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OSCILLATORY PROPERTIES OF SOLUTIONS OF SECOND ORDER NONLINEAR NEUTRAL DIFFERENTIAL INEQUALITIES WITH OSCILLATING COEFFICIENTS

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Abstract. This paper deals with the second order nonlinear neutral differential inequalities (A_ν): $(-1)^\nu x(t)\{z''(t)+(-1)^\nu q(t)f(x(h(t)))\}\le 0,\ \ t\ge t_0\ge 0$, where $\nu=0$ or $\nu=1$, $z(t)=x(t)+p(t)x(t-\tau)$, $0<\tau=$ const, $p,q,h:[t_0,\infty)\to R$ $f:R\to R$ are continuous functions. There are proved sufficient conditions under which every bounded solution of (A_ν) is either oscillatory or $\liminf\limits_{t\to\infty}|x(t)|=0$.

1. Introduction

Consider the second order nonlinear neutral differential inequalities

$$(A_\nu)\qquad (-1)^\nu x(t)\{z''(t)+(-1)^\nu q(t)f(x(h(t)))\}\le 0,\quad t\ge t_0\ge 0,$$

where $\nu=0$ or $\nu=1$, $z(t)=x(t)+p(t)x(t-\tau)$, $0<\tau=$ const, $p,q,h:[t_0,\infty)\to R$ are continuous functions, $\lim\limits_{t\to\infty}h(t)=\infty$, $q(t)$ it allowed to oscillate on $[t_0,\infty)$ and $p,q\not\equiv 0$ on any subinterval of half line $[t_0,\infty)$, $f:R\to R$ is continuous, $u\,f(u)>0$ for $u\ne 0$.

Recently several authors have been studying the oscillatory properties of solutions of neutral delay differential equations of the first and higher order. Among numerous of interesting results of this type can be found in the papers [1–8] and to the references obtained therein.

On the end of the paper [5] it is written: When q is allowed to oscillate the problem is far more difficult, and any results, even for linear equations, would be of interest.

In this paper we give some new aspects in the study of the oscillatory properties of solutions of the inequalities (A_ν) with the oscillatory coefficient q .

Let $T_0>t_0$ be such that $T=min\{inf_{t\ge T_0}h(t),\ T_0-\tau\}\ge t_0$. A function $x:[T,\infty)\to R$ is a solution of (A_ν) on $[T_0,\infty)$ if $x(t)$ is

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continuous on $[T,\infty)$, the function $z(t)$ is two times continuously differentiable on $[T_0,\infty)$ and $x(t)$ satisfies (A_ν) on $[T_0,\infty)$.

We consider solutions of (A_ν) only such that $sup\{\,|x(t)| : t\in[t_x,\infty)\,\}>0$ for any $t_x\ge T_0$. Such a solution is called nonoscillatory if it is eventually of constant sign. Otherwise it is called oscillatory.

2. Main results

In addition we suppose that:

(C_1) There exits a sequence of intervals $\{(a_n,b_n)\}_{n=1}^{\infty}$ such that
$\bigcup\limits_{n=1}^{\infty}(a_n,b_n)\subset[t_0,\infty),\quad \lim\limits_{n\to\infty}a_n=\infty,$
and for any $n\in N: b_n-a_n>\tau,\ b_n<a_{n+1},\quad a_{n+1}-a_n\le M<\infty.$

(C_2) $q(t)>0$ for all $t\in\bigcup\limits_{n=1}^{\infty}(a_n,b_n)$ and $\liminf\limits_{t\to\infty}q(t)=0.$

Denote $J_n=(a_n,b_n),\quad A_k=\bigcup\limits_{n=k}^{\infty}J_n$ for any $k\in N.$

Let there exist constants p_1,p_2 such that the following holds:

(C_3) $p_1\le p(t)\le p_2,\ t\in[t_0,\infty).$

Lemma 1. *Let $x(t)$ be a bounded solution of (A_ν) on $[T_0,\infty)$ and let (C_3) hold. Then the function $z(t)=x(t)+p(t)\,x(t-\tau)$ is bounded.*

Proof. The proof of Lemma is evident. □

Theorem 1. *Let (C_1), (C_2), (C_3) hold. If*

$$(C_4)\qquad \lim_{n\to\infty}\int_{a_n}^{b_n}q(s)\,ds=\infty,$$

then every bounded solution of (A_0) is either oscillatory or $\liminf\limits_{t\to\infty}|x(t)|=0$.

Proof. Let $x(t)$ be a nonoscillatory bounded solution of (A_0) on $[T_0,\infty)$, Without loss of generality we suppose that $x(t-\tau)>0$ and $x(h(t))>0$ on $[\,t_1,\infty)$, $t_1\ge T_0+\tau$. Let $\{J_n\}_{n=1}^{\infty}$ be a sequence of intervals defined by (C_1). Since $q(t)>0$ for any $t\in A_1\cap[t_1,\infty)$, then from (A_0) we get that $z'(t)$ is decreasing and $z(t)$ is monotone on $A_1\cap[t_1,\infty)$.

In view of that $x(t)$ (>0) is bounded on $[t_1,\infty)$, there exist a constant K and $T_1\ge t_1$ such that $|f(x(h(t)))|\le K$ for all $t\ge T_1$. With regard to (C_2) for any $\delta>0$ there exists a $T_2\ge T_1$ such that

$$(1)\qquad q(t)\ge-\delta/K\,M\quad\text{for}\quad t\ge T_2.$$

If $x(t)>0$ for $t\in[t_1,\infty)$, then from (A_ν) we get

$$(\bar A_\nu)\qquad (-1)^\nu\{z''(t)\,+(-1)^\nu\,q(t)\,f(x(h(t)))\}\le 0,\quad t\ge t_1$$

Then from $(\bar{A}_0)$ with regard to (1) we have $z''(t) \le \delta/M$ for $t \ge T_2$. Integrating the last inequality from b_n to a_{n+1} $(b_n \ge T_2, n \in N)$ we have

$$z'(a_{n+1}) \le z'(b_n) + \delta, \quad b_n \ge T_2, \ n \in N. \tag{2}$$

I) Let there exists a $n_0 \ge 1$ such that $z'(t) < 0$ for all $t \in A_{n_0} \cap [T_2, \infty)$. Integrating $(\bar{A}_0)$ from a_n to b_n, $n \ge n_0$ and using that $z'(t) < 0$ we obtain

$$\int_{a_n}^{b_n} q(t)\, f(x(h(t)))\, dt \le z'(a_n) - z'(b_n) \le -z'(b_n). \tag{3}$$

a) Let $\inf_{n \ge n_0}\{z'(b_n)\} > -\infty$, then from (3) we have

$$\int_{a_n}^{b_n} q(t)\, f(x(h(t)))\, dt < \infty, \quad a_{n_0} \ge T_2, \quad n \ge n_0. \tag{4}$$

The last inequality with regard to (C_4) and the property of the function f and h implies $\liminf_{t \to \infty} x(t) = 0$.

b) Let $\inf_{n \ge n_0}\{z'(b_n)\} = -\infty$. Then in view of (2) and that $z'(t)$ (< 0) is decreasing on $A_{n_0} \cap [T_2, \infty)$, we get that $z(t)$ is unbounded below. Then this, in view of (C_3) and of Lemma 1 we get that $x(t)$ is unbounded, which is a contradition to the assumption that $x(t)$ is bounded on $[T_0, \infty)$.

II) Let there exists a sequence $\{m_k\}_{k=1}^{\infty}$, $m_k \in N$ such that $z'(t) > 0$ and $z'(t)$ is decreasing for all $t \in A_{m_1} \subset [t_0, \infty)$. Then integrating $(\bar{A}_0)$ from a_{m_k} to b_{m_k}, $k \ge 1$, we have

$$\int_{a_{m_k}}^{b_{m_k}} q(t)\, f(x(h(t)))\, dt \le z'(a_{m_k}) - z'(b_{m_k}) \le z'(a_{m_k}). \tag{5}$$

Because $x(t)$ (> 0) is bounded on $[T_0, \infty)$, by Lemma 1 we get that $z(t)$ is bounded on $[T_0, \infty)$. Therefore with regard to (2) and the monotonicity of $z(t)$, $z'(t)$ we have $\sup_{m_k \ge m_1}\{z'(a_{m_k})\} < \infty$. Thus from (5) we get (4), which implies as in the case *Ia*) that $\liminf_{t \to \infty} x(t) = 0$.

The proof of Theorem 1 is complete. □

Theorem 2. *Let (C_1), (C_2), (C_3) and (C_4) hold. Then every bounded solution of (A_1) is either oscillatory or $\liminf_{t \to \infty} |x(t)| = 0$.*

Proof. Let $x(t)$ be a nonoscillatory bounded solution of (A_1) on $[T_0, \infty)$, Without loss of generality we suppose that $x(t - \tau) > 0$ and $x(h(t)) > 0$ on $[t_1, \infty)$, $t_1 \ge T_0 + \tau$. Let $\{J_n\}_{n=1}^{\infty}$ be a sequence of intervals defined by (C_1). If $q(t) > 0$ for any $t \in A_1 \cap [t_1, \infty)$, then from (A_1) we get that $z'(t)$ is increasing and $z(t)$ is monotone on $A_1 \cap [t_1, \infty)$.

Analogously as in the proof of Theorem 1 we have (1). Then from $(\bar{A}_1)$ in view of (1) we have $z''(t) \geq -\delta/M$ for $t \geq T$. Integrating the last inequality from b_n to a_{n+1}, $b_n \geq T, n \in N$, we obtain

$$z'(a_{n+1}) \geq z'(b_n) + \delta, \quad t \in A_1 \cap [T, \infty). \tag{6}$$

I) Let there exist a $n_0 \geq 1$ such that $z'(t) > 0$ for all $t \in A_{n_0}$, $a_{n_0} \geq T$. Integrating $(\bar{A}_1)$ from a_n to b_n, for any $n \geq n_0$ we obtain

$$\int_{a_n}^{b_n} q(t)\, f(x(h(t)))\, dt \leq z'(b_n) - z'(a_n) \leq +z'(b_n). \tag{7}$$

a) Let $\sup_{n \geq n_0}\{z'(b_n)\} < \infty$, then from (7) in view of (C_4) and the property of the functions f and h, we have $\liminf_{t\to\infty} x(t) = 0$.

b) Let $\sup_{n \geq n_0}\{z'(b_n)\} = \infty$, then in view of (6) and the fact that $z'(t)$ (> 0) is increasing for all $t \in A_{n_0}$, we have that $z(t)$ is unbounded above. Then in view of (C_3) and of Lemma 1 we get that $x(t)$ is unbounded, which is a contradition.

II) Let there exists a sequence $\{m_k\}_{k=1}^{\infty}$, $m_k \in N$ such that $z'(t) < 0$ and $z'(t)$ is increasing for all $t \in A_{m_1} \subset [T_0, \infty)$. Then integrating $(\bar{A}_1)$ from a_{m_k} to b_{m_k}, $k \geq 1$, we obtain

$$\int_{a_{m_k}}^{b_{m_k}} q(t)\, f(x(h(t)))\, dt \leq z'(b_{m_k}) - z'(a_{m_k}) \leq -z'(a_{m_k}). \tag{8}$$

In view of Lemma 1 and that $x(t)$ (> 0) is bounded on $[T_0, \infty)$, we have that $z(t)$ is bounded on $[T_0, \infty)$. Then with regard to (6) and the monotonicity of $z(t)$, $z'(t)$ we get that $\sup_{m_k \geq m_1}\{-z'(a_{m_k})\} < \infty$. Therefore from (8) we get

$$\int_{a_{m_k}}^{b_{m_k}} q(t)\, f(x(h(t)))\, dt < \infty.$$

The last relation in view of (C_4) and the property of the function f and h we get that $\liminf_{t\to\infty} x(t) = 0$.

The proof of Theorem 2 is complete. □

Now denote

$$q_+(t) = \max\{0, q(t)\}, \quad q_-(t) = \max\{0, -q(t)\}, \quad t \in [t_0, \infty). \tag{9}$$

Then $q(t) = q_+(t) - q_-(t)$.

Lemma 2. [6, Lemma 1.5.2] *Let $f, g, p \in C([t_0, \infty), R)$ and $c \in R$ be such that $f(t) = g(t) + p(t)\, g(t - c)$, $t \geq t_0 + \max\{0, c\}$. Assume that there exist numbers $p_1, p_2, p_3, p_4 \in R$ such that $p(t)$ is one of the following ranges:*

i) $p_1 \leq p(t) \leq 0$,

ii) $0 \leq p(t) \leq p_2 < 1$,

iii) $1 < p_3 \leq p(t) \leq p_4$.

Suppose that $g(t) > 0$ for $t \geq t_0$, $\liminf_{t\to\infty} g(t) = 0$ and that $\lim_{t\to\infty} f(t) = L \in R$ exists. Then $L = 0$.

Lemma 3. *Let* $f, g, p \in C([t_0, \infty), R)$ *and* $c \in (0, \infty)$ *be such that* $f(t) = g(t) + p(t)\, g(t-c)$ *for* $t \geq t_0 + c$. *Assume that* $0 < g(t) \leq g_0 < \infty$, $\lim\limits_{t\to\infty} f(t) = 0$. *In addition we suppose that there exists constant* p_1, p_2 *such that either*

$$-1 < p_1 \leq p(t) \leq 0, \text{ or } \quad 0 \leq p(t) \leq |p_1| < 1, \tag{10}$$

or

$$p(t) \leq p_2 < -1. \tag{11}$$

Then $\lim\limits_{t\to\infty} g(t) = 0$.

Proof. i) Let (10) hold. Then

$$g(t) = f(t) - p(t)\, g(t-c) \leq f(t) + |p_1|\, g(t-c), \quad t \geq t_0 + c.$$

By iteration for sufficiently large t we have

$$g(t) \leq f(t) + |p_1|\, f(t-c) + |p_1|^2\, f(t-2\,c) + \cdots + |p_1|^{n-1}\, f(t-(n-1)\,c) + |p_1|^n\, g(t-n\,c).$$

The last relation we can writte in the form

$$0 < g(t+n\,c) \leq f(t+n\,c) + |p_1|\, f(t+(n-1)\,c) + |p_1|^2\, f(t+(n-2)\,c) + \cdots + \\ + |p_1|^{n-1}\, f(t+c) + |p_1|^n\, g(t),$$

for sufficiently large t. In view of $\lim\limits_{t\to\infty} f(t) = 0$, for any $\varepsilon_1 > 0$ there exists sufficiently large T such that $|f(t)| < \varepsilon_1$ for $t \geq T$. Then

$$|g(t+n\,c)| < \varepsilon_1 \frac{1}{1-|p_1|} + |p_1|^n\, g_0, \quad t \geq T. \tag{12}$$

Therefore for any $\varepsilon > 0$ there exist ε_1 and $n = n_0$ such that

$$\frac{\varepsilon_1}{1+p_1} + |p_1|^{n_0}\, g_0 < \varepsilon.$$

Then from (12) in view of the last relation we have $\lim\limits_{t\to\infty} g(t) = 0$.

ii) Let (11) hold. Then from $p(t)\, g(t-c) = f(t) - g(t)$ with regard to (11) we get

$$g(t) \leq \frac{1}{p_2}\,(f(t+c) - g(t+c)), \quad t \geq t_0 + 2\,c.$$

By iteration for sufficiently large t we have

$$g(t) \leq \frac{1}{p_2}\, f(t+c) - \frac{1}{p_2^2}\, f(t+2\,c) + \cdots + (-1)^{n-1}\, \frac{1}{p_2^n}\, f(t+n\,c) + (-1)^n\, \frac{1}{p_2^n}\, g(t+n\,c).$$

In view of $\lim\limits_{t\to\infty} f(t) = 0$, for any $\varepsilon_1 > 0$ there exists sufficiently large T such that $|f(t)| < \varepsilon_1$, for $t \geq T$. Then

$$|g(t)| \;\leq\; \frac{\varepsilon_1}{|p_2| - 1} \;+\; \frac{g_0}{|p_2|^n}.$$

Then analogously as in case i) we obtain $\lim\limits_{t\to\infty} g(t) = 0$. $\square$

Lemma 4. [9, Lemma 2] *Let* $w \in C([t_0,\infty))$, $v \in C^1([t_0,\infty))$ *and there exists* $\lim_{t\to\infty}[w(t)\,v'(t) + v(t)]$ *in the extended real line* $R^{\#}$. *Then* $\lim_{t\to\infty} v(t)$ *exists in* $R^{\#}$.

Theorem 3. *Let* (C_3) *hold. In addition we suppose that*

$$\int_{t_0}^{\infty} q_+(t)\,dt = \infty \quad \text{and} \tag{12}$$

$$\int_{t_0}^{\infty} q_-(t)\,dt < \infty. \tag{13}$$

Then every bounded solutions of (A_0) *is oscillatory, or* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Let $x(t)$ be a bounded nonoscillatory solution of (A_0) on $[T_0,\infty)$. Without loss of generality we suppose that $x(t-\tau) > 0$, $x(h(t))) > 0$ on $[t_1,\infty)\,, t_1 \ge T_0+\tau$. Analogously as in the proof of Theorem 1 there exist $K > 0$ and $t_2 \ge t_1$ such that $|f(x(h(t)))| \le K$ for all $t \ge t_2$. Then the inequality $(\bar{A}_0)$ in view of (9) we can writte in the form

$$z''(t) + q_+(t)\,f(x(h(t))) - K\,q_-(t) \le 0, \quad \text{for} \quad t \ge t_2. \tag{14}$$

With regard to (13) there exists a $L > 0$ such that $\int_{t_2}^{\infty} q_-(t)\,dt = L$. Then (14) via the estimation (9) we have $z'(t) \le z'(t_2) + K\,L$, i.e. $z'(t)$ is bounded above. If $\int_{t_0}^{\infty} q_+(t)\; f(x(h(t)))\; dt = \infty$, then the estimation (14) implies that $\lim_{t\to\infty} z'(t) = -\infty$ and therefore $\lim_{t\to\infty} z(t) = -\infty$. This in view of Lemma 1 and (C_3) contradicts the fact that $x(t)$ is bounded on $[T_0,\infty)$. Therefore

$$\int_{t_0}^{\infty} q_+(t)\; f(x(h(t)))\; dt < \infty. \tag{15}$$

Then (15) in view of (12) and the properties of functions f and h implies that

$$\liminf_{t\to\infty} x(t) = 0. \tag{16}$$

The proof of Theorem 3 is complete. □

Now we consider the equation

$$z''(t) + q(t)\,f(x(h(t))) = 0, \quad t \ge t_0 > 0, \tag{E}$$

as a special case of (A_0).

Theorem 4. *Let either (10) or*

$$-\infty < p_3 \le p(t) \le p_2 < -1. \tag{17}$$

hold. In addition we suppose that

$$\int_{t_0}^{\infty} t\, q_+(t)\, dt = \infty \quad \text{and} \tag{18}$$

$$\int_{t_0}^{\infty} t\, q_-(t)\, dt < \infty. \tag{19}$$

Then every bounded solutions of (E) is either oscillatory, or $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} z^{(k)}(t) = 0,\ k = 0, 1.$

Proof. Let $x(t)$ be a bounded and positive solution of (E) on $[T_0, \infty)$. Without loss of the generality we suppose that $x(t-\tau) > 0$ and $x(h(t)) > 0$ on $[t_1, \infty)$, $t_1 \ge T_0 + \tau$. Multiplying (E) by t and then integratimg from t_2 to s, we have

$$u(s) = \int_{t_2}^{s} t\, z''(t)\, dt \ = \int_{t_2}^{s} t\, q_-(t)\, f(x(h(t)))\, dt - \int_{t_2}^{s} t\, q_+(t)\, f(x(h(t)))\, dt. \tag{20}$$

If $\int_{t_2}^{\infty} t\, q_+(t)\, f(x(h(t)))\, dt = \infty$, then in view of (19) and the boundedness of $x(t)$, from (20) we get $\lim_{s\to\infty} u(s) = -\infty$. By Lemma 4 there exists $\lim_{s\to\infty} z(s) = z_0 \in R^{\#}$. Let $|z_0| < \infty$. Then $\lim_{s\to\infty} u(s) = -\infty$ implies $\lim_{s\to\infty} s\, z'(s) = -\infty$. From this relations we get that $\lim_{s\to\infty} z(s) = -\infty$, which contradicts the fact that $|z_0| < \infty$. Therefore $\lim_{s\to\infty} |z(s)| = \infty$. This in view of Lemma 1 gives a contradiction to the fact that $x(t)$ is bounded. Therefore

$$\int_{t_2}^{\infty} t\, q_+(t)\, f(x(h(t)))\, dt < \infty. \tag{21}$$

Then (21) in view of (18) and the property of f and h implies that (16) holds.

Now, letting $s \to \infty$ in (20), then using the boundedness of $x(t)$, (19),(21) and the property of f, we have

$$\lim_{s\to\infty} (s\, z'(s) - z(s)) = L_1, \quad |L_1| < \infty. \tag{22}$$

With regard to Lemma 4 and the fact that $z(t)$ is bounded we obtain that $\lim_{t\to\infty} z(t) = L$, $|L| < \infty$. Then if we use either (10) or (17), (16) and Lemma 2 we obtain that $L = 0$. From (22) in view of $L = 0$ we get that $\lim_{t\to\infty} z'(t) = 0$. We proved that $\lim_{t\to\infty} z^{(k)}(t) = 0$, $k = 0, 1$. Then if we use Lemma 3 we hawe $\lim_{t\to\infty} x(t) = 0$.

The proof of Theorem 4 is complete. □

References

[1] Bainov, D. D., Mishev, D. P., *Oscillation Theory for Neutral Equations with Delay*, Adam Hilger IOP Pablisching Ltd. (1991) 288pp..

[2] Grammatikopoulos, M. K., Grove, E. A., Ladas, G., *Oscillation and asymptotic behavior of second order neutral differential equations with deviating arguments*, Canad. Math. Soc. V8 (1967) 153–161.

[3] Graef, J. R., Grammatikopoulos, M. K., Spikes, P. W., *Asymptotic Properties of Solutions of Neutral Delay Differential Equations of the Second Order*, Radovi Matematički V_4 (1988) 113 –149.

[4] Graef, J. R., Grammatikopoulos, M. K., Spikes, P. W., *On the Asymptitic Behavior of Solutions of Second Order Nonlinear Neutral Delay Differential Equations,*, Journal Math. Anal. Appl. V156 N_1 (1991) 23–39.

[5] Graef, J. R., Grammatikopoulos, M. K., Spikes, P. W., *Asymptotic Behavior of Nonoscillatory Solutions of Neutral Delay Differential Equations of Arbitrary Order*, Nonlinear Analysis, Theory, Math., Appl. V21, N1 (1993) 23–42.

[6] Györi, I., Ladas, G., *Oscillation Theory of Delay Differential Equations*, Clear. Press., Oxford (1991) 368pp.

[7] Jaroš, J., Kussano, T., *Sufficient conditions for oscillations of higher order linear functional differential equations of neutral type*, Japan J. Math.15 (1989) 415–432.

[8] Jaroš, J., Kusano, T., *Oscillation properties of first order nonlinear functional differential equations of neutral type*, Diff. and Int. Equat. (1991) 425–436.

[9] Kusano, T., Onose, H., *Nonoscillation theorems for differential equation with deviating argument*, Pacific J. Math. 63, N_1 (1976) 185–192.

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