \quad t \in T ,$$

where $f \in L^2(H), f(t) \in F(t, x(t), \lambda)$ *a.e.* So for every $t \in T$, we have

$$\|x(t)\| \leq \sup_{\lambda \in \Lambda} \|z_\lambda\|_\infty + \int_0^t (a(s) + c(s)\|x(s)\|)\, ds\,.$$

Note that because of hypothesis H_0' $\sup_{\lambda\in\Lambda} \|z_\lambda\|_\infty < \infty$ (see Yotsutani [31]).

Hence by Gronwall's inequality, we get that there exists $M_1 > 0$ such that for all $t \in T$ and all $x(\cdot) \in S(\lambda)$, $\lambda \in \Lambda$ we have $\|x(t)\| \leq M_1$. Then by considering $F(t, r_{M_1}(x), \lambda)$ instead of $F(t, x, \lambda)$ (here $r_{M_1} : H \to H$ denotes the M_1-radial retraction on H), we may assume without any loss of generality that $|F(t,x,\lambda)| \leq \psi(t)$ *a.e.* with $\psi(\cdot) \in L_+^2$ (in fact, we can have $\psi(t) = a(t) + c(t)M_1$). So in this section we will assume that $|F(t,x,\lambda)| \leq \psi(t)$ *a.e.* for all $(x,\lambda) \in H \times \Lambda$.

Also in the proof of theorem 5.1 below, we will need the following simple continuity result, concerning the solution map $q : L^2(H) \to C(T,H)$. Recall that $q(\cdot)$ assigns to every $h \in L^2(H)$ the unique strong solution $q(h)(\cdot) \in C(T,H)$ of the Cauchy problem

$$\left\{ \begin{array}{c} -\dot{z}(t) \in \partial\varphi(t, z(t)) + h(t) \\ z(0) = x_0 \in \overline{\operatorname{dom}\varphi(0,\cdot)} \end{array} \right\}.$$

In the sequel by $\|\cdot\|_w$ we will denote the (weak) norm on $L^1(T,H)$ defined by

$$\|h\|_w = \sup_{0\leq t\leq b} \|\int_0^t h(s)\, ds\|$$

Convergence in this norm will be denoted by $\xrightarrow{\|\cdot\|_w}$.

Lemma. *If hypothesis $H(\varphi)$ holds, $\{h_n, h\}_{n\geq 1} \subseteq L^2(T,H)$, $\|h_n(t)\|$, $\|h(t)\| \leq \psi(t)$ a.e. with $\psi(\cdot) \in L_+^2$ and $h_n \xrightarrow{\|\cdot\|_w} h$, then $q(h_n) \to q(h)$ in $C(T,H)$.*

Proof. First we will show that $h_n \xrightarrow{w} h$ in $L^2(T,H)$ Since step functions are dense in $L^2(H)$, it is enough to show that $(h_n, s)_{L^2(T,H)} \to (h,s)_{L^2(T,H)}$ for all $s(t) = \sum_{k=1}^{N} \chi_{(t_{k-1},t_k)}(t) v_k^*$ with $0 \leq t_{k-1} < t_k \leq b$ and $v_k^* \in H$ (here $(\cdot,\cdot)_{L^2(T,H)}$ stands for the inner product in $L^2(T,H)$). We have:

$$\begin{aligned} |(h_n - h, s)_{L^2(H)}| &= |\sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} (h_n(s) - h(s), v_k^*)\, ds| \\ &\leq \sum_{k=1}^{N} \|\int_{t_{k-1}}^{t_k} (h_n(s) - h(s))\, ds\| \|v_k^*\| \\ &\leq \|h_n - h\|_w \cdot \sum_{k=1}^{N} \|v_k^*\| \to 0 \quad \text{as } n \to \infty\,. \end{aligned}$$

So indeed $h_n \xrightarrow{w} h$ in $L^2(T,H)$.

Next since $\|h_n(t)\|, \|h(t)\| \leq \psi(t)$ *a.e.*, with $\psi \in L^2(T,H)$, invoking theorem 3.1 of Papageorgiou [23], we know that $\{q(h_n)(\cdot)\}_{n\geq 1} \subseteq C(T,H)$ is relatively compact. Hence we may assume that $q(h_n) \to v$ in $C(T,H)$ as $n \to \infty$. We have:

$$\frac{1}{2}\|q(h_n)(t) - q(h)(t)\|^2 \leq \int_0^t (h_n(s) - h(s), q(h_n)(s) - q(h)(s))\, ds$$

$$= \int_0^t (h_n(s) - h(s), q(h_n)(s) - v(s))\, ds + \int_0^t (h_n(s) - h(s), v(s) - q(h)(s))\, ds$$

$$\leq \int_0^t 2\psi(s)\|q(h_n)(s) - v(s)\|\, ds + \int_0^t (h_n(s) - h(s), v(s) - q(h)(s))ds \to 0$$

$$\text{as} \quad n \to \infty \quad \text{and so} \quad q(h_n) \to q(h) \quad \text{in} \quad C(T,H) \quad \text{as} \quad n \to \infty\,.$$

$\square$

Now we can state and prove our parametric relaxation theorem. For this, we need to assume that Λ is also separable (i.e. Λ is a Polish space).

Theorem 5.1. *If hypotheses $H(\varphi)$, $H(F)_2$, H_0 hold, $\lambda \to [y(\lambda), g(\lambda)]$ is a continuous selector of the multifunction $\lambda \to P_r(\lambda)$ and $\epsilon > 0$, then there exists $\lambda \to x(\lambda)$ a continuous map from Λ into $C(T,H)$ such that for all $\lambda \in \Lambda$, $x(\lambda) \in S(\lambda)$ and $\|x(\lambda) - y(\lambda)\|_\infty < \epsilon$.*

Proof. From the lemma we know that we can find $\delta > 0$ such that if $h \in L^2(H)$, $\|h(t)\| \leq \psi(t)$ *a.e.* and $\|g(\lambda) - h\|_w \leq \delta$, then $\|y(\lambda) - q_\lambda(h)\|_\infty \leq \frac{\epsilon}{4M\widehat{b}}$ where $M = \exp(\theta(b))$ and $\widehat{b} = \max[b, 1]$. $\square$

Partition $T = [0,b]$ into intervals $T_k = [t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, N$ such that $\int\limits_{T_k} \psi(s)ds < \frac{\delta}{4}$ (it can be done because of the absolute continuity of the Lebesgue integral). Let $\eta(\lambda)(t) = \int\limits_0^t g(\lambda)(s)\, ds$. Then $\eta(\lambda)(\cdot) \in C(T,H)$ and by hypothesis $\lambda \to \eta(\lambda)(t_{k+1}) - \eta(\lambda)(t_k)$ is a continuous selector of the parametric Aumann (set-valued) integral $\int\limits_{T_k} \overline{\text{conv}} F(t, y(\lambda)(t), \lambda)\, dt$. From corollary 4.3 of Hiai-Umegaki [14], we know that $cl \int\limits_{T_k} \overline{\text{conv}} F(t, y(\lambda)(t), \lambda) dt = cl \int\limits_{T_k} F(t, y(\lambda)(t), \lambda) dt$, while from the corollary on p. 188 of Papageorgiou [18], we know that $cl \int\limits_{T_k} \overline{\text{conv}} F(t, y(\lambda)(t), \lambda) dt = \int\limits_{T_k} \overline{\text{conv}} F(t, y(\lambda)(t), \lambda) dt \in P_{wkc}(H)$. Consider the multifunction $R_k : \Lambda \to P_f(L^1(T_k, H))$ defined by $R_k(\lambda) = S^1_{F(\cdot, y(\lambda)(\cdot), \lambda)}$. From theorem 4.1 of [20], we know that $R_k(\cdot)$ is *l.s.c.* and clearly has decomposable values. So apply theorem 1 of Fryszkowski-Rzezuchowski [13] and get $r_k : \Lambda \to L^1(T_k, H)$ $k \in \{0, 1, \ldots, N\}$ continuous maps such that for all $\lambda \in \Lambda$, $r_k(\lambda) \in R_k(\lambda)$ and $\|\int\limits_{t_k}^{t_{k+1}} g(\lambda)(t)\, dt - \int\limits_{t_k}^{t_{k+1}} r_k(\lambda)(t)\, dt\| < \frac{\delta}{2N}$. Let $f(\lambda)(\cdot) = \sum\limits_{k=0}^{N} \chi_{T_k}(\cdot) r_k(\lambda)(\cdot) \in L^2(T,H)$, $\|f(\lambda)(t)\| \leq$

$\psi(t)$ *a.e.* and set $z(\lambda) = q_\lambda(f(\lambda))$. We claim that $\|g(\lambda) - f(\lambda)\|_w < \delta$. Indeed by definition, we have

$$\|g(\lambda) - f(\lambda)\|_w = \sup\left[\|\int_0^t (g(\lambda)(s) - f(\lambda)(s))\,ds\|, t \in T\right].$$

Let $t \in T_m$ for some $m \in \{0, 1, \dots, N\}$. We have

$$\begin{aligned}
&\|\int_0^t (g(\lambda)(s) - f(\lambda)(s))ds\| \\
&= \|\int_{\cup_{k=0}^{m-1} T_k} (g(\lambda)(s) - f(\lambda)(s))\,ds\| + \|\int_{t_m}^t (g(\lambda)(s) - f(\lambda)(s))\,ds\| \\
&< \frac{\delta}{2} + \int_{t_m}^t 2\psi(s)\,ds < \frac{\delta}{2} + \frac{\delta}{2} = \delta\,.
\end{aligned}$$

Since $t \in T$ was arbitrary, we conclude that

$$\|g(\lambda) - f(\lambda)\|_w \le \delta\,.$$

Therefore $\|y(\lambda) - z(\lambda)\|_\infty \le \frac{\epsilon}{4M\hat{b}}$. Note that

$$\begin{aligned}
&f(\lambda)(t) \in F(t, y(\lambda)(t), \lambda) \quad a.e. \\
\text{so} \quad &d(f(\lambda)(t), F(t, z(\lambda)(t), \lambda)) \le k(t)\frac{\epsilon}{4M\hat{b}}\,.
\end{aligned}$$

Apply theorem 4.1 (the "Filippov-Gronwall" inequality), with $p_f(\lambda)(\cdot) = = k(\cdot)\frac{\epsilon}{4M\hat{b}}$. Then according to that result, we get a continuous map $\lambda \to x(\lambda)$ from Λ into $C(T, H)$ such that for all $\lambda \in \Lambda$, $x(\lambda) \in S(\lambda)$ and

$$\begin{aligned}
\|x(\lambda) - z(\lambda)\|_\infty &\le b\frac{\epsilon}{4M\hat{b}}M + \frac{\epsilon}{4M\hat{b}}M\int_0^t k(s)\exp(-\theta(s))\,ds \\
&\le \frac{\epsilon}{4} + \frac{\epsilon}{4}(1 - e^{-\theta(t)})(\quad \text{since} \quad \hat{b} \ge 1) \\
&\le \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}\,.
\end{aligned}$$

Therefore, finally we have

$$\begin{aligned}
\|x(\lambda) - y(\lambda)\|_\infty &\le \|x(\lambda) - z(\lambda)\|_\infty + \|z(\lambda) - y(\lambda)\|_\infty \\
&\le \frac{\epsilon}{4M\hat{b}} + \frac{\epsilon}{2} \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.
\end{aligned}$$

□

6. An application to control systems

In this section we illustrate the abstract results obtained in this paper, with an application to parabolic distributed parameter control systems. Specifically we will prove a parametric version of the "bang-bang principle".

So let $T = [0, b]$ and $Z \subseteq \mathbb{R}^N$ a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial Z = \Gamma$. We consider the following parametric control system:

$$(4) \qquad \left\{ \begin{array}{l} \dfrac{\partial x}{\partial t} - a(t) \displaystyle\sum_{k=1}^{N} D_k(|D_k x|^{p-2} D_k x) + x|x|^{p-2} \\ = f(t, z, x(t, z), \lambda) u(t, z) \quad a.e. \text{ on } \quad T \times Z \\ x|_{T\times\Gamma=0, x(0,z)=v(z,\lambda), u(t,z)\in U(t,z) \quad a.e. \ , p\geq 2.} \end{array} \right\}$$

In conjunction with (4) above, we also consider its "convexified" version

$$(5) \qquad \left\{ \begin{array}{l} \dfrac{\partial x}{\partial t} - a(t) \displaystyle\sum_{k=1}^{N} D_k(|D_k x|^{p-2} D_k x) + x|x|^{p-2} \\ = f(t, z, x(t, z), \lambda) u(t, z) \quad a.e. \text{ on } T \times Z \\ x|_{T\times\Gamma=0, x(0,z)=v(z,\lambda), u(t,z)\in \overline{\text{conv}} U(t,z) \quad a.e. \ , p\geq 2.} \end{array} \right\}$$

Note that as always $D_k = \frac{\partial}{\partial z_k}$, $k = 1, 2, \ldots, N$.

We will need the following hypotheses on the data:

$\underline{H(a)}$:$0 < c \leq a(t)$ and $|a(t') - a(t)| \leq \ell |t' - t|$ with $\ell > 0$

$\underline{H(f)}$:$f : T \times Z \times \mathbb{R} \times \Lambda \to \mathbb{R}$ is a function such that

(1) $(t, z) \to f(t, z, x, \lambda)$ is measurable,

(2) $|f(t, z, x, \lambda) - f(t, z, x', \lambda)| \leq k(t, z)|x - x'|$ $a.e.$ for all $\lambda \in \Lambda$, with $k(\cdot, \cdot) \in L^1(T \times Z)$,

(3) $|f(t, z, x, \lambda)| \leq a(t, z) + c(t, z)|x|$ $a.e.$ for all $\lambda \in \Lambda$, with $a \in L^2(T \times Z)$, $c \in L^\infty(T \times Z)$,

(4) $\lambda \to f(t, z, x, \lambda)$ is continuous.

$\underline{H(U)}$:$U : T \times Z \to P_f(\mathbb{R})$ is a measurable multifunction such that $|U(t, z)| \leq M$ for all $(t, z) \in T \times Z$.

$\underline{H(v)}$:$\lambda \to v(\cdot, \lambda)$ is a continuous and bounded map from Λ into $L^2(Z)$ with $v(\cdot, \lambda) \in W_0^{1,p}(Z)$ for all $\lambda \in \Lambda$.

As in section 5, we assume that Λ is a Polish space.

Theorem 6.1. *If hypotheses $H(a)$, $H(f)$, $H(U)$, $H(v)$ hold, $\lambda \to y(\lambda)$ is a continuous map from Λ into $C(T, L^2(Z))$, for every $\lambda \in \Lambda$, $y(\lambda)$ is a solution of (5) and $\epsilon > 0$, then there exists a map $\lambda \to x(\lambda)$ which is continuous from Λ into $C(T, L^2(Z))$, such that for every $\lambda \in \Lambda$, $x(\lambda)$ is a solution of (4) and*

$$\sup_{t\in T} \int_Z |x(\lambda)(t, z) - y(\lambda)(t, z)|^2 \, dz < \epsilon \, .$$

Proof. Let $H = L^2(Z)$ and let $\varphi : T \times H \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be defined by

$$\varphi(t,x) = \begin{cases} a(t)\frac{1}{p}\int_Z \sum_{k=1}^N D_k x(z)|^p dz + \frac{1}{p}\int_Z |x(z)|^p\, dz & \text{if} \quad x \in W_0^{1,p}(Z) \\ +\infty & \text{otherwise.} \end{cases}$$

Note that for all $t \in T$, $\varphi(t,\cdot) \in \Gamma_0(H)$ and *dom* $\varphi(t,\cdot) = W_0^{1,p}(Z)$. Since $W_0^{1,p}(Z)$ embeds compactly into $L^2(Z)$, we get that $\varphi(t,\cdot)$ is of compact type. Also using hypothesis $H(a)$ and the fact that $\|x\| = (\int\limits_Z \sum\limits_{k=1}^N |D_k x(z)|^p dz)^{1/p}$ is an equivalent norm on $W_0^{1,p}(Z)$, we easily check that hypothesis $H(\varphi)$ is satisfied. Furthermore using Green's identity we can see that

$$\partial\varphi(t,x) = -a(t)\Delta_p x + x|x|^{p-2}$$

with $\Delta_p x = -\sum\limits_{k=1}^N D_k(|D_k x|^{p-2} D_k x)$ (pseudo-Laplacian) and dom$\Delta_p = D(\Delta_p) = \{y \in W_0^{1,p}(Z) : \Delta_p y \in L^2(Z)\}$.

Next let $F : T \times H \times \Lambda \to P_f(H)$ be defined by

$$F(t,x,\lambda) = \{v \in L^2(Z) : v(z) = f(t,z,x(z),\lambda)u(z), u(z) \in U(t,z) \ a.e. \quad \text{on} \quad Z\}$$

Let $v \in H = L^2(Z)$. Then using theorem 2.2 of Hiai-Umegaki [14], we get

$$\begin{aligned} &d(v, F(t,x,\lambda)) \\ &= \inf\Big[\int_Z |v(z) - f(t,z,x(z),\lambda)u(z)|dz : u \in L^2(Z), u(z) \in U(t,z) a.e.\Big] \\ &= \int_Z \inf\Big[|v(z) - f(t,z,x(z),\lambda)u| : u \in U(t,z)\Big]\, dz\,. \end{aligned}$$

Because of hypothesis $H(U)$, we can find a sequence of measurable functions $u_n : T \times Z \to \mathbb{R}$ such that $U(t,z) = \overline{\{u_n(t,z)\}}_{n\geq 1}$ for all $(t,z) \in T \times Z$ (see Wagner [28], theorem 4.2). Hence we have

$$\begin{aligned} &\inf\left[|v(z) - f(t,z,x(z),\lambda)u| : u \in U(t,z)\right] \\ &\quad = \inf_{n\geq 1} |v(z) - f(t,z,x(z),\lambda)u_n(t,z)| \\ &\quad \Rightarrow t \to d(v, F(t,x,\lambda)) \quad \text{is measurable} \\ &\quad \Rightarrow t \to F(t,x,\lambda) \quad \text{is a measurable multifunction.} \end{aligned}$$

Also let $\widehat{f} : T \times L^2(Z) \times \Lambda \to L^2(Z)$ be the Nemitsky (superposition) operator corresponding to the function f; i.e. $\widehat{f}(t,x,\lambda)(\cdot) = f(t,\cdot,x(\cdot),\lambda)$. We have

$$\begin{aligned} h(F(t,x,\lambda), F(t,x',\lambda)) &= h(\widehat{f}(t,x,\lambda)S^1_{U(t,\cdot)}, \widehat{f}(t,x',\lambda)S^1_{U(t,\cdot)}) \\ &\leq \|\widehat{f}(t,x,\lambda) - \widehat{f}(t,x',\lambda)\|_2 M \\ &\leq \widehat{k}(t) M \|x - x'\|_2 \end{aligned}$$

with $\widehat{k}(t) = \|k(t,\cdot)\|_1 \in L^1_+$.

Furthermore, note that

$$h(F(t,x,\lambda), F(t,x,\lambda')) \leq M\|\widehat{f}(t,x,\lambda) - \widehat{f}(t,x,\lambda')\|_2$$

$\Rightarrow \lambda \to F(t,x,\lambda)$ is h-continuous (see hypothesis $H(f)(4)$), a fortiori then $l.s.c.$

Finally, note that because of hypotheses $H(f)$ (3) and $H(U)$, we have that

$$|F(t,x,\lambda)| \leq \widehat{a}(t) + \widehat{c}\|x\|_2 \ ae.$$

with $\widehat{a} \in L^2_+$, $\widehat{c} > 0$.

Observe that $\overline{\text{conv}}F(t,x,\lambda) = \overline{\text{conv}}\widehat{f}(t,x,\lambda)S^1_{U(t,\cdot)} = \widehat{f}(t,x,\lambda)\overline{\text{conv}}S^1_{U(t,\cdot)} = \widehat{f}(t,x,\lambda)$, $S^1_{\overline{\text{conv}}U(t,\cdot)}$ (see Hiai-Umegaki [14]). So we can rewrite systems (4) and (5) in the following equivalent subdifferential inclusion forms:

$$(4') \qquad \left\{ \begin{array}{c} -\dot{x}(t) \in \partial\varphi(t,x(t)) + F(t,x(t),\lambda) \quad a.e. \\ x(0) = \widehat{v}(\lambda) \end{array} \right\}$$

and

$$(5') \qquad \left\{ \begin{array}{c} -\dot{x}(t) \in \partial\varphi(t,x(t)) + \overline{\text{conv}}F(t,x(t),\lambda) \quad a.e. \\ x(0) = \widehat{v}(\lambda) \end{array} \right\}$$

Here $\widehat{v}(\lambda) = v(\cdot,\lambda) \in W^{1,p}_0(Z) = \text{dom}\,\varphi(0,\cdot)$ and $\lambda \to \widehat{v}(\lambda)$ is continuous from Λ into $L^2(Z)$ (see hypothesis $H(v)$). So we have satisfied hypothesis H'_0.

Let $u(t,z)$ be the control generating the state $y(\lambda)(\cdot,\cdot)$. Clearly then $\lambda \to [y(\lambda)(\cdot),\ \widehat{f}(\cdot,y(\lambda)(\cdot))\widehat{u}(\cdot)]$ $(\widehat{u}(t)(\cdot) = u(t,\cdot))$ is a continuous map from Λ into $C(T,H) \times L^2(H)$ such that for all $\lambda \in \Lambda$, the pair belongs in $P_r(\lambda)$ for problem $(5')$. Apply theorem 5.1 to get $\lambda \to x(\lambda)$ continuous from Λ into $C(T, L^2(Z))$ such that for all $\lambda \in \Lambda$, $x(\lambda)(\cdot,\cdot)$ solves $(4')$, hence (4) too, and $\|x(\lambda) - y(\lambda)\|^2_{C(T,H)} < \epsilon \Rightarrow \underline{t \in T} \to \sup \int_Z |x(\lambda)(t,z) - y(\lambda)(t,z)|^2 dz < \epsilon$. □

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