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NATURAL FUNCTIONS ON $T^*T^{(r)}$ AND T^*T^{r*}

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Abstract. We determine all natural functions on $T^*T^{(r)}$ and T^*T^{r*}.

All manifolds and maps are assumed to be infinitely differentiable.

1. Let $\mathcal{M}f_n$ be the category of n-dimensional manifolds and their local diffeomorphisms. Consider a natural bundle F over n-manifolds, [2].

Definition 1. A natural function g on F is a system of functions

$$g_M : FM \to \mathbf{R}$$

for every n-manifold M satisfying

$$g_M = g_N \circ Ff$$

for all $f : M \to N$ from $\mathcal{M}f_n$.

Example 1. Let us remark that for every vector bundle $E \to M$, $x \in M$ and $y \in E_x$ we have a natural linear isomorphism between E_x and $V_yE := T_yE_x$ given by

$$v \to \frac{d}{dt}|_{t=0}(y + tv) \ .$$

For any vector space W we have $< \ , \ >: W^* \times W \to \mathbf{R}$, $< a, v >= a(v)$.

Let $T^{(r)} = (J^r(., \mathbf{R})_0)^*$ be the linear r-th order tangent bundle functor and let $T^{r*} = J^r(., \mathbf{R})_0$ be the r-th order cotangent bundle functor, cf. [2]. For any n-manifold M and $s \in \{1, ..., r\}$ we define $\lambda_M^{<s>} : T^*T^{(r)}M \to \mathbf{R}$ by

$$\lambda_M^{<s>}(a) :=< (A^{<s>} \circ \pi)(a), q(a) > \ ,$$

where $q : T^*T^{(r)}M \to T^{(r)}M$ is the cotangent bundle projection,

$$A^{<s>} : (T^{(r)}M)^* \tilde{=} T^{r*}M \to T^{r*}M \tilde{=} (T^{(r)}M)^*$$

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is a fibre bundle morphism over id_M given by

$$A^{<s>}(j^r_x\gamma) := j^r_x(\gamma^s),\ \gamma : M \to \mathbf{R},\ \gamma(x) = 0,\ x \in M\ ,$$

and $\pi : T^*T^{(r)}M \to (T^{(r)}M)^*$ is a fibre bundle morphism over id_M given by

$$\pi(a) := a|V_{q(a)}T^{(r)}M \tilde{=} T^{(r)}_x M,\ a \in (T^*T^{(r)})_x M,\ x \in M\ .$$

Furthermore we define $\mu^{<s>}_M : T^*T^{r*}M \to \mathbf{R}$ by

$$\mu^{<s>}_M(a) :=< (A^{<s>} \circ q)(a), \overline{\pi}(a) >\ ,$$

where $q : T^*T^{r*}M \to T^{r*}M$ is the cotangent bundle projection, $A^{<s>} : T^{r*}M \to T^{r*}M$ is as above and $\overline{\pi} : T^*T^{r*}M \to (T^{r*}M)^*$ is a fibre bundle morphism over id_M given by

$$\overline{\pi}(a) := a|V_{q(a)}T^{r*}M \tilde{=} T^{r*}_x M,\ (a : T_{q(a)}T^{r*}M \to \mathbf{R}) \in (T^*T^{r*})_x M,\ x \in M\ .$$

Clearly, $\{\lambda^{<s>}_M\}$ is a natural function on $T^*T^{(r)}|\mathcal{M}f_n$ and $\{\mu^{<s>}_M\}$ is a natural function on $T^*T^{r*}|\mathcal{M}f_n$.

In [1], I. Kolář has described all natural functions on T^*F for F from a large class of natural bundles. The method presented in [1] can not be applied in the cases $F = T^{(r)}|\mathcal{M}f_n$ (if $r \geq 2$) and $F = T^{r*}|\mathcal{M}f_n$ because of the following reasons:
(a) If the assumptions (I), (II), (III) of [1] were satisfied for $F = T^{(r)}|\mathcal{M}f_n$, then using the results of [3] we could deduce that any natural function on $T^*T^{(r)}|\mathcal{M}f_n$ is of the form $f \circ \lambda^{<1>}_M$, where $f \in C^\infty(\mathbf{R}, \mathbf{R})$. This contradicts to Theorem 1.
(b) It follows from [4] that $F = T^{r*}|\mathcal{M}f_n$ do not satisfy Condition (I) of [1].

In this paper we determine all natural functions on $T^*T^{(r)}|\mathcal{M}f_n$ and $T^*T^{r*}|\mathcal{M}f_n$. We are going to prove

Theorem 1. *All natural functions on $T^*T^{(r)}|\mathcal{M}f_n$ are of the form*

$$\left\{ f \circ (\lambda^{<1>}_M, \ldots, \lambda^{<r>}_M) \right\}\ ,$$

where $f \in C^\infty(\mathbf{R}^r)$ is a smooth function of r variables.

Theorem 2. *All natural functions on $T^*T^{r*}|\mathcal{M}f_n$ are of the form*

$$\left\{ f \circ (\mu^{<1>}_M, \ldots, \mu^{<r>}_M) \right\}\ ,$$

where $f \in C^\infty(\mathbf{R}^r)$ is a smooth function of r variables.

In the case $r = 1$ both theorems are equivalent because of a natural isomorphism $T^*T \tilde{=} T^*T^*$, cf. [2].

2. The proofs of Theorems 1 and 2 will be given in Item 3. In this item we prove some lemmas.

Let $q, \pi, \overline{\pi}, \lambda^{<s>}_M$ and $\mu^{<s>}_M$ be as in Example 1. The usual coordinates on $\mathbf{R}^n$ are denoted by $x^1, \ldots, x^n$ and the canonical vector fields induced by $x^1, \ldots, x^n$ on $\mathbf{R}^n$ by $\partial_1, \ldots, \partial_n$. For any vector field X on M the complete lift of X to a natural bundle FM is denoted by FX.

It is clear that $T^{(r)}((x^1)^r\partial_1)$ and $T^{r*}((x^1)^r\partial_1)$ are vertical over 0. We start with the proof of the following lemma.

Lemma 1. *The sets*

$$\{y \in T_0^{(r)}\mathbf{R}^n :< T^{(r)}((x^1)^r\partial_1)(y), j_0^r(x^1) > \neq 0\}$$

and

$$\{y \in T_0^{(r)}\mathbf{R}^n :< T^{r*}((x^1)^r\partial_1)(j_0^r(x^1)), y > \neq 0\}$$

are dense in $T_0^{(r)}\mathbf{R}^n$, *provided the following identifications are used:*

$$j_0^r(x^1) \in T_0^{r*}\mathbf{R}^n \tilde{=} (V_y T^{(r)}\mathbf{R}^n)^* \text{ and}$$

$$(T_0^{(r)}\mathbf{R}^n)^* \tilde{=} V_{j_0^r(x^1)}T^{r*}\mathbf{R}^n$$

for any $y \in T_0^{(r)}\mathbf{R}^n$.

Proof. Let φ_t be the flow of $(x^1)^r\partial_1$ near 0. Then we have

$$\begin{aligned} < T^{(r)}((x^1)^r\partial_1)(y), j_0^r(x^1) > &=< \frac{d}{dt}|_{t=0} T_0^{(r)}\varphi_t(y), j_0^r(x^1) > \\ &= \frac{d}{dt} < T^{(r)}\varphi_t(y), j_0^r(x^1) > |_{t=0} \\ &= \frac{d}{dt} < y, j_0^r(x^1 \circ \varphi_t^{-1} > |_{t=0} \\ &=< y, j_0^r(\frac{\partial}{\partial t}(x^1 \circ \varphi_t^{-1})_{t=0}) > \\ &= - < y, j_0^r((x^1)^r) > \end{aligned}$$

and similarly

$$< T^{r*}((x^1)^r\partial_1)(j_0^r(x^1)), y >= - < y, j_0^r((x^1)^r) >$$

for any $y \in T_0^{(r)}\mathbf{R}^n$. This implies our lemma. □

Now we prove the following lemma.

Lemma 2. *Let* g, h *be natural functions on* $T^*T^{(r)}|\mathcal{M}f_n$ *(or on* $T^*T^{r*}|\mathcal{M}f_n$*). Suppose that*

$$g_{\mathbf{R}^n}(a) = h_{\mathbf{R}^n}(a)$$

for all $a \in (T^*T^{(r)})_0\mathbf{R}^n$ *(or for all* $a \in (T^*T^{r*})_0\mathbf{R}^n$*) with*

$$\pi(a) = j_0^r(x^1) \qquad (\text{or } q(a) = j_0^r(x^1)\ ). \tag{2.1}$$

Then $g = h$.

Proof. Consider $a \in (T^*T^{(r)})_0\mathbf{R}^n$ (or $a \in (T^*T^{r*})_0\mathbf{R}^n$). Using the invariancy of g and h it suffices to show that $g_{\mathbf{R}^n}(a) = h_{\mathbf{R}^n}(a)$.

Suppose that $\pi(a) = j_0^r(\gamma)$ (or $q(a) = j_0^r(\gamma)$) for some $\gamma : \mathbf{R}^n \to \mathbf{R}$ with $\gamma(0) = 0$ and $d_0\gamma \neq 0$. By the rank theorem there is an embedding $\varphi : \mathbf{R}^n \to \mathbf{R}^n$, $\varphi(0) = 0$, such that

$$T^{r*}\varphi(j_0^r(\gamma)) = j_0^r(x^1)\ .$$

Then

$$\pi(T^*T^{(r)}\varphi(a)) = j_0^r(x^1)\ (\text{or } q(T^*T^{r*}\varphi(a)) = j_0^r(x^1)\ ).$$

Now, using the invariancy of g and h with respect to φ and the assumption of the lemma we deduce that $g_{\mathbf{R}^n}(a) = h_{\mathbf{R}^n}(a)$. Thus $g_{\mathbf{R}^n} = h_{\mathbf{R}^n}$ on some dense subset in $(T^*T^{(r)})_0\mathbf{R}^n$ (or in $(T^*T^{r*})_0\mathbf{R}^n$). Since $g_{\mathbf{R}^n}$ and $h_{\mathbf{R}^n}$ are both of class C^∞, it holds $g_{\mathbf{R}^n} = h_{\mathbf{R}^n}$ over 0. □

Using Lemma 2 we prove the following lemma.

Lemma 3. *Let g, h be natural functions on $T^*T^{(r)}|\mathcal{M}f_n$ (or on $T^*T^{r*}|\mathcal{M}f_n$). Suppose that*

$$g_{\mathbf{R}^n}(a) = h_{\mathbf{R}^n}(a)$$

for all $a \in (T^*T^{(r)})_0\mathbf{R}^n$ (or for all $a \in (T^*T^{r*})_0\mathbf{R}^n$) satisfying the conditions (2.1) and

$$(2.2) \qquad < a, T^{(r)}\partial_i(q(a)) >= 0 \qquad (\text{or } < a, T^{r*}\partial_i(q(a)) >= 0)$$

for $i = 3, ..., n$. Then $g = h$.

Proof. Consider $a \in (T^*T^{(r)})_0\mathbf{R}^n$ with $\pi(a) = j_0^r(x^1)$ (or $a \in (T^*T^{r*})_0\mathbf{R}^n$ with $q(a) = j_0^r(x^1)$). Using Lemma 2 it is sufficient to show that $g_{\mathbf{R}^n}(a) = h_{\mathbf{R}^n}(a)$.

Define $\Theta \in T_0^*\mathbf{R}^n$ by

$$< \Theta, Z(0) >=< a, T^{(r)}Z(q(a)) >\ \ (\text{or } < \Theta, Z(0) >=< a, T^{r*}Z(q(a)) >)$$

for all constant vector fields Z on $\mathbf{R}^n$. There is a linear isomorphism $\psi : \mathbf{R}^n \to \mathbf{R}^n$ such that $x^1 \circ \psi = x^1$ and

$$T_0^*\psi(\Theta) = \alpha d_0x^1 + \beta d_0x^2$$

for some $\alpha, \beta \in \mathbf{R}$. Let $\overline{a} = T^*T^{(r)}\psi(a)$ (or $\overline{a} = T^*T^{r*}\psi(a)$). Since $T^{r*}\psi(j_0^r(x^1)) = j_0^r(x^1)$, $\overline{a}$ satisfies the condition (2.1) with a replaced by $\overline{a}$. Moreover,

$$\begin{aligned} < \overline{a}, T^{(r)}\partial_i(q(\overline{a})) > &=< a, T^{(r)}((\psi^{-1})_*\partial_i)(q(a)) > \\ &=< \Theta, ((\psi^{-1})_*\partial_i)(0) > \\ &=< T^*\psi(\Theta), \partial_i(0) >= 0 \end{aligned}$$

for $i = 3, ..., n$. (Similarly,

$$< \overline{a}, T^{r*}\partial_i(q(\overline{a})) >= 0$$

for $i = 3, ..., n$.) Then by the assumption of the lemma $g_{\mathbf{R}^n}(\overline{a}) = h_{\mathbf{R}^n}(\overline{a})$. Thus by the invariancy of g and h with respect to ψ we obtain $g_{\mathbf{R}^n}(a) = h_{\mathbf{R}^n}(a)$. □

Lemmas 1 and 3 imply the following assertion.

Lemma 4. *Let g, h be natural functions on $T^*T^{(r)}|\mathcal{M}f_n$ (or on $T^*T^{r*}|\mathcal{M}f_n$). Suppose that*

$$g_{\mathbf{R}^n}(a) = h_{\mathbf{R}^n}(a)$$

for all $a \in (T^*T^{(r)})_0\mathbf{R}^n$ (or for all $a \in (T^*T^{r*})_0\mathbf{R}^n$) satisfying the conditions (2.1) and (2.2) for $i = 2, ..., n$. Then $g = h$.

Proof. Consider $a \in (T^*T^{(r)})_0\mathbf{R}^n$ (or $a \in (T^*T^{r*})_0\mathbf{R}^n$) with (2.1) and (2.2) for $= 3, ..., n$. By Lemma 3 it suffices to show that $g_{\mathbf{R}^n}(a) = h_{\mathbf{R}^n}(a)$.

Using the density argument and Lemma 1 we can additionally assume that

$$< T^{(r)}((x^1)^r\partial_1)(q(a)), j^r_0(x^1) >= \frac{1}{\alpha}$$

$$(\text{or } < T^{r*}((x^1)^r\partial_1)(j^r_0(x^1)), \overline{\pi}(a) >= \frac{1}{\alpha})$$

for some $\alpha \in \mathbf{R}$.

Let $< a, T^{(r)}\partial_2(q(a)) >= \beta$ (or $< a, T^{r*}\partial_2(q(a)) >= \beta$). Since

$$j^{r-1}_0(\partial_2 - \alpha\beta(x^1)^r\partial_1) = j^{r-1}_0(\partial_2) ,$$

there exists an embedding $\varphi : \mathbf{R}^n \to \mathbf{R}^n$, $\varphi(0) = 0$, such that:

$$j^r_0(\varphi) = j^r_0(id) ,$$

$$germ_0(T\varphi \circ (\partial_2 - \alpha\beta(x^1)^r\partial_1)) = germ_0(\partial_2 \circ \varphi) \text{ and}$$

$$germ_0(T\varphi \circ \partial_i) = germ_0(\partial_i \circ \varphi)$$

for $i = 3, ..., n$, cf. [2].

Let $\overline{a} = T^*T^{(r)}\varphi(a)$ (or $\overline{a} = T^*T^{r*}\varphi(a)$). Since φ preserves both $j^r_0(x^1)$ and ∂_i for $i = 3, ..., n$, then $\overline{a}$ satisfies the conditions (2.1) and (2.2) for $i = 3, ..., n$. Moreover,

$$\begin{aligned} < \overline{a}, T^{(r)}\partial_2(q(\overline{a})) > &=< a, T^*T^{(r)}\varphi^{-1}(T^{(r)}\partial_2(q(\overline{a}))) > \\ &=< a, T^{(r)}\partial_2(q(a)) - \alpha\beta T^{(r)}((x^1)^r\partial_1)(q(a)) > \\ &= \beta - \alpha\beta\frac{1}{\alpha} = 0 \end{aligned}$$

$$(\text{or } < \overline{a}, T^{r*}\partial_2(q(\overline{a})) >= 0).$$

Then by the assumption of the lemma $g_{\mathbf{R}^n}(\overline{a}) = h_{\mathbf{R}^n}(\overline{a})$. Now, by the invariancy of g and h with respect to φ we obtain that $g_{\mathbf{R}^n}(a) = h_{\mathbf{R}^n}(a)$. □

Similarly, one can prove the following assertion.

Lemma 5. *Let g, h be natural functions on $T^*T^{(r)}|\mathcal{M}f_n$ (or on $T^*T^{r*}|\mathcal{M}f_n$). Suppose that*

$$g_{\mathbf{R}^n}(a) = h_{\mathbf{R}^n}(a)$$

for any $a \in (T^*T^{(r)})_0\mathbf{R}^n$ (or for any $a \in (T^*T^{r*})_0\mathbf{R}^n$) satisfying the conditions (2.1) and (2.2) for $i = 1, ..., n$. Then $g = h$.

Proof. The proof is a replica of the proof of Lemma 4. (In the text of the proof of Lemma 4 we replace ∂_2 by ∂_1, Lemma 3 by Lemma 4 and $i = 3, ..., n$ by $i = 2, ..., n$.) □

Now, we prove the main lemma.

Lemma 6. *Let g, h be natural functions on $T^*T^{(r)}|\mathcal{M}f_n$ (or on $T^*T^{r*}|\mathcal{M}f_n$). Suppose that*

$$g_{\mathbf{R}^n}(a) = h_{\mathbf{R}^n}(a)$$

for every $a \in (T^*T^{(r)})_0\mathbf{R}^n$ (or for every $a \in (T^*T^{r*})_0\mathbf{R}^n$) satisfying the conditions (2.1), (2.2) for $i = 1, ..., n$ and

$$(2.3) \qquad < q(a), j^r_0(x^\alpha) >= 0 \qquad (or\ < \overline{\pi}(a), j^r_0(x^\alpha) >= 0\ )$$

for all $\alpha = (\alpha_1, ..., \alpha_n) \in (\mathbf{N} \cup \{0\})^n$ with $1 \le |\alpha| \le r$ and $\alpha_2 + ... + \alpha_n \ge 1$. Then $g = h$.

Proof. Consider $a \in (T^*T^{(r)})_0\mathbf{R}^n$ (or $a \in (T^*T^{r*})_0\mathbf{R}^n$) satisfying the conditions (2.1) and (2.2) for $i = 1, ..., n$. By Lemma 5 it is sufficient to show that $g_{\mathbf{R}^n}(a) = h_{\mathbf{R}^n}(a)$.

Let $c_t := (x^1, tx^2, ..., tx^n) : \mathbf{R}^n \to \mathbf{R}^n$, $t \neq 0$. It is easy to see that

$$T^*T^{(r)}c_t(a) \to a^o \qquad (\text{or } T^*T^{r*}c_t(a) \to a^o\ )$$

as $t \to 0$ for some a^o satisfying (2.1), (2.2) for $i = 1, ..., n$, and (2.3) for all $\alpha = (\alpha_1, ..., \alpha_n) \in (\mathbf{N} \cup \{0\})^n$ with $1 \le |\alpha| \le r$ and $\alpha_2 + ... + \alpha_n \ge 1$. Then using the invariancy of g and h with respect to c_t we deduce that $g_{\mathbf{R}^n}(a) = g_{\mathbf{R}^n}(a^o) = h_{\mathbf{R}^n}(a^o) = h_{\mathbf{R}^n}(a)$. □

3. We are now in position to prove both theorems. Let g be a natural function on $T^*T^{(r)}|\mathcal{M}f_n$ (or on $T^*T^{r*}\mathcal{M}f_n$). Define $f : \mathbf{R}^r \to \mathbf{R}$ by

$$f(\xi) = g_{\mathbf{R}^n}(a_\xi),$$

where $\xi = (\xi_1, ..., \xi_r) \in \mathbf{R}^r$ and $a_\xi \in (T^*T^{(r)})_0\mathbf{R}^n$ (or $a_\xi \in (T^*T^{r*})_0\mathbf{R}^n$) is the unique form satisfying the conditions:
(2.1), (2.2) for $i = 1, ..., n$, (2.3) for all $\alpha = (\alpha_1, ..., \alpha_n) \in (\mathbf{N} \cup \{0\})^n$ with $1 \le |\alpha| \le r$ and $\alpha_2 + ... + \alpha_n \ge 1$, and

$$(2.4) \qquad < q(a_\xi), j^r_0((x^1)^s) >= \xi_s \qquad (\text{or } < \overline{\pi}(a_\xi), j^r_0((x^1)^s) >= \xi_s)$$

for $s = 1, ..., r$.
It is clear that f is smooth. We see that

$$g_{\mathbf{R}^n}(a_\xi) = f(\lambda^{<1>}_{\mathbf{R}^n}(a_\xi), ..., \lambda^{<r>}_{\mathbf{R}^n}(a_\xi))$$
$$(\text{or } g_{\mathbf{R}^n}(a_\xi) = f(\mu^{<1>}_{\mathbf{R}^n}(a_\xi), ..., \mu^{<r>}_{\mathbf{R}^n}(a_\xi))\ )$$

for all $\xi \in \mathbf{R}^r$. Hence by Lemma 6 we obtain

$$g_M = f \circ (\lambda^{<1>}_M, ..., \lambda^{<r>}_M)\ (\text{or } g_M = f \circ (\mu^{<1>}_M, ..., \mu^{<r>}_M)\ )\ .$$

□

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