ARCHIVUM MATHEMATICUM (BRNO)
Tomus 31 (1995), 43 – 54

A MULTIPLICATION OF E –VARIETIES OF ORTHODOX SEMIGROUPS

Martin Kuřil*

Abstract. We define semantically a partial multiplication on the lattice of all e–varieties of regular semigroups. In the case that the first factor is an e–variety of orthodox semigroups we describe our multiplication syntactically in terms of biinvariant congruences.

1. Introduction

We will investigate here an operator on the lattice of all e–varieties of regular semigroups. We define semantically a partial multiplication on this lattice: $\mathcal{U}\square\mathcal{V}$ is defined if $\mathcal{U}$ is an e–variety of regular semigroups and $\mathcal{V}$ is an e–variety of inverse semigroups. In the case that $\mathcal{U}$ is an e–variety of orthodox semigroups we can also describe our multiplication syntactically in terms of biinvariant congruences.

The motivation for us was Polák's paper [5]. It deals with a multiplication on the lattice of varieties of $*$-regular semigroups. The semantical definition is based on a certain semidirect product $\mathcal{S}\times_\varphi\mathcal{T}$, where $\mathcal{S}=(S,\cdot,')$ is a $*$-regular semigroup, $\mathcal{T}=(T,\cdot,')$ is a locally inverse $*$-regular semigroup and $\varphi:(T,\cdot)\longrightarrow(\mathrm{End}\,(S,\cdot,'),\circ)$ is a homomorphism. We adopt this definition in the context of regular semigroups (without an explicite unary operation). Seeing that the multiplication in $S\times_\varphi\mathcal{T}$ for $*$-regular semigroups $\mathcal{S}$ and $\mathcal{T}$ is defined using the unary operation in $(T,\cdot,')$, we introduce the semidirect product $\mathcal{S}\times_\varphi\mathcal{T}$ for regular semigroups $\mathcal{S}$ and $\mathcal{T}$ only when $\mathcal{T}$ is an inverse semigroup – then the unary operation is implicitly given. So, for an e–variety $\mathcal{U}$ of regular semigroups and for an e–variety $\mathcal{V}$ of inverse semigroups we define $\mathcal{U}\square\mathcal{V}$ as the e–variety generated by all semidirect products of members of $\mathcal{U}$ and $\mathcal{V}$. We also touch the question about the relation between $\mathcal{U}\square\mathcal{V}$ and the Malcev product of $\mathcal{U}$ and $\mathcal{V}$.

In order to describe our multiplication syntactically, we use the notion of biinvariant congruence introduced in the paper [4] by Kaďourek and Szendrei and the

---

1991 *Mathematics Subject Classification*: 20M07, 20M17.
Key words and phrases: regular semigroup, orthodox semigroup, inverse semigroup, e–variety, biinvariant congruence.
Received January 4, 1994.
*The author acknowledges the support of the Grant No. 201/93/2121 of the Grant Agency of Czech Republic

one-to-one correspondence between all e–varieties of orthodox semigroups and all biinvariant congruences given in the same paper. Thus the syntactical description of $\mathcal{U}\Box\mathcal{V}$ is restricted to the case when $\mathcal{U}$ is an e–variety of orthodox semigroups. We define the multiplication of biinvariant congruences formally in the same way as Polák does in [5] for fully invariant congruences. Our paper culminates with Theorem 5.1, which clarifies the connections between syntax and semantics.

2. A semidirect product of regular semigroups

For basic notions in the theory of semigroups the reader is referred to [3].

Let $\mathcal{S} = (S, \cdot)$ be a semigroup. Let $a \in S$, $b \in S$. The element b is an inverse of a if $aba = a$, $bab = b$. We will denote by $V(a)$ the set of all inverses of a in S.

If for every element a of S there exists some $x \in S$ such that $axa = a$ then $\mathcal{S}$ is regular. Equivalently, $\mathcal{S}$ is regular if and only if $V(a) \neq \emptyset$ for all $a \in S$.

Let $E_{\mathcal{S}} = \{a \in S / a \cdot a = a\}$ denote the set of all idempotents of $\mathcal{S}$.

A semigroup $\mathcal{S}$ is called an orthodox semigroup if it is regular and $E_{\mathcal{S}}$ is a subsemigroup in $\mathcal{S}$ and $\mathcal{S}$ is an inverse semigroup if card $(V(a)) = 1$ for any $a \in S$. The unique inverse of a is denoted by a'. It holds: $(a')' = a$, $(ab)' = b'a'$ for all $a \in S$, $b \in S$. It is well known that $\mathcal{S}$ is an inverse semigroup if and only if it is regular and any two idempotents of $\mathcal{S}$ commute.

The set of all endomorphisms of $\mathcal{S}$ is denoted by End $\mathcal{S}$.

Let $\mathcal{T} = (T, \cdot)$ be an inverse semigroup, let $\varphi : (T, \cdot) \to (\mathrm{End}\,(S, \cdot), \circ)$, where $\circ$ is the composition $(\alpha \circ \beta)(s) = \alpha(\beta(s))$ $(\alpha, \beta \in \mathrm{End}\ \mathcal{S},\ s \in S)$, be a homomorphism.

Put $S \times_\varphi T = \{(s,t) \in S \times T / \varphi(tt')(s) = s\}$ and define

$$(s,t) \cdot (u,v) = (\varphi(tvv't')(s) \cdot \varphi(t)(u), t \cdot v)$$

for (s,t), $(u,v) \in S \times_\varphi T$.
The structure $(S \times_\varphi T, \cdot)$ will be denoted by $\mathcal{S} \times_\varphi \mathcal{T}$.

2.1. Lemma. *$\mathcal{S} \times_\varphi \mathcal{T}$ is a semigroup.*

Proof.
1. $S \times_\varphi T \neq \emptyset$:
   Let $s \in S, t \in T$. Put $r = \varphi(tt')(s) \in S$. We see that
   $(r,t) \in S \times_\varphi T$:
   $\varphi(tt')(r) = \varphi(tt')(\varphi(tt')(s)) = \varphi(tt'tt')(s) = \varphi(tt')(s) = r.$
2. Correctness of the multiplication:
   Let $S \in t)$, $(u,v) \in S \times_\varphi T$. Then $\varphi(tv(tv)')(\varphi(tvv't')(s) \cdot \varphi(t)(u))$
   $= \varphi(\underline{tv(tv)'}tvv't')(s) \cdot \varphi(tv(tv)'t)(u) = \varphi(tvv't')(s) \cdot \varphi(tv(tv)'t)(\varphi(vv')(u))$
   $= \varphi(\overline{tvv't')(s)} \cdot \varphi(\underline{tv(tv)'tv}v')(u) = \varphi(tvv't')(s) \cdot \varphi(tvv')(u)$
   $= \varphi(tvv't')(s) \cdot \varphi(\overline{t)(\varphi(vv'})(u)) = \varphi(tvv't')(s) \cdot \varphi(t)(u).$
3. Associativity of the multiplication:
   Let (p,q), (s,t), $(u,v) \in S \times_\varphi T$. Then
   $((p,q) \cdot (s,t)) \cdot (u,v) = (\varphi(qtt'q')(p) \cdot \varphi(q)(s), qt) \cdot (u,v)$
   $= (\varphi(qtvv'(qt)')(\varphi(qtt'q')(p) \cdot \varphi(q)(s)) \cdot \varphi(qt)(u), qtv)$
   $= (\varphi(qtvv'\underline{t'q'qtt'q'})(p) \cdot \varphi(q\underline{tvv't'q'q})(s) \cdot \varphi(qt)(u), qtv)$

$= (\varphi(qtvv't'q')(p) \cdot \varphi(\underline{qq'q}tvv't')(s) \cdot \varphi(qt)(u), qtv)$
$= (\varphi(qtvv't'q')(p) \cdot \varphi(qtvv't')(s) \cdot \varphi(qt)(u), qtv)$
since $tvv't'$, $q'q \in E_{\mathcal{T}}$ and idempotents in $\mathcal{T}$ commute, and
$(p,q) \cdot ((s,t) \cdot (u,v))$
$= (p,q) \cdot (\varphi(tvv't')(s) \cdot \varphi(t)(u), tv)$
$= (\varphi(qtvv't'q')(p) \cdot \varphi(q)(\varphi(tvv't')(s) \cdot \varphi(t)(u)), qtv)$
$= (\varphi(qtvv't'q')(p) \cdot \varphi(qtvv't')(s) \cdot \varphi(qt)(u), qtv)$.

2.2. Lemma. *If $\mathcal{S}$ is regular, then $\mathcal{S} \times_\varphi \mathcal{T}$ is also regular.*

Proof. Let $(s,t) \in S \times_\varphi T$. There is $p \in S$, $sps = s$. Then
1. $(\varphi(t')(p), t') \in S \times_\varphi T$:
$\varphi(t't)(\varphi(t')(p)) = \varphi(t'tt')(p) = \varphi(t')(p)$.
2. $(s,t) \cdot (\varphi(t')(p), t') \cdot (s,t) = (s,t)$:
$(s,t)(\varphi(t')(p), t')(s,t) = (\varphi(tt'tt')(s) \cdot \varphi(t)(\varphi(t')(p)), tt')(s,t)$
$= (s \cdot \varphi(tt')(p), tt')(s,t) = (\varphi(tt'tt'tt')(s \cdot \varphi(tt')(p)) \cdot \varphi(tt')(s), tt't)$
$= (\varphi(tt')(s) \cdot \varphi(tt')(p) \cdot \varphi(tt')(s), t) = (\varphi(tt')(s), t) = (s,t)$.

2.3. Lemma. *Let $(s,t) \in S \times_\varphi T$. Then (s,t) is an idempotent in $\mathcal{S} \times_\varphi \mathcal{T}$ if and only if $s \in E_{\mathcal{S}}$ and $t \in E_{\mathcal{T}}$.*

Proof.
1. Let (s,t) be an idempotent in $\mathcal{S} \times_\varphi \mathcal{T}$.
Then $(s,t) = (s,t) \cdot (s,t) = (\varphi(ttt't')(s) \cdot \varphi(t)(s), t^2)$.
We see that $t \in E_{\mathcal{T}}$. Further
$\varphi(ttt't')(s) \cdot \varphi(t)(s) = s$, i.e.
$s = \varphi(tt')(s) \cdot \varphi(t)(\varphi(tt')(s)) = s \cdot \varphi(t^2t')(s) = s \cdot \varphi(tt')(s) = s \cdot s$.
So, $s \in E_{\mathcal{S}}$.
2. Let $s \in E_{\mathcal{S}}$ and $t \in E_{\mathcal{T}}$.
Then $(s,t) \cdot (s,t) = (\varphi(ttt't')(s) \cdot \varphi(t)(s), t^2)$
$= (\varphi(tt')(s) \cdot \varphi(t)(\varphi(tt')(s)), t)$
$= (s \cdot \varphi(t^2t')(s), t) = (s \cdot \varphi(tt')(s), t) = (s^2, t) = (s,t)$.

2.4. Lemma. *If $\mathcal{S}$ is orthodox, then $\mathcal{S} \times_\varphi \mathcal{T}$ is also orthodox.*

Proof. Let (s,t), (u,v) be idempotents in $\mathcal{S} \times_\varphi \mathcal{T}$. Since $s \in E_{\mathcal{S}}$ (see 2.3) and $\varphi(tvv't') \in \mathrm{End}\,(S,\cdot)$, we have $\varphi(tvv't')(s) \in E_{\mathcal{S}}$.
Similarly, $\varphi(t)(u) \in E_{\mathcal{S}}$.
Now $\varphi(tvv't')(s) \cdot \varphi(t)(u) \in E_{\mathcal{S}}$, since $\mathcal{S}$ is orthodox. By 2.3 also $t \in E_{\mathcal{T}}$, $v \in E_{\mathcal{T}}$, thus $t \cdot v \in E_{\mathcal{T}}$ ($\mathcal{T}$ is inverse). Using 2.3 we obtain: $(s,t) \cdot (u,v) = (\varphi(tvv't')(s) \cdot \varphi(t)(u), t \cdot v)$ is an idempotent in $\mathcal{S} \times_\varphi \mathcal{T}$.

2.5. Lemma. *Let $I \neq \emptyset$. Let $\mathcal{S}_i = (S_i, \cdot)$ be a semigroup for $i \in I$. Let $\mathcal{T}_i = (T_i, \cdot)$ be an inverse semigroup for $i \in I$. Finally, let $\varphi_i : (T_i, \cdot) \to (\mathrm{End}\,(S_i, \cdot), \circ)$ be a homomorphism for $i \in I$. Then*

$$\prod_{i \in I} (\mathcal{S}_i \times_{\varphi_i} \mathcal{T}_i) \cong \prod_{i \in I} \mathcal{S}_i \times_\varphi \prod_{i \in I} \mathcal{T}_i\,,$$

where the homomorphism

$$\varphi : \Big(\prod_{i\in I} T_i, \cdot\Big) \to \Big(\mathrm{End}\,\Big(\prod_{i\in I} S_i, \cdot\Big), \circ\Big)$$

is given by $\varphi((t_i)_{i\in I})((s_i)_{i\in I}) = (\varphi_i(t_i)(s_i))_{i\in I}$. *The isomorphism is given by* $((s_i, t_i))_{i\in I} \mapsto ((s_i)_{i\in I}, (t_i)_{i\in I})$.

Proof. It is an easy exercise due to the fact that φ is defined componentwise.

3. A multiplication of e–varieties of regular semigroups

For any class $\mathcal{V}$ of regular semigroups, we will denote by $H(\mathcal{V})$, $S_r(\mathcal{V})$ and $P(\mathcal{V})$, respectively, the classes of all homomorphic images, regular subsemigroups and direct products of semigroups in $\mathcal{V}$.

We adopt the following notations for classes of regular semigroups:
R - the class of all regular semigroups
O - the class of all orthodox semigroups
I - the class of all inverse semigroups.

A class $\mathcal{V} \subseteq R$ satisfying $H(\mathcal{V}) \subseteq \mathcal{V}$, $S_r(\mathcal{V}) \subseteq \mathcal{V}$ and $P(\mathcal{V}) \subseteq \mathcal{V}$ is called an e–variety. The classes R, O, I are examples of e–varieties. The concept of e–variety was introduced by Hall in [2]. Simultaneously and independently Kaďourek and Szendrei in [4] have considered e–varieties of orthodox semigroups, which they call bivarieties of orthodox semigroups.

Denote by $\langle\mathcal{V}\rangle$ the least e–variety of regular semigroups containing the class $\mathcal{V} \subseteq R$.

Let $\mathcal{U} \subseteq R$ and $\mathcal{V} \subseteq I$ be e–varieties. Now we define a multiplication $\Box$ in the following way:

$$\begin{aligned}\mathcal{U}\Box\mathcal{V} = &\langle\{\mathcal{S} \times_\varphi \mathcal{T} / \mathcal{S} \in \mathcal{U}, \mathcal{T} \in \mathcal{V}, \\ &\varphi : (T, \cdot) \to (\mathrm{End}\,(S, \cdot), \circ) \ \text{ is a homomorphism}\}\rangle\end{aligned}$$

The correctness of the definition is based on 2.2.

3.1. Result ([4], Corollary 1.3., Proposition 1.4.)
Let $\mathcal{V} \subseteq O$. Then
(i) $S_r H(\mathcal{V}) \subseteq H S_r(\mathcal{V})$
(ii) $\langle\mathcal{V}\rangle = H S_r P(\mathcal{V})$.

3.2. Lemma. *Let* $\mathcal{U} \subseteq O$ *and* $\mathcal{V} \subseteq I$ *be e–varieties. Then*

$$\begin{aligned}\mathcal{U}\Box\mathcal{V} = &H S_r(\{\mathcal{S} \times_\varphi \mathcal{T} / \mathcal{S} \in \mathcal{U}, \mathcal{T} \in \mathcal{V}, \\ &\varphi : (T, \cdot) \to (\mathrm{End}\,(S, \cdot), \circ) \ \textit{is a homomorphism}\}).\end{aligned}$$

Proof. Put $\mathcal{W} = \{\mathcal{S} \times_\varphi \mathcal{T} / \mathcal{S} \in \mathcal{U}, \mathcal{T} \in \mathcal{V},$
$\varphi : (T, \cdot) \to (\mathrm{End}\,(S, \cdot), \circ)$ is a homomorphism $\}$.

It follows from 2.4 that $\mathcal{W} \subseteq O$. Then $\mathcal{U}\Box\mathcal{V} = HS_rP(\mathcal{W})$, by 3.1 (ii). It is clear that $HS_r(\mathcal{W}) \subseteq HS_rP(\mathcal{W})$. Further, $P(\mathcal{W}) \subseteq H(\mathcal{W})$, by 2.5. Then $HS_rP(\mathcal{W}) \subseteq HS_rH(\mathcal{W})$ and this together with 3.1 (i) gives $HS_rP(\mathcal{W}) \subseteq HS_r(\mathcal{W})$.

Now we consider the Malcev product of e–varieties. Recall that for any two classes $\mathcal{U} \subseteq R$ and $\mathcal{V} \subseteq R$, the Malcev product $\mathcal{U} \circ \mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$ is defined to be the class

$$\{\mathcal{S} \in R / \quad \text{there exists a congruence} \quad \kappa \text{ on } \mathcal{S} \text{ such that} \\ e\kappa \in \mathcal{U} \text{ for every } \quad e \in E_{\mathcal{S}} \text{ and } \mathcal{S}/\kappa \in \mathcal{V}\}.$$

3.3. Lemma. *Let $\mathcal{U} \subseteq R$ and $\mathcal{V} \subseteq I$ be e–varieties. Let $\mathcal{S} = (S,\cdot) \in \mathcal{U}$, $\mathcal{T} = (T,\cdot) \in \mathcal{V}$ and $\varphi : (T,\cdot) \to (\mathrm{End}\,(S,\cdot),\circ)$ be a homomorphism. Then $\mathcal{S}\times_\varphi\mathcal{T} \in \mathcal{U}\circ\mathcal{V}$.*

Proof. Clearly, the second projection $\pi : S \times_\varphi T \to T$ is a homomorphism of $\mathcal{S} \times_\varphi \mathcal{T}$ into $\mathcal{T}$. Choose $t \in T$ and $s \in S$. Then

$$\varphi(tt')(\varphi(t)(s)) = \varphi(t)(s)$$

gives $(\varphi(t)(s), t) \in S \times_\varphi T$, which means that π is surjective. Thus $\mathcal{S} \times_\varphi \mathcal{T} / \ker\pi \cong \mathcal{T}$ and $\mathcal{S} \times_\varphi \mathcal{T} / \ker\pi \in \mathcal{V}$.

Let (s,t) be an idempotent in $\mathcal{S} \times_\varphi \mathcal{T}$. We will show that $(s,t)\ker\pi \in \mathcal{U}$.

Put $S_t = \{p \in S / \varphi(tt')(p) = p\}$. Clearly, $(s,t)\ker\pi = S_t \times \{t\}$.

S_t is a regular subsemigroup in $\mathcal{S}$:

1. $S_t \neq \emptyset$, since $s \in S_t$.
2. Let $p, q \in S_t$. Then $\varphi(tt')(p\cdot q) = \varphi(tt')(p) \cdot \varphi(tt')(q) = p \cdot q$, which means that $p \cdot q \in S_t$.
3. S_t is regular:

Let $p \in S_t$. There is $x \in S$, $pxp = p$. Put $y = \varphi(tt')(x)$. Then $\varphi(tt')(y) = \varphi(tt')(\varphi(tt')(x)) = \varphi(tt')(x) = y$ and $p \cdot y \cdot p = \varphi(tt')(p) \cdot \varphi(tt')(x) \cdot \varphi(tt')(p) = \varphi(tt')(pxp) = \varphi(tt')(p) = p$.

So, $y \in S_t$ and $p \cdot y \cdot p = p$. Finally, $(s,t)\ker\pi \cong S_t$:

The mapping $\alpha : S_t \to (s,t)\ker\pi$, defined by

$$\alpha(p) = (p,t)$$

for all $p \in S_t$, is bijective. Further, $\alpha(p\cdot q) = \alpha(p)\cdot\alpha(q)$ for all $p, q \in S_t$:

$$\begin{aligned} \alpha(p\cdot q) &= (p\cdot q, t)\,, \\ \alpha(p)\cdot\alpha(q) &= (p,t)\cdot(q,t) = (\varphi(ttt't')(p)\cdot\varphi(t)(q), t^2) \\ &= (\varphi(tt')(p)\cdot\varphi(t)(\varphi(tt')(q)), t) \\ &= (\varphi(tt')(p)\cdot\varphi(tt')(q), t) = (p\cdot q, t) \end{aligned}$$

(we have used that $t \in E_{\mathcal{T}}$ by 2.3).

So, $(s,t)\ker\pi \in \mathcal{U}$, since S_t is a regular subsemigroup in $\mathcal{S}$ and $(s,t)\ker\pi \cong S_t$.

3.4. Corollary. *Let $\mathcal{U} \subseteq R$ and $\mathcal{V} \subseteq I$ be e–varieties. Then $\mathcal{U}\Box\mathcal{V} \subseteq \langle\mathcal{U}\circ\mathcal{V}\rangle$.*

4. A multiplication of biinvariant congruences

Let Y be a non-empty set, $Y' = \{y'/y \in Y\}$ be a disjoint copy of Y. We denote by $F'(Y)$ the free semigroup over $Y \cup Y'$.

For any word $f = z_1 z_2 \dots z_n (z_1, z_2, \dots, z_n \in Y \cup Y')$ we put $f' = z_n' \dots z_2' z_1'$ with $(y')' = y$ for $y \in Y$.

A mapping $\vartheta : Y \cup Y' \to S$ into a regular semigroup $(S, \cdot)$ such that $\vartheta(y') \in V(\vartheta(y))$ for all $y \in Y$ is called a matched mapping.

By a biidentity over Y we mean any pair $f \hat{=} g$ of words $f,\ g \in F'(Y)$. We will say that a biidentity $f \hat{=} g$ is satisfied in a regular semigroup $(S, \cdot)$ if, for any matched mapping $\vartheta : Y \cup Y' \to S$, we have $\Theta(f) = \Theta(g)$, where $\Theta : F'(Y) \to S$ is the homomorphism extending ϑ. We say that a biidentity is satisfied in a class $\mathcal{V} \subseteq R$ if it is satisfied in each member of $\mathcal{V}$.

For any class $\mathcal{V} \subseteq R$, denote by $\varrho(\mathcal{V}, Y)$ – the set of all biidentities over Y which are satisfied in $\mathcal{V}$.

Let Y be a non-empty set. We write $f(y_1, y_1', \dots, y_n, y_n')$ to indicate that only elements $y_1, \dots, y_n \in Y$, $y_1', \dots, y_n' \in Y'$ may occur in $f \in F'(Y)$.
By $f(p_1, q_1, \dots, p_n, q_n)$ we denote the element of $F'(Y)$ obtained by substituting $p_1, q_1, \dots, p_n, q_n \in F'(Y)$ for $y_1, y_1', \dots, y_n, y_n'$ into f.

A congruence ϱ on $F'(Y)$ is termed biinvariant if $\varrho(O, Y) \subseteq \varrho$ and it has the following property: whenever $f,\ g,\ p_1,\ q_1, \dots, p_n, q_n \in F'(Y)$ such that

$$f(y_1, y_1', \dots, y_n, y_n') \varrho\ g(y_1, y_1', \dots, y_n, y_n')$$

and

$$p_i q_i p_i \varrho(O, Y) p_i,\ \ q_i p_i q_i \varrho(O, Y) q_i \quad \text{for} \quad i = 1, 2, \dots, n$$

then also

$$f(p_1, q_1, \dots, p_n, q_n) \varrho\ g(p_1, q_1, \dots, p_n, q_n).$$

The set of all biinvariant congruences on $F'(Y)$ will be denoted by $BIC\ F'(Y)$.

Let $X = \{x_1, x_2, \dots\}$ be a set of variables, $\varrho \in BIC\ F'(X)$.
Define a new alphabet

$$E_\varrho = \{(f\varrho, x) \in F'(X)/\varrho \times X / f x x' \varrho f\}.$$

Putting

$$E_\varrho' = \{(g\varrho, x') \in F'(X)/\varrho \times X' / g x' x \varrho g\},$$

we see that $E_\varrho \cap E_\varrho' = \emptyset$ and that the mappings

$$(f\varrho, x) \longmapsto (fx\varrho, x')$$
$$(g\varrho, x') \longmapsto (gx'\varrho, x)$$

are mutually inverse bijections between E_ϱ and E_ϱ'. Denote both by $'$.

Now define $\pi_\varrho : F'(X) \to F'(E_\varrho)$ by

$$\begin{aligned} y_1 \dots y_m \longmapsto & (y_1 \dots y_m y'_m \dots y'_1 \varrho, y_1) \cdot (y_1 \dots y_m y'_m \dots y'_2 \varrho, y_2) \\ & \dots (y_1 \dots y_m y'_m \varrho, y_m) \end{aligned}$$

where $y_1, \dots, y_m \in X \cup X'$.

Given $\sigma \in BIC\ F'(E_\varrho)$, define

$$\sigma \Box \varrho \subseteq F'(X) \times F'(X)$$

by

$$f(\sigma \Box \varrho) g \Leftrightarrow f \varrho g \text{ and } \pi_\varrho(f) \sigma \pi_\varrho(g) \quad (f, g \in F'(X)).$$

Remark. If $\varrho \in BIC\ \ F'(X)$, $\sigma \in BIC\ \ F'(E_\varrho)$ and $\varrho \supseteq \varrho(I, X)$, then $\sigma \Box \varrho \in BIC\ \ F'(X)$. We will prove it in 5.1 (i).

Put

$$\begin{aligned} h * (f\varrho, x) &= (hf\varrho, x), \\ h * (g\varrho, x') &= (hg\varrho, x'), \\ h * e_1 \dots e_m &= (h * e_1) \dots (h * e_m) \end{aligned}$$

for $\varrho \in BIC\ F'(X)$, $(f\varrho, x) \in E_\varrho$, $(g\varrho, x') \in E'_\varrho$, $h \in F'(X)$, $e_1, \dots, e_m \in F'(E_\varrho)$.

4.1. Lemma. *Let $\varrho \in BIC\ \ F'(X)$, $\varrho \supseteq \varrho(I, X)$, f, $g \in F'(X)$. Then*

$$\pi_\varrho(fg) = (fgg'f' * \pi_\varrho(f)) \cdot (f * \pi_\varrho(g)).$$

Proof. Let $f = y_1 \dots y_m$, $g = z_1 \dots z_n$, $y_1, \dots, y_m$, $z_1, \dots, z_n \in X \cup X'$. Then

$$\begin{aligned} \pi_\varrho(y_1 \dots y_m z_1 \dots z_n) &= (fgg'y'_m \dots y'_1 \varrho, y_1) \dots (fgg'y'_m \varrho, y_m) \\ & \quad \cdot (fgz'_n \dots z'_1 \varrho, z_1) \dots (fgz'_n \varrho, z_n) \\ &= (fgg'y'_m \dots y'_1 \varrho, y_1) \dots (fgg'y'_m \varrho, y_m) \\ & \quad \cdot (f * \pi_\varrho(g)). \end{aligned}$$

Now $fgg'y'_m \dots y'_i \varrho fgg'f'fy'_m \dots y'_i$ for $i = 1, \dots, m$, since $gg'f'f\varrho(I, X)f'fgg'$ and $ff'f\varrho(I, X)f$.

4.2. Lemma. *Let $(S, \cdot) \in R$, $(T, \cdot) \in I$, $\varphi : (T, \cdot) \to (\operatorname{End}(S, \cdot), \circ)$ be a homomorphism, $\vartheta : X \cup X' \to S \times_\varphi T$ be a matched mapping, such that*

$$\vartheta(x_i) = (s_i, t_i),\ \vartheta(x'_i) = (p_i, q_i) \quad (\text{for} \quad i = 1, 2, 3, \dots).$$

Let $\varrho \in BIC\ F'(X)$, $\varrho \supseteq \varrho(I, X)$, $\varrho \subseteq \varrho(\{(T, \cdot)\}, X)$. Let $\overset{=}{\vartheta} : X \cup X' \to T$ be given by

$$x_i \longmapsto t_i\,,\ x'_i \longmapsto q_i\,.$$

Let $\bar{\vartheta} : E_\varrho \cup E'_\varrho \to S$ *be given by*

$$(f\varrho, x_i) \longmapsto \varphi(\bar{\bar{\Theta}}(f))(s_i) \quad (\textit{for } \ (f\varrho, x_i) \in E_\varrho),$$
$$(g\varrho, x'_i) \longmapsto \varphi(\bar{\bar{\Theta}}(g))(p_i) \quad (\textit{for } \ (g\varrho, x'_i) \in E'_\varrho)$$

where $\bar{\bar{\Theta}} : F'(X) \to T$ *is the homomorphism extending* $\bar{\bar{\vartheta}}$. *Finally, let* $\Theta : F'(X) \to S \times_\varphi T$ $(\bar{\Theta} : F'(E_\varrho) \to S)$ *be the homomorphism extending* ϑ $(\bar{\vartheta})$. *Then*

(i) the mappings $\bar{\vartheta}$, $\bar{\bar{\vartheta}}$ *are matched*

(ii) $\bar{\Theta}$ $(f * a) = \varphi(\bar{\bar{\Theta}}(f))(\bar{\Theta}(a))$ *for all* $f \in F'(X)$, $a \in F'(E_\varrho)$

(iii) $\Theta(f) = (\bar{\Theta}(\pi_\varrho(f)), \bar{\bar{\Theta}}(f))$ *for all* $f \in F'(X)$.

Proof. (i) $(s_i, t_i)(p_i, q_i)(s_i, t_i) = (s_i, t_i)$, $(p_i, q_i)(s_i, t_i)(p_i, q_i) = (p_i, q_i)$ since ϑ is a matched mapping.
Further, $(s_i, t_i)(p_i, q_i)(s_i, t_i)$
$= (\varphi(t_i q_i t_i t'_i q'_i t'_i)(s_i) \cdot \varphi(t_i q_i t_i t'_i q'_i t'_i t_i)(p_i) \cdot \varphi(t_i q_i)(s_i), t_i q_i t_i)$, $(p_i, q_i)(s_i, t_i)(p_i, q_i)$
$= (\varphi(q_i t_i q_i q'_i t'_i q'_i)(p_i) \cdot \varphi(q_i t_i q_i q'_i t'_i q'_i q_i)(s_i) \cdot \varphi(q_i t_i)(p_i), q_i t_i q_i)$.
So, $t_i = t_i q_i t_i$ and $q_i = q_i t_i q_i$. Thus $q_i = t'_i$, $t_i = q'_i$. We see, that the mapping $\bar{\bar{\vartheta}}$ is matched. This together with $\varrho \subseteq \varrho(\{(T, \cdot)\}, X)$ gives the correctness of the definition of $\bar{\vartheta}$. Further,
$s_i = \varphi(t_i t'_i t_i t'_i t_i t'_i)(s_i) \cdot \varphi(t_i t'_i t_i t'_i t_i t'_i t_i)(p_i) \cdot \varphi(t_i t'_i)(s_i) = s_i \cdot \varphi(t_i)(p_i) \cdot s_i$,
$p_i = p_i \cdot \varphi(t'_i)(s_i) \cdot p_i$, $\varphi(t_i)(p_i) = \varphi(t_i)(p_i) \cdot s_i \cdot \varphi(t_i)(p_i)$. Let $(f\varrho, x_i) \in E_\varrho$. We have to show that $\bar{\vartheta}((f\varrho, x_i)) \cdot \bar{\vartheta}((f\varrho, x_i)') \cdot \bar{\vartheta}((f\varrho, x_i)) = \bar{\vartheta}((f\varrho, x_i))$ and $\bar{\vartheta}((f\varrho, x_i)') \cdot \bar{\vartheta}((f\varrho, x_i)) \cdot \bar{\vartheta}((f\varrho, x_i)') = \bar{\vartheta}((f\varrho, x_i)')$, i.e. $\varphi(\bar{\bar{\Theta}}(f))(s_i) \cdot \varphi(\bar{\bar{\Theta}}(fx_i))(p_i) \cdot \varphi(\bar{\bar{\Theta}}(f))(s_i) = \varphi(\bar{\bar{\Theta}}(f))(s_i)$ and $\varphi(\bar{\bar{\Theta}}(fx_i))(p_i) \cdot \varphi(\bar{\bar{\Theta}}(f))(s_i) \cdot \varphi(\bar{\bar{\Theta}}(fx_i))(p_i) = \varphi(\bar{\bar{\Theta}}(fx_i))(p_i)$. We use the following fact
$$\begin{aligned}\varphi(\bar{\bar{\Theta}}(fx_i))(p_i) &= \varphi(\bar{\bar{\Theta}}(f)\bar{\bar{\vartheta}}(x_i))(p_i)\\ &= \varphi(\bar{\bar{\Theta}}(f) \cdot t_i)(p_i)\\ &= \varphi(\bar{\bar{\Theta}}(f))(\varphi(t_i)(p_i))\end{aligned}$$
and already proved equalities $s_i = s_i \cdot \varphi(t_i)(p_i) \cdot s_i$, $\varphi(t_i)(p_i) = \varphi(t_i)(p_i) \cdot s_i \cdot \varphi(t_i)(p_i)$.

(ii) Let $(h\varrho, x_i) \in E_\varrho$. Then
$$\begin{aligned}\bar{\Theta}(f * (h\varrho, x_i)) &= \bar{\Theta}((fh\varrho, x_i)) = \bar{\vartheta}((fh\varrho, x_i))\\ &= \varphi(\bar{\bar{\Theta}}(fh))(s_i) = \varphi(\bar{\bar{\Theta}}(f)\bar{\bar{\Theta}}(h))(s_i)\\ &= \varphi(\bar{\bar{\Theta}}(f))(\varphi(\bar{\bar{\Theta}}(h))(s_i)) = \varphi(\bar{\bar{\Theta}}(f))(\bar{\vartheta}((h\varrho, x_i)))\\ &= \varphi(\bar{\bar{\Theta}}(f))(\bar{\Theta}((h\varrho, x_i))).\end{aligned}$$
Similarly, $\bar{\Theta}(f * (g\varrho, x'_i)) = \varphi(\bar{\bar{\Theta}}(f))(\bar{\Theta}((g\varrho, x'_i)))$ for $(g\varrho, x'_i) \in E'\varrho$.
Let $a_1, \ldots, a_m \in E_\varrho \cup E'\varrho$. Then

$$\begin{aligned}(\bar{\Theta}(f * a_1 \dots a_m) &= \bar{\Theta}((f * a_1)\dots(f * a_m)) = \bar{\Theta}(f * a_1)\dots\bar{\Theta}(f * a_m)\\ &= \varphi(\bar{\bar{\Theta}}(f))(\bar{\Theta}(a_1)) \ \dots \ \varphi(\bar{\bar{\Theta}}(f))(\bar{\Theta}(a_m))\\ &= \varphi(\bar{\bar{\Theta}}(f))(\bar{\Theta}(a_1) \ \dots \ \bar{\Theta}(a_m)) = \varphi(\bar{\bar{\Theta}}(f))(\bar{\Theta}(a_1 \dots a_m)) .\end{aligned}$$

(iii) By induction with respect to the length of f : $\Theta(x_i) = (s_i, t_i)$,

$$\begin{aligned}(\bar{\Theta}(\pi_\varrho(x_i)), \bar{\bar{\Theta}}(x_i)) &= (\bar{\vartheta}((x_i x_i'\varrho, x_i)), \bar{\bar{\vartheta}}(x_i)) = (\varphi(\bar{\bar{\Theta}}(x_i x_i'))(s_i), t_i)\\ &= (\varphi(\bar{\bar{\vartheta}}(x_i)\cdot\bar{\bar{\vartheta}}(x_i'))(s_i), t_i) = (\varphi(t_i \cdot q_i)(s_i), t_i)\\ &= (\varphi(t_i \cdot t_i')(s_i), t_i) = (s_i, t_i) .\end{aligned}$$

Similarly, $\Theta(x_i') = (\bar{\Theta}(\pi_\varrho(x_i')), \bar{\bar{\Theta}}(x_i'))$. Let $g, h \in F'(X)$. Then

$$\begin{aligned}\Theta(g\cdot h) = \Theta(g)\cdot\Theta(h) &= (\bar{\Theta}(\pi_\varrho(g)), \bar{\bar{\Theta}}(g))\cdot(\bar{\Theta}(\pi_\varrho(h)), \bar{\bar{\Theta}}(h))\\ &= (\varphi(\bar{\bar{\Theta}}(g)(\bar{\bar{\Theta}}(h)\bar{\bar{\Theta}}(h')\bar{\bar{\Theta}}(g'))(\bar{\Theta}(\pi_\varrho(g)))\cdot\varphi(\bar{\bar{\Theta}}(g))(\bar{\Theta}(\pi_\varrho(h))), \bar{\bar{\Theta}}(g\cdot h))\\ &= (\varphi(\bar{\bar{\Theta}}(ghh'g'))(\bar{\Theta}(\pi_\varrho(g)))\cdot\varphi(\bar{\bar{\Theta}}(g))(\bar{\Theta}(\pi_\varrho(h))), \bar{\bar{\Theta}}(g\cdot h))\\ &= (\bar{\Theta}(ghh'g' * \pi_\varrho(g))\cdot\bar{\Theta}(g * \pi_\varrho(h)), \bar{\bar{\Theta}}(g\cdot h)) = (\bar{\Theta}(\pi_\varrho(g\cdot h)), \bar{\bar{\Theta}}(g\cdot h)) ,\end{aligned}$$

since $\varphi(\bar{\bar{\Theta}}(ghh'g'))(\bar{\Theta}(\pi_\varrho(g))) = \bar{\Theta}(ghh'g' * \pi_\varrho(g))$ and $\varphi(\bar{\bar{\Theta}}(g))(\bar{\Theta}(\pi_\varrho(h))) = \bar{\Theta}(g * \pi_\varrho(h))$ by the part (ii), $(ghh'g' * \pi_\varrho(g))\cdot(g * \pi_\varrho(h)) = \pi_\varrho(g\cdot h)$ by 4.1. We have also used the equalities $(\bar{\bar{\Theta}}(h))' = \bar{\bar{\Theta}}(h')$ and $(\bar{\bar{\Theta}}(g))' = \bar{\bar{\Theta}}(g')$.

4.3. Lemma. *Let $\varrho \in BIC\ F'(X)$, $\varrho \supseteq \varrho(I, X)$, $\sigma \in BIC\ F'(E_\varrho)$. Let $\mathcal{S} = (S, \cdot) \in R$, $\mathcal{T} = (T, \cdot) \in I$, $\varrho \subseteq \varrho(\{\mathcal{T}\}, X)$, $\sigma \subseteq \varrho(\{\mathcal{S}\}, E_\varrho)$. Finally, let $\varphi : (T, \cdot) \to (\mathrm{End}\,(S, \cdot), \circ)$ be a homomorphism. Then $\sigma\Box\varrho \subseteq \varrho(\{\mathcal{S} \times_\varphi \mathcal{T}\}, X)$.*

Proof. Let $f, g \in F'(X)$, $f(\sigma\Box\varrho)g$, $\vartheta : X \cup X' \to S \times_\varphi T$ be a matched mapping. We have to show that $\Theta(f) = \Theta(g)$. We know that $f\varrho g$, $\pi_\varrho(f)\sigma\pi_\varrho(g)$. By 4.2 (i) $\bar{\bar{\Theta}}(f) = \bar{\bar{\Theta}}(g)$, $\bar{\Theta}(\pi_\varrho(f)) = \bar{\Theta}(\pi_\varrho(g))$. Thus $\Theta(f) = \Theta(g)$ (by 4.2 (iii)).

4.4. Result ([4], Corollary 1.12.)
For any infinite set Y, the rules $\mathcal{V} \longmapsto \varrho(\mathcal{V}, Y)$ and $\varrho \longmapsto |\varrho|$ (where $|\varrho|$ denotes the class of all orthodox semigroups in which all biidentities from ϱ are satisfied) determine a one–to–one correspondence between all e–varieties of orthodox semigroups and all biinvariant congruences on $F'(Y)$.

We will denote the one–to–one correspondence from 4.4 by the symbol $\longleftrightarrow$.

4.5. Result ([1], Lemma 1)
Let $\psi : (Q, \cdot) \to (W, \cdot)$ be a surjective homomorphism of regular semigroups and let $a, b \in W$, $b \in V(a)$. Then there exist $c, d \in Q$ such that $d \in V(c)$ and $\psi(c) = a$, $\psi(d) = b$.

4.6. Lemma. *Let $\varrho \in BIC\ F'(X)$, $\varrho \supseteq \varrho(I, X)$, $\sigma \in BIC\ F'(E_\varrho)$. Let $\mathcal{U} \subseteq O$, $\mathcal{V} \subseteq I$ be e–varieties such that $\mathcal{U} \longleftrightarrow \sigma$, $\mathcal{V} \longleftrightarrow \varrho$. Then*

$$\sigma\Box\varrho \subseteq \varrho(\mathcal{U}\Box\mathcal{V}, X) .$$

Proof. Let $f, g \in F'(X)$, $f(\sigma\Box\varrho)g$, $(W, \cdot) \in \mathcal{U}\Box\mathcal{V}$ and let $\vartheta : X \cup X' \to W$ be a matched mapping. We will show that $\Theta(f) = \Theta(g)$. It follows from 3.2 that there exist $(S,\cdot) \in \mathcal{U}$, $(T,\cdot) \in \mathcal{V}$, a homomorphism $\varphi : (T,\cdot) \to (\mathrm{End}\,(S,\cdot),\circ)$, a regular subsemigroup $(Q,\cdot)$ in $(S,\cdot) \times_\varphi (T,\cdot)$ and a surjective homomorphism $\psi : (Q,\cdot) \to (W,\cdot)$. By 4.5, there is a matched mapping $\widehat{\vartheta} : X \cup X' \to Q$ such that $\psi(\widehat{\vartheta}(y)) = \vartheta(y)$ for all $y \in X \cup X'$. It is clear that $\psi(\widehat{\Theta}(f)) = \Theta(f)$ for all $f \in F'(X)$. Now we use 4.3. We have $\widehat{\Theta}(f) = \widehat{\Theta}(g)$. Thus $\psi(\widehat{\Theta}(f)) = \psi(\widehat{\Theta}(g))$, $\Theta(f) = \Theta(g)$.

4.7. Lemma. *Let $\varrho \in BICF'(X)$, $\sigma \in BIC\ F'(E\varrho)$. Then the mapping $\varphi : F'(X)/\varrho \longrightarrow (\mathrm{End}\,(F'(E_\varrho)/\sigma),\circ)$ given by*

$$\varphi(f g)(a\sigma) = (f * a)\sigma$$

$(f \in F'(X), a \in F'(E_\varrho))$ is a correctly defined homomorphism.

Proof. 1. correctness of the definition:
Let $f,\ g \in F'(X)$, $a,\ b \in F'(E_\varrho)$, $f\varrho g$, $a\sigma b$. We will show that $f * a\sigma g * b$. Choose $(h\varrho, y) \in E_\varrho \cup E'_\varrho$. Then

$$\begin{aligned} f * (h\varrho, y) &= (fh\varrho, y) = ((f\varrho)(h\varrho), y) \\ &= ((g\varrho)(h\varrho), y) = (gh\varrho, y) = g * (h\varrho, y)\,. \end{aligned}$$

This implies $f * a = g * a$. It is clear that for $(h\varrho, x) \in E_\varrho$

$$g * (h\varrho, x) = (gh\varrho, x), g * (h\varrho, x)' = g * (hx\varrho, x') = (ghx\varrho, x') = (gh\varrho, x)'$$

and

$$(gh\varrho, x) \cdot (gh\varrho, x)' \cdot (gh\varrho, x)\varrho(O, E_\varrho)(gh\varrho, x),$$
$$(gh\varrho, x)' \cdot (gh\varrho, x) \cdot (gh\varrho, x)'\varrho(O, E_\varrho)(gh\varrho, x)'\,.$$

Since $\sigma \in BIC\ F'(E_\varrho)$, we get $g * a\sigma g * b$. So, $f * a\sigma g * b$.

2. $\varphi(f\varrho) : F'(E_\varrho)/\sigma \longrightarrow F'(E_\varrho)/\sigma$ is a homomorphism (for any $f \in F'(X)$):
Let $a,\ b \in F'(E_\varrho)$. Then

$$\begin{aligned} \varphi(f\varrho)((a\sigma)\cdot(b\sigma)) &= \varphi(f\varrho)(ab\sigma) = (f * (ab))\sigma = (f * a)(f * b)\sigma \\ &= ((f * a)\sigma) \cdot ((f * b)\sigma) = \varphi(f\varrho)(a\sigma) \cdot \varphi(f\varrho)(b\sigma)\,. \end{aligned}$$

3. φ is a homomorphism:
Let $f,\ g \in F'(X)$, $a \in F'(E_\varrho)$. Then

$$\begin{aligned} \varphi(f\varrho)(\varphi(g\varrho)(a\sigma)) &= \varphi(f\varrho)((g * a)\sigma) = (f * (g * a))\sigma = ((f \cdot g) * a)\sigma \\ &= \varphi(fg\varrho)(a\sigma) = \varphi((f\varrho)(g\varrho))(a\sigma)\,. \end{aligned}$$

Now we recall the notion of a bifree object. Let $\mathcal{V}$ be a class of regular semigroups. By a bifree object in $\mathcal{V}$ on a non-empty set Y, we mean a pair $(\mathcal{S}, i)$, where $\mathcal{S} = (S,\cdot) \in \mathcal{V}$ and $i : Y \cup Y' \longrightarrow S$ is a matched mapping such that the following universal property is satisfied: for any semigroup $\mathcal{T} = (T,\cdot) \in \mathcal{V}$ and any matched mapping $\vartheta : Y \cup Y' \longrightarrow T$, there exists a unique homomorphism $\psi : \mathcal{S} \to \mathcal{T}$ such that $\psi i = \vartheta$. In cases when the mapping i is obvious, we omit it and we term simply $\mathcal{S}$ to be a bifree object in $\mathcal{V}$ on Y.

4.8. Result ([4], Theorem 1.9.)
In any class $\mathcal{V} \subseteq O$ satisfying $S_r(\mathcal{V}) \subseteq \mathcal{V}$ and $P(\mathcal{V}) \subseteq \mathcal{V}$ there exists a bifree object on any non-empty set Y, and it is isomorphic to $F'(Y)/\sigma(\mathcal{V}, Y)$.

4.9. Lemma. *Let $\varrho \in BIC\ F'(X)$, $\varrho \supseteq \varrho(I,X)$, $\sigma \in BIC\ F'(E_\varrho)$. Let $\varphi : F'(X)/\varrho \longrightarrow (\mathrm{End}\,(F'(E_\varrho)/\sigma), \circ)$ be the homomorphism from 4.7. Finally, let $\vartheta : X \cup X' \longrightarrow F'(E_\varrho)/\sigma \times_\varphi F'(X)/\varrho$ be given by*

$$y \longmapsto (\pi_\varrho(y)\sigma, y\varrho) \quad (y \in X \cup X')\,.$$

Then
(i) ϑ is a matched mapping
(ii) $\Theta(f) = (\pi_\varrho(f)\sigma, f\varrho)$ for all $f \in F'(X)$.

Proof. Note that $F'(X)/\rho \in I$ and $F'(E_\varrho)/\sigma \in O$ (see 4.8).

(i) ϑ is correctly defined:

$$\begin{aligned}\varphi((y\varrho)(y\varrho)')(\pi_\varrho(y)\sigma) &= \varphi((y\varrho)(y'\varrho))((yy'\varrho, y)\sigma)\\ &= \varphi(yy'\varrho)((yy'\varrho, y)\sigma) = (yy' * (yy'\varrho, y))\sigma\\ &= (yy'yy'\varrho, y)\sigma = (yy'\varrho, y)\sigma = \pi_\varrho(y)\sigma\,.\end{aligned}$$

We have used the following fact: $\varrho \in BIC\ F'(X)$ and $\varrho \supseteq \varrho(I,X)$ implies $(f\varrho)' = f'\varrho$ for any $f \in F'(X)$. ϑ is a matched mapping: Choose $x \in X$. Then

$$\begin{aligned}&\vartheta(x) = ((xx'\varrho, x)\sigma), x\varrho),\ \vartheta(x') = ((x'x\varrho, x')\sigma, x'\varrho),\ \vartheta(x)\vartheta(x')\vartheta(x)\\ &\quad= (\varphi(xx'xx'xx'\varrho)((xx'\varrho, x)\sigma) \cdot \varphi(xx'xx'xx'x\varrho)((x'x\varrho, x')\sigma)\cdot\\ &\quad\cdot \varphi(xx'\varrho)((xx'\varrho, x)\sigma),\ xx'x\varrho) = (\varphi(xx'\varrho)((xx'\varrho, x)\sigma)\cdot\\ &\quad\cdot \varphi(xx'\varrho)((xx'x\varrho, x')\sigma) \cdot \varphi(xx'\varrho)((xx'\varrho, x)\sigma), x\varrho)\\ &\quad= (\varphi(xx'\varrho)((xx'\varrho, x)(xx'\varrho, x)'(xx'\varrho, x)\sigma), x\varrho) = (\varphi(xx'\varrho)((xx'\varrho, x)\sigma), x\varrho)\\ &\quad= ((xx'\varrho, x)\sigma, x\varrho) = \vartheta(x).\end{aligned}$$

Similarly, $\vartheta(x')\ \vartheta(x)\ \vartheta(x') = \vartheta(x')$.

(ii) By induction with respect to the length of f:
Let $y \in X \cup X'$. Then $\Theta(y) = \vartheta(y) = (\pi_\varrho(y)\sigma, y\varrho)$. Let $g,\ h \in F'(X)$. Then

$$\begin{aligned}\Theta(g \cdot h) = \Theta(g) \cdot \Theta(h) &= (\pi_\varrho(g)\sigma, g\varrho) \cdot (\pi_\varrho(h)\sigma, h\varrho)\\ &= (\varphi((g\varrho)(h\varrho)(h\varrho)'(g\varrho)')(\pi_\varrho(g)\sigma) \cdot \varphi(g\varrho)(\pi_\varrho(h)\sigma), (g\varrho)\cdot(h\varrho))\\ &= (\varphi(ghh'g'\varrho)(\pi_\varrho(g)\sigma) \cdot \varphi(g\varrho)(\pi_\varrho(h)\sigma),\ gh\varrho)\\ &= ((ghh'g' * \pi_\varrho(g)) \cdot (g * \pi_\varrho(h))\sigma, gh\varrho)\\ &= (\pi_\varrho(gh)\sigma, gh\varrho),\end{aligned}$$

since $(ghh'g' * \pi_\varrho(g)) \cdot (g * \pi_\varrho(h)) = \pi_\varrho(gh)$ by 4.1.

4.10. Lemma. *Let $\varrho \in BIC\ F'(X)$, $\varrho \supseteq \varrho(I,X)$, $\varrho \in BIC\ F'(E_\varrho)$. Let $\mathcal{U} \subseteq O$, $\mathcal{V} \subseteq I$ be e–varieties such that $\mathcal{U} \leftrightarrow \sigma$, $\mathcal{V} \leftrightarrow \varrho$. Then*

$$\sigma\Box\varrho \supseteq \varrho(\mathcal{U}\Box\mathcal{V}, X)\,.$$

Proof. Let $f,\, g \in F'(X)$, $f\varrho(\mathcal{U}\Box\mathcal{V}, X)g$. We will show that $f(\sigma\Box\varrho)g$, i.e. $f\varrho g$ and $\pi_\varrho(f)\sigma\pi_\varrho(g)$. Note that $F'(E_\varrho)/\sigma \in \mathcal{U}$, $F'(X)/\varrho \in \mathcal{V}$ (see 4.4 and 4.8). We use the homomorphism $\varphi : F'(X)/\varrho \rightarrow (\mathrm{End}\,(F'(E_\varrho)/\sigma), \circ)$ from 4.7 and the matched mapping $\vartheta : X \cup X' \rightarrow F'(E_\varrho)/\sigma \times_\varphi F'(X)/\varrho$ from 4.9. Clearly $F'(E_\varrho)/\sigma \times_\varphi F'(X)/\varrho \in \mathcal{U}\Box\mathcal{V}$. Thus the biidentity $f \hat{=} g$ is satisfied in $F'(E_\varrho)/\sigma \times_\varphi F'(X)/\varrho$ and therefore $\Theta(f) = \Theta(g)$. Hence , by 4.9 (ii), $(\pi_\varrho(f)\sigma, f\varrho) = (\pi_\varrho(g)\sigma, g\varrho)$.

5. The multiplication of e–varieties corresponds to the multiplication of biinvariant congruences

Recall that the symbol $\leftrightarrow$ denotes the one-to-one correspondence between all e–varieties of orthodox semigroups and all biinvariant congruences (see 4.4).

5.1. Theorem. *Let $\varrho \in BIC\ F'(X)$, $\varrho \supseteq \varrho(I,X)$, $\sigma \in BIC\ F'(E_\varrho)$. Let $\mathcal{U} \subseteq O$, $\mathcal{V} \subseteq I$ be e–varieties such that $\mathcal{U} \leftrightarrow \sigma$, $\mathcal{V} \leftrightarrow \varrho$. Then*

(i) $\sigma\Box\varrho \in BICF'(X)$
(ii) $\mathcal{U}\Box\mathcal{V} \leftrightarrow \sigma\Box\varrho$
(iii) The mapping $\psi : F'(X)/\sigma\Box\varrho \to F'(E_\varrho)/\sigma \times_\varphi F'(X)/\varrho$ defined by $\psi(f(\sigma\Box\varrho)) = (\pi_\varrho(f)\sigma, f\varrho)$, where φ is the homomorphism from 4.7, is an embedding.

Proof. (i) and (ii): Note that $\mathcal{U}\Box\mathcal{V} \subseteq O$ (see 2.4). Using 4.6 and 4.10 we have $\sigma\Box\varrho = \varrho(\mathcal{U}\Box\mathcal{V}, X)$. Thus $\mathcal{U}\Box\mathcal{V} \leftrightarrow \sigma\Box\varrho$ by 4.4.

(iii): Note that $(\pi_\varrho(f)\sigma, f\varrho) \in F'(E_\varrho)/\sigma \times_\varphi F'(X)/\varrho$ (see 4.9). It follows immediately from the definition of $\sigma\Box\varrho$ that ψ is a correctly defined injective mapping. ψ is a homomorphism:
Let $f,\ g \in F'(X)$. Then $\psi((f(\sigma\Box\varrho)) \cdot (g(\sigma\Box\varrho))) = \psi(fg(\sigma\Box\varrho)) = (\pi_\varrho(fg)\sigma, fg\varrho)$ $= ((fgg'f' * \pi_\varrho(f)) \cdot (f * \pi_\varrho(g))\sigma, fg\varrho)$, since $\pi_\varrho(fg) = (fgg'f' * \pi_\varrho(f)) \cdot (f * \pi_\varrho(g))$ by 4.1. Further,
$\psi(f(\sigma\Box\varrho)) \cdot \psi(g(\sigma\Box\varrho)) = (\pi_\varrho(f)\sigma, f\varrho) \cdot (\pi_\varrho(g)\sigma, g\varrho)$
$= (\varphi((f\varrho)(g\varrho)(g\varrho)'(f\varrho)')(\pi_\varrho(f)\sigma) \cdot \varphi(f\varrho)(\pi_\varrho(g)\sigma), (f\varrho)(g\varrho))$
$= (\varphi(fgg'f'\varrho)(\pi_\varrho(f)\sigma) \cdot \varphi(f\varrho)(\pi_\varrho(g)\sigma), fg\varrho) = ((fgg'f' * \pi_\varrho(f)) \cdot (f * \pi_\varrho(g))\sigma, fg\varrho)$.

5.2. Remark. Theorem 5.1 together with Result 4.8 show that bifree objects in $\mathcal{U}\Box\mathcal{V}$ are isomorphic to some subsemigroups in suitable semidirect products of bifree objects in $\mathcal{U}$ by bifree objects in $\mathcal{V}$, for any e–varieties $\mathcal{U} \subseteq O$ and $\mathcal{V} \subseteq I$.

References

[1] Hall, T. E., *Congruences and Green's relations on regular semigroups*, Glasgow Math. J. **13** (1972), 167-175.
[2] Hall, T. E., *Identities for existence varieties of regular semigroups*, Bull. Austral. Math. Soc. **40** (1989), 59-77.
[3] Howie, J. M., *An Introduction to Semigroup Theory*, Academic Press, London, 1976.
[4] Kaďourek, J., Szendrei, M. B., *A new approach in the theory of orthodox semigroups*, Semigroup Forum **40** (1990), 257-296.
[5] Polák, L., *A multiplication on the lattice of varieties of ∗-regular semigroups*, to appear in Proceedings of Luino International Conference on Semigroups.

Martin Kuřil
Department of Mathematics
J. E. Purkyně University
České mládeže 8
400 96 Ústí nad Labem, CZECH REPUBLIC