ARCHIVUM MATHEMATICUM (BRNO)
Tomus 31 (1995), 65 – 74

DIAMOND IDENTITIES FOR RELATIVE CONGRUENCES

Gábor Czédli

Abstract. For a class $\mathcal{K}$ of structures and $A \in \mathcal{K}$ let $\mathrm{Con}^*(A)$ resp. $\mathrm{Con}^{\mathcal{K}}(A)$ denote the lattices of $*$- congruences resp. $\mathcal{K}$-congruences of A, cf. Weaver [25]. Let $\mathrm{Con}^*(\mathcal{K}) := I\{\mathrm{Con}^*(A)\colon\ A \in \mathcal{K}\}$ where I is the operator of forming isomorphic copies, and $\mathrm{Con}^r(\mathcal{K}) := I\{\mathrm{Con}^{\mathcal{K}}(A)\colon\ A \in \mathcal{K}\}$. For an ordered algebra A the lattice of order congruences of A is denoted by $\mathrm{Con}^{\leq}(A)$, and let $\mathrm{Con}^{\leq}(\mathcal{K}) := I\{\mathrm{Con}^{\leq}(A)\colon\ A \in \mathcal{K}\}$ if $\mathcal{K}$ is a class of ordered algebras. The operators of forming subdirect squares and direct products are denoted by Q^s and P, respectively. Let λ be a lattice identity and let Σ be a set of lattice identities. Let $\Sigma \models_c \lambda\ (r; Q^s, P)$ denote that for every class $\mathcal{K}$ of structures which is closed under Q^s and P if Σ holds is $\mathrm{Con}^r(\mathcal{K})$ then so does λ. The consequence relations $\Sigma \models_c \lambda\ (*; Q^s)$, $\Sigma \models_c \lambda\ (\leq; Q^s)$ and $\Sigma \models_c \lambda\ (H, S, P)$ are defined analogously; the latter is the usual consequence relation in congruence varieties (cf. Jónsson [19]), so it will also be denoted simply by $\models_c$. If $\Sigma \not\models \lambda$ (in the class of all lattices) then the above-mentioned consequences are called nontrivial. The present paper shows that if $\Sigma \models$ modularity and $\Sigma \models_c \lambda$ is a known result in the theory of congruence varieties then $\Sigma \models_c \lambda\ (*; Q^s)$, $\Sigma \models_c \lambda\ (\leq; Q^s)$ and $\Sigma \models_c \lambda\ (r; Q^s, P)$ as well. In most of these cases λ is a diamond identity in the sense of [3].

1. Introduction

For a class $\mathcal{K}$ of algebras let $\mathrm{Con}(\mathcal{K})$ denote $I\{\mathrm{Con}(A)\colon\ A \in \mathcal{K}\}$, i.e. the class of lattices isomorphic to congruence lattices of algebras in $\mathcal{K}$. If $\mathcal{K}$ is a variety then the lattice variety generated by $\mathrm{Con}(\mathcal{K})$ is called a *congruence variety*, cf. Jónsson [19]. For a lattice identity λ and a set of lattice identities Σ, Σ is said to imply λ in congruence varieties, in notation $\Sigma \models_c \lambda$, if every congruence variety that satisfies (every member of) Σ also satisfies λ. If, in addition, Σ does not imply λ in all lattices, in notation $\Sigma \not\models \lambda$, then the consequence relation $\Sigma \models_c \lambda$ is called nontrivial. Many nontrivial results of the form $\{\sigma\} \models_c \lambda$ have appeared so far, cf., e.g., Nation [22], Day and Freese [8], Freese, Herrmann and Huhn [11] and Jónsson [19]; for a more detailed list and a survey cf. Jónsson [19]. These results

1991 *Mathematics Subject Classification*: Primary 08B10, Secondary 06C05.
Key words and phrases: Congruence variety, relative congruence, ordered algebra, von Neumann frame, lattice identity.
Research partially supported by Hungarian National Foundation for Scientific Research (OTKA) grant no. 1903
Received February 4, 1994.

state that certain lattice identities are equivalent to the modular or distributive law in congruence varieties. Another kinds of $\models_c$ results are given in [4], where infinitely many nontrivial $\{\sigma_i\} \models_c \lambda_i$ are established such that the λ_i are pairwise non-equivalent even in congruence varieties.

The aim of the present paper is to generalize these results for more general situations. Therefore we will consider structures (i.e., nonempty sets equipped with operations and relations, cf. Weaver [25] for an overview), not only algebras. The operators of forming subdirect squares, direct products and isomorphic copies will be denoted by Q^s, P and I, respectively. The relations on direct products are defined by componentwise, while the relations for substructures (or subdirect products) are obtained via restriction to their base set. Another way of generalization is to consider Q^s-closed classes $\mathcal{K}$ instead of varieties. Let $\mathcal{K}$ be a class of similar structures and $A, B \in \mathcal{K}$. A map φ: $A \to B$ is called a *homomorphism* if it commutes with the fundamental operations and for any relation symbol ρ and arbitrary $a_1, \dots, a_n \in A$ if $\rho_A(a_1, \dots, a_n)$ then $\rho_B(\varphi(a_1), \dots, \varphi(a_n))$. Given $A \in \mathcal{K}$, the kernels of homomorphisms from A into other structures in $\mathcal{K}$ are called $\mathcal{K}$-congruences or *relative congruences* of A. Let $\mathrm{Con}^{\mathcal{K}}(A)$ be the set of $\mathcal{K}$-congruences of A. The proof of Theorem 3 in Weaver [25] shows that $\mathrm{Con}^{\mathcal{K}}(A)$ is a lattice (with respect to inclusion) provided $\mathcal{K}$ is closed under direct products. Therefore $\mathrm{Con}^r(\mathcal{K}) := I\{\mathrm{Con}^{\mathcal{K}}(A)\colon A \in \mathcal{K}\}$ is a class of lattices when $\mathcal{K}$ is P-closed. Considering $\mathrm{Con}^r(\mathcal{K})$ instead of $\mathrm{Con}(\mathcal{K})$ offers us the third way of generalization. For a structure A an equivalence relation Θ of A is called a $*$-congruence of A if Θ is a congruence in the algebraic sense and for any k-ary relation symbol ρ and any $\langle a_1, b_1\rangle, \dots, \langle a_k, b_k\rangle \in \Theta$ we have

$$\rho_A(a_1, \dots, a_k) \iff \rho_A(b_1, \dots, b_k),$$

cf. Weaver [25]. The $*$-congruences of A form a sublattice of the equivalence lattice of A; this lattice will be denoted by $\mathrm{Con}^*(A)$. For an *algebra* A we have $\mathrm{Con}^*(A) = \mathrm{Con}(A)$.

A triple $\langle A; F; \leq\rangle$ is called an *ordered algebra* if $\langle A; F\rangle$ is an algebra, $\langle A; \leq\rangle$ is a partially ordered set, and every $f \in F$ is monotone with respect to $\leq$. Varieties of ordered algebras were studied e.g. in Bloom [1]. In case of ordered algebras, the monotone and operation-preserving maps are called *homomorphisms*, and their kernels are called *order congruences*. Given an ordered algebra A, the set $\mathrm{Con}^{<}(A)$ of of order congruences of A is an algebraic lattice. (This was proved in [6], where an inner definition of order congruences and a description of their join is also given.) For a class $\mathcal{K}$ of ordered algebras let $\mathrm{Con}^{<}(\mathcal{K}) := I\{\mathrm{Con}^{<}(A)\colon A \in \mathcal{K}\}$. For a class $\mathcal{K}$ of ordered algebras and $B \in \mathcal{K}$ the lattices $\mathrm{Con}^{<}(B)$, $\mathrm{Con}^*(B)$ and $\mathrm{Con}^{\mathcal{K}}(B)$ are pairwise different in general, even if $\mathcal{K}$ is closed under P and Q^s.

We will investigate three further consequence relations among lattice identities. Let λ be a lattice identity and let Σ be a set of lattice identities. Let $\Sigma \models_c \lambda\ (r; Q^s, P)$ resp. $\Sigma \models_c \lambda\ (*; Q^s)$ resp. $\Sigma \models_c \lambda\ (\leq; Q^s)$ denote that for every class $\mathcal{K}$ of structures which is closed under Q^s and P resp. every Q^s-closed class $\mathcal{K}$ of structures resp. every Q^s-closed class $\mathcal{K}$ of ordered algebras if Σ holds in

$\mathrm{Con}^r(\mathcal{K})$ resp. $\mathrm{Con}^*(\mathcal{K})$ resp. $\mathrm{Con}^<(\mathcal{K})$ then so does λ. According to the notations above, $\models_c$ could be denoted by $\models_c \ (H, S, P)$. The reader will certainly notice by the end of the paper that the Q^s-closedness of $\mathcal{K}$ could be replaced by the following weaker assumption: "if $A \in \mathcal{K}$ and α is a congruence (of the respective type) of A then α, as a subalgebra of A^2, belongs to $\mathcal{K}$.

Clearly, $\Sigma \models_c \lambda$ follows from any of the above-defined three consequence relations. Our goal is to prove the converse under reasonable restrictions. I.e., we want to turn a lot of $\Sigma \models_c \lambda$ results into $\Sigma \models_c \lambda \ (r; Q^s, P)$, $\Sigma \models_c \lambda \ (*; Q^s)$ and $\Sigma \models_c \lambda \ (\leq; Q^s)$ statements. The proofs of the classical $\Sigma \models_c \lambda$ results often involve particular tools. For example, free algebras are used in Day and Freese [8, p. 1156] or Jónsson [19, p. 379]; Mal'cev conditions are used in Day [7] and Mederly [21], and even commutator theory is required in [3]. The scope of these tools is often extended far beyond varieties of algebras. There are free structures and there are Mal'cev conditions for $*$-congruences, cf. Weaver [25]. Free ordered algebras and some Mal'cev conditions are available for ordered algebras (cf. Bloom [1] and [6]). The methods used in [2] and [5] also indicate that certain $\models_c$ results can be generalized. Even commutator theory has been developed for relative congruences of quasivarieties of algebras and some Mal'cev-like conditions are also available, cf. Kearnes and McKenzie [20], Dziobiak [9] and Nurakunov [23]. However, all these recent developments are insufficient for our purposes as they require much stronger closedness assumption on $\mathcal{K}$.

Fortunately, some of the known $\Sigma \models_c \lambda$ results, namely those in Freese and Jónsson [12] and Freese, Herrmann and Huhn [11], are in fact $\Sigma \models_c \lambda \ (Q^s)$ results, and we will not have much difficulty in generalizing them. In presence of modularity, the rest of the known $\Sigma \models_c \lambda$ results can be, at least in principle, deduced from [3]. Since [3] relies on commutator theory, our main achievement is the generalizing [3] so that commutator theory be avoided.

Let dist resp. mod stand for the distributive resp. modular law. Although the usage of "known" hurts mathematical rigorousity below, it is time to indicate that our aim is to prove the following

Proposition 1. *Suppose $\Sigma \models_c \lambda$ is a known result in the theory of congruence varieties and $\Sigma \models$ mod . Then $\Sigma \models_c \lambda \ (r; Q^s, P)$, $\Sigma \models_c \lambda \ (*; Q^s)$ and $\Sigma \models_c \lambda \ (\leq; Q^s)$.*

We do not know if $\Sigma \models$ mod can be omitted or "known result" can be replaced by "true statement" in Proposition 1. If the answer were affirmative in both cases then the congruence varieties would form a lattice, cf. [5].

2. Preliminary lemmas and main results

For structures A and B a homomorphism $\varphi\colon\ A \to B$ is said to be *$*$-homomorphism* if for every relation symbol ρ and $a_1, \dots, a_k \in A$ we have

$$\rho_A(a_1, \dots, a_k) \iff \rho_B(\varphi(a_1), \dots, \varphi(a_k)).$$

It is easy to see, cf. Weaver [25], that $*$-congruences are precisely the kernels of $*$-homomorphisms. A homomorphism resp. $*$-homomorphism $\varphi\colon\ A \to A$ is called a *retraction* resp. $*$-*retraction* if $\varphi \circ \varphi = \varphi$. The retraction of an ordered algebra is defined analogously; then φ must be monotone, of course. If $\varphi\colon A \to A$ is a retraction then $B := \varphi(A)$ is called a retract of A. (The relations on B are defined as the restrictions of the relations on A.) Associated with this φ we have a map $\hat\varphi$ from the set of equivalences of B into the set of equivalences of A defined by $\hat\varphi(\Theta) = \{\langle a, b\rangle \in A^2\colon \langle \varphi(a), \varphi(b)\rangle \in \Theta\}$. In the sequel, the restriction of $\hat\varphi$ to $\mathrm{Con}^*(B)$, $\mathrm{Con}^r(B)$ or $\mathrm{Con}^<(B)$ will also be denoted by $\hat\varphi$.

Lemma 1. *Suppose $\varphi\colon A \to A$ is a retraction, $A \in \mathcal{K}$, and $B = \varphi(A)$.*

(A) *If φ is a $*$-retraction then $\hat\varphi\colon \mathrm{Con}^*(B) \to \mathrm{Con}^*(A)$*
(B) *If $B \in \mathcal{K}$ and $\mathcal{K}$ is P-closed then $\hat\varphi\colon \mathrm{Con}^{\mathcal{K}}(B) \to \mathrm{Con}^{\mathcal{K}}(A)$*
(C) *If A is an ordered algebra and φ is monotone then $\hat\varphi\colon \mathrm{Con}^<(B) \to \mathrm{Con}^<(A)$*

is a lattice embedding.

Proof. Since the meet coincides with the intersection, it is evident that $\hat\varphi$ is a meet-homomorphism in all the three cases. If Θ is an equivalence on B and $a, b \in B$ then $\langle a, b\rangle \in \hat\varphi(\Theta) \iff \langle \varphi(a), \varphi(b)\rangle \in \Theta \iff \langle a, b\rangle = \langle \varphi(a), \varphi(b)\rangle \in \Theta$, thus $\hat\varphi$ is injective. The treatment for joins is more or less the same for all the three cases, thus we detail (B) only. Assume that for $C, D, E \in \mathcal{K}$ and homomorphisms $\alpha\colon B \to C$, $\beta\colon B \to D$ and $\gamma\colon B \to E$ we have $\mathrm{Ker}\alpha \vee \mathrm{Ker}\beta = \mathrm{Ker}\gamma$ in $\mathrm{Con}^{\mathcal{K}}(B)$. Then $\hat\varphi(\mathrm{Ker}\alpha) = \mathrm{Ker}(\alpha\circ\varphi)$, $\hat\varphi(\mathrm{Ker}\beta) = \mathrm{Ker}(\beta\circ\varphi)$ and $\hat\varphi(\mathrm{Ker}\gamma) = \mathrm{Ker}(\gamma\circ\varphi)$. Since $\hat\varphi$ is monotone, $\hat\varphi(\mathrm{Ker}\alpha) \le \hat\varphi(\mathrm{Ker}\gamma)$ and $\hat\varphi(\mathrm{Ker}\beta) \le \hat\varphi(\mathrm{Ker}\gamma)$. Now let $\delta\colon A \to F$ be an arbitray homomorphism such that $F \in \mathcal{K}$, $\mathrm{Ker}\delta \supseteq \hat\varphi(\mathrm{Ker}\alpha) = \mathrm{Ker}(\alpha\circ\varphi)$ and $\mathrm{Ker}\delta \supseteq \hat\varphi(\mathrm{Ker}\beta) = \mathrm{Ker}(\beta\circ\varphi)$; we have to show that $\mathrm{Ker}\delta \supseteq \mathrm{Ker}(\gamma\circ\varphi)$. Suppose $\langle a, b\rangle \in \mathrm{Ker}(\gamma\circ\varphi)$ for some $a, b \in A$. Since $\langle \varphi(\varphi(a)), \varphi(a)\rangle = \langle \varphi(a), \varphi(a)\rangle \in \mathrm{Ker}\alpha$, we have $\langle \varphi(a), a\rangle \in \mathrm{Ker}(\alpha\circ\varphi) \subseteq \mathrm{Ker}\delta$. Similarly, $\langle \varphi(b), b\rangle \in \mathrm{Ker}\delta$. Now consider the restriction $\delta|_B\colon B \to F$, which is a homomorphism. If $c, d \in B$ and $\langle c, d\rangle \in \mathrm{Ker}\alpha$ then $\langle c, d\rangle = \langle \varphi(c), \varphi(d)\rangle \in \mathrm{Ker}(\alpha\circ\varphi) \subseteq \mathrm{Ker}\delta$. Thus $\mathrm{Ker}\alpha \subseteq \mathrm{Ker}(\delta|_B)$, and $\mathrm{Ker}\beta \subseteq \mathrm{Ker}(\delta|_B)$ comes similarly. Therefore $\mathrm{Ker}\gamma \subseteq \mathrm{Ker}(\delta|_B)$. From $\langle a, b\rangle \in \mathrm{Ker}(\gamma\circ\varphi)$ we infer $\langle \varphi(a), \varphi(b)\rangle \in \mathrm{Ker}\gamma \subseteq \mathrm{Ker}(\delta|_B) \subseteq \mathrm{Ker}\delta$, and $\langle a, b\rangle \in \mathrm{Ker}\delta$ follows by transitivity. Therefore $\hat\varphi$ is a $\vee$-homomorphism, and (B) is proved.

The arguments for (A) resp. (C) are quite analogous: we have to use $*$-homomorphisms resp. monotone homomorphisms, and A, B, C, D, E, F will be arbitrary structures resp. arbitrary ordered algebras, not necessarily in $\mathcal{K}$. $\square$

The amalgamation property we are going to consider first appeared in Freese and Jónsson [12], and played a central role in Freese, Herrmann and Huhn [11].

Definition. A class $\mathcal{C}$ of lattices is said to satisfy the Freese—Jónsson amalgamation property, in short FJAP, if for each $L \in \mathcal{C}$ and $a \in L$ there exists an $M \in \mathcal{C}$ and embeddings φ_1, φ_2 of L in M such that

(a) $\varphi_1(x) = \varphi_2(x)$ for all $x \ge a$ in L,
(b) $\varphi_1(x) \vee \varphi_2(x) = \varphi_1(a)$ for all $x \le a$ in L, and
(c) $\varphi_i(y) \vee (\varphi_1(x) \wedge \varphi_2(x)) = \varphi_i(x)$ for all $y \le x$ in L and $i = 1, 2$.

Lemma 2. *Let $\mathcal{C}$ be one of the following classes:*

(A) $\mathrm{Con}^*(\mathcal{K})$ *where $\mathcal{K}$ is a Q^s-closed class of structures;*
(B) $\mathrm{Con}^r(\mathcal{K})$ *where $\mathcal{K}$ is a class of structures closed under P and Q^s;*
(C) $\mathrm{Con}^<(\mathcal{K})$ *where $\mathcal{K}$ is a Q^s-closed class of ordered algebras.*

Then $\mathcal{C}$ satisfies FJAP.

Proof. The construction needed by the proof of this lemma is the same as that for a Q^s-closed class of algebras (cf. Freese and Jónsson [12] or Hagemann and Herrmann [13]). We give the details in case (B) only. Suppose $C \in \mathcal{K}$ and $\alpha \in \mathrm{Con}^{\mathcal{K}}(C)$. Let $A := \{\langle x, y\rangle \in C^2\colon\ x\alpha y\}$. Since A is a subdirect square of C, it belongs to $\mathcal{K}$. Let ι denote the embedding $C \to A$, $x \mapsto \langle x, x\rangle$, and denote $\iota(C)$ by B. Then $\iota\colon\ C \to B$ is an isomorphism, which induces an isomorphism, also denoted by ι, from $\mathrm{Con}^r(C)$ to $\mathrm{Con}^r(B)$. Let ψ_i be the retraction $A \to B$, $\langle x_1, x_2\rangle \mapsto \langle x_i, x_i\rangle$. Then $\hat{\psi}_i\colon\ \mathrm{Con}^{\mathcal{K}}(B) \to \mathrm{Con}^{\mathcal{K}}(A)$ is an embedding by Lemma 1. Therefore $\hat{\psi}_i \circ \iota\colon\ \mathrm{Con}^{\mathcal{K}}(C) \to \mathrm{Con}^{\mathcal{K}}(A)$, $\Theta \mapsto \Theta_i := \{\langle\langle x_1, x_2\rangle, \langle y_1, y_2\rangle\rangle \in A^2\colon\ x_i \Theta y_i\}$ is a lattice embedding for $i = 1, 2$. For $\Theta \geq \alpha$, $\Theta_1 = \Theta_2$ is obvious. For $\Theta \leq \alpha$ it is easy to see that $\Theta_1 \circ \Theta_2 \supseteq \alpha_1 = \alpha_2$, thus we obtain that $\alpha_1 \subseteq \Theta_1 \circ \Theta_2 \subseteq \Theta_1 \vee_{\mathcal{K}} \Theta_2 \subseteq \alpha_1 \vee_{\mathcal{K}} \alpha_1 = \alpha_1$, showing (b) in the definition of FJAP. (Here $\vee_{\mathcal{K}}$ stands for the join taken in $\mathrm{Con}^{\mathcal{K}}(A)$.) Now let $i \in \{1, 2\}$ and $\Theta \subseteq \Psi \in \mathrm{Con}^{\mathcal{K}}(C)$. Then $\Psi_i \subseteq 0_i \circ (\Psi_1 \cap \Psi_2) \circ 0_i$ where 0 denotes the smallest (relative) congruence of C. Indeed, e.g. for $i = 1$, if $\langle x_1, x_2\rangle \Psi_1 \langle y_1, y_2\rangle$ then $\langle x_1, x_2\rangle 0_1 \langle x_1, x_1\rangle \Psi_1 \cap \Psi_2 \langle y_1, y_1\rangle 0_1 \langle y_1, y_2\rangle$. Therefore $\Psi_i \subseteq 0_i \circ (\Psi_1 \cap \Psi_2) \circ 0_i \subseteq 0_i \vee_{\mathcal{K}} (\Psi_1 \wedge \Psi_2) \subseteq \Theta_i \vee_{\mathcal{K}} (\Psi_1 \wedge \Psi_2) \subseteq \Psi_i \vee_{\mathcal{K}} (\Psi_i \wedge \Psi_i) = \Psi_i$, proving (c) in the definition of FJAP. This completes the proof of (B). The arguments for (A) resp. (C) are analogous, for ψ_i becomes a $*$-retraction resp. monotone retraction. □

Given a ring R with 1, let $H\mathcal{L}(R)$ denote the class of homomorphic images of lattices embeddable in the submodule lattice of (unitary left) R-modules. $H\mathcal{L}(R)$ is just the congruence variety $HSP(\mathrm{Con}(R\text{-}\mathbf{Mod}))$. For integers $m \geq 0$ and $n \geq 1$ let $D(m, n)$ denote the ring sentence $(\exists r)(m \cdot r = n \cdot 1)$. (Here 1 is the ring unit and $k \cdot x = x + x + \ldots + x$, k times.) $D(m, n)$ is called a divisibility condition. In [18] an algorithm is given which associates a pair $\langle m_\varepsilon, n_\varepsilon\rangle$ of integers, $m_\varepsilon \geq 0$, $n_\varepsilon \geq 1$, with an arbitrary lattice identity ε such that for any R we have

Theorem A. *ε holds in $H\mathcal{L}(R)$ iff $D(m_\varepsilon, n_\varepsilon)$ holds in R.*

Let $V(0) := H\mathcal{L}(\mathbf{Q})$, i.e., the lattice variety generated by the rational projective geometries. For $k > 0$ let $V(k) := H\mathcal{L}(\mathbf{Z}_k)$ where $\mathbf{Z}_k$ is the factor ring of integers modulo k. For a nonnegative integer k and a prime p let $\mathrm{expt}(k, p)$ denote the largest integer $i \geq 0$ for which $p^i \mid k$; by $\mathrm{expt}(0, p)$ we mean the smallest infinite ordinal ∞. From [18, Prop. 1] we invoke

Theorem B. *$D(m, n)$ holds in a ring R iff for any prime p with $\mathrm{expt}(m, p) > \exp(n, p)$ R satisfies $D(p^{\mathrm{expt}(n,p)+1}, p^{\mathrm{expt}(n,p)})$ and, in addition, $m = 0$ implies that the characteristic of R is not 0. In case the the characteristic of R is $k > 0$ then $D(m, n)$ holds in R iff $(m, k) \mid n$.*

For technical reasons, in connection with Theorem B, we define $G(m,n) := \{p^{i+1}\colon\ p \text{ prime},\ i = \operatorname{expt}(n,p) < \operatorname{expt}(m,p)\} \cup \{i\colon\ i = 0 = m\}$, $\quad m \geq 0,\ n \geq 1$. Note that $\{i\colon\ i = 0 = m\}$ is $\{0\}$ or $\emptyset$, and $G(m,n) = \emptyset$ if m divides n.

For $n \geq 2$, an n-diamond in a modular lattice L is defined to be an $(n+1)$-tuple $\vec{a} = \langle a_0, a_1, \dots, a_n\rangle \in L^{n+1}$ satisfying $\bigvee_{i\neq j}^{0,n} a_i = 1_{\vec{a}}$ and $a_\ell \wedge \bigvee_{i\neq k,\ell}^{0,n} a_i = 0_{\vec{a}}$ for all j and all $k \neq \ell$, where $1_{\vec{a}} = \bigvee_i^{0,n} a_i$ and $0_{\vec{a}} = \bigwedge_i^{0,n} a_i$. This concept is due to András Huhn [16], [15] (who calls it an $(n-1)$-diamond.) Let λ: $p(x_1,\dots,x_t) = q(x_1,\dots,x_t)$ be a lattice identity. We call λ a *diamond identity*, cf. [3] and [4], if λ implies modularity and, in addition, there are $(n+1)$-ary lattice terms $c_1(y_0,y_1,\dots,y_n),\dots,c_t(y_0,y_1,\dots,y_n)$ for some $n \geq 2$ such that for an arbitrary modular lattice L if $p(c_1(\vec{a}),\dots,c_t(\vec{a})) = q(c_1(\vec{a}),\dots,c_t(\vec{a}))$ for every n-diamond $\vec{a}$ in L then λ holds in L. Some examples are listed in [3, p. 291].

Our main result is the following

Theorem 1. *Let Σ be a set of lattice identities with $\Sigma \models$ modularity and let λ be a diamond identity. Then the following five conditions are equivalent*

(i) $\Sigma \models_c \lambda$,
(ii) $\Sigma \models_c \lambda\ (*; Q^s)$,
(iii) $\Sigma \models_c \lambda\ (r; Q^s, P)$,
(iv) $\Sigma \models_c \lambda\ (\leq; Q^s)$,
(v) $\{0\} \cap \{m_\lambda\} \subseteq \{m_\varepsilon\colon\ \varepsilon \in \Sigma\}$, *and for any prime p if* $\operatorname{expt}(m_\lambda,p) > \operatorname{expt}(n_\lambda,p)$ *then* $\operatorname{expt}(n_\lambda,p) \geq \operatorname{expt}(n_\varepsilon,p) < \operatorname{expt}(m_\varepsilon,p)$ *holds for some* $\varepsilon \in \Sigma$.

The equivalence of (i) and (v) was established in [3]. To unify the treatment for several kinds of congruences, another consequence relation is worth introducing. Let T be a "set" of lattice varieties. We say that $\Sigma \models^T \lambda$ if for every $U \in T$ if Σ holds in U then so does λ. Now, in virtue of Lemma 2, Theorem 1 will clearly follow from

Theorem 2. *Let Σ be a set of lattice identities with $\Sigma \models$ modularity and let λ be a diamond identity. Let T be a set of lattice varieties such that each U in T is generated by a class satisfying FJAP and $V(k) \in T$ for all $k \geq 0$. Then $\Sigma \models^T \lambda$ if and only if (v) of Theorem 1 holds.*

The key to this theorem is the following generalization of Freese [10] (when λ is the distributive law, cf. also Freese, Herrmann and Huhn [11, Cor. 14]) and [3, Thm. 1].

Theorem 3. *Let T be as in Theorem 2, and let $U \in T$. Suppose that a diamond identity λ does not hold in U and U consists of modular lattices. Then there is an h in $G(m_\lambda, n_\lambda)$ such that $V(h)$ is a subvariety of U.*

3. Further tools and proving the main results

For a prime power p^k let $R(p,k)$ denote $\mathbf{Z}_{p^k}$, the factor ring of integers modulo p^k. Let $R(p,\infty)$ denote the ring of rational numbers whose denominator is not divisible by p, and let $R(0,1) := \mathbf{Q}$, the ring of rational numbers. For any of these rings $R(u,v)$, let $L(u,v,n)$ be the lattice of submodules of ${}_{R(u,v)}R(u,v)^n$. One of the main tools we need is taken from Herrmann [14]:

Theorem C. *Every subdirectly irreducible modular lattice which is generated by an n-diamond is isomorphic or dually isomorphic to one of the following lattices: $L(p,k,n)$ for a prime power p^k, $L(p,\infty,n)$ for a prime p, or $L(0,1,n)$.*

Note that an important particular case of this theorem was proved in Herrmann and Huhn [15], which also could be used for our purposes in virtue of Freese, Herrmann and Huhn [11, Prop. 12].

Proof of Theorem 3. Suppose the assumptions of the Theorem hold, and let U_0 be a class of lattices which satisfies FJAP and generates the variety U. For a lattice identity ε let ε^{d} denote the dual of ε. For a prime p let $V(p^\infty) := H\mathcal{L}(R(p,\infty)$-$\mathbf{Mod})$. Then $V(p^k) = H\mathcal{L}(R(p,k)$-$\mathbf{Mod})$ for every $p \in \{0\} \cup \{\text{primes}\}$ and $1 \leq k \leq \infty$. Since λ fails in U, there is an integer $f > 1$, an $M = M_f \in U_0$, and an f-diamond $\vec{a}$ in M such that λ fails in the sublattice $L = L_f$ generated by (the elements $a_0, a_1, \ldots, a_f$ of) $\vec{a}$. By Freese, Herrmann and Huhn [11, Lemma 11], by the equivalence of n-diamonds with dual n-diamonds (cf. Huhn [17]) and by the equivalence of von Neumann n-frames with n-diamonds (cf. Herrmann and Huhn [15, (1.7)]) we obtain that for any integer $g \geq f$ there is a lattice $M_g \in U_0$, a sublattice L_g generated by a g-diamond in M_g and an embedding $\varphi\colon M \to M_g$ such that the restriction $\varphi|_L$ of φ is an $L \to L_g$ embedding. Clearly, for every $g \geq f$, λ fails in L_g and $L_g \in U$. Decomposing L_g as a subdirect product of subdirectly irreducible lattices, every factor will be generated by a g-diamond, namely by the image of the original diamond under the natural projection. These subdirect factors belong to U and at least one them fails λ. Therefore (up to notational changes) we may assume that the $L_g \in U$ are subdirectly irreducible.

By Hutchinson's duality result [18, Thm. 7] the congruence varieties $H\mathcal{L}(R)$ are selfdual lattice varieties. Therefore, thanks to congruence permutability and strong Mal'cev conditions associated with an arbitrary lattice identity ε and its dual (cf. Wille [26] or Pixley [24], or for a more explicit form [18, Thm. 1]),

(1) there is an integer $r(\varepsilon)$ such that, for any ring R, ε holds in $H\mathcal{L}(R)$ iff ε holds in $\mathrm{Con}({}_R R^n)$ for some $n \geq r(\varepsilon)$ iff ε^{d} holds in $\mathrm{Con}({}_R R^n)$ for some $n \geq r(\varepsilon)$.

For $b \in \{0,1,2,\ldots\} \cup \{p^\infty\colon\ p \text{ prime}\}$ and $a \in \{0,1,2,\ldots\}$ we define the "generalized greatest common divisor" as follows:

$$(a,b)' := \begin{cases} 0, & \text{if } b = 0 \text{ and } a = 0 \\ 1, & \text{if } b = 0 \text{ and } a > 0 \\ (a,b), & \text{if } b \in \{1,2,3,\ldots\} \\ p^{\mathrm{expt}(a,p)}, & \text{if } b = p^\infty \text{ and } a > 0 \\ p^\infty, & \text{if } b = p^\infty \text{ and } a = 0. \end{cases}$$

Note that $(-,-)'$ is not a commutative operation, and p^∞ divides no positive integer. Combining (1) and Theorems A and B we obtain for any $p \in \{0\} \cup \{\text{primes}\}$ and any $1 \le k \le \infty$:

(2) Suppose $n \ge r(\varepsilon)$. Then ε holds in $V(p^k)$ iff ε holds in $L(p,k,n)$ iff ε holds in the dual of $L(p,k,n)$ iff $D(m_\varepsilon, n_\varepsilon)$ holds in $R(p,k)$ iff $(m_\varepsilon, p^k)' \mid n_\varepsilon$.

By Theorem C, each of the $L_g \in U \quad (g \ge f)$ is of the form $L(p_g, k_g, g)^{u_g}$ where $p_g \in \{0\} \cup \{\text{primes}\}$, $1 \le k_g \le \infty$, $u_g \in \{0,1\}$, and $k_g = 1$ when $p_g = 0$. Here $L(p_g,k_g,g)^1 := L(p_g,k_g,g)$ and $L(p_g,k_g,g)^0 := L(p_g,k_g,g)^{\mathrm{d}}$, the dual of $L(p_g,k_g,g)$. Since λ fails in L_g, we conclude from (2) that

(3) For $g \ge f$ we have $(m_\lambda, p_g^{k_g})'$ does not divide n_λ.

For $q \in \{0\} \cup \{\text{primes}\}$ let $J_q := \{g\colon\ g \ge f \text{ and } p_g = q\}$. Now the proof ramifies depending on m_λ.

Assume first that $m_\lambda = 0$. Suppose J_0 is infinite, and let ε be an identity which holds in U. Then ε holds in $L(0,1,g)^{u_g}$ for infinitely many g. (2) yields that $m_\varepsilon > 0$, whence ε holds in $V(0)$ by (2). Thus $V(0) \subseteq U$, and $0 \in G(m_\lambda, n_\lambda)$.

Suppose J_q is infinite for some $q > 0$ and let $i := \operatorname{expt}(n_\lambda, q)$. Then $k_g > i$ for $g \in J_q$ and $q^{i+1} \in G(m_\lambda, n_\lambda)$ by (3). Suppose an identity ε holds in U. Taking a sufficiently large $g \in J_q$ we conclude from (2) that ε holds in $V(q^{k_g})$. But $(m_\varepsilon, q^{k_g})' \mid n_\varepsilon$ implies $(m_\varepsilon, q^{i+1})' \mid n_\varepsilon$, whence ε holds in $V(q^{i+1})$ by (2). This shows that $V(q^{i+1}) \subseteq U$.

Suppose now that J_q is finite for every $q \in \{0\} \cup \{\text{primes}\}$. Then $\{p_g\colon\ g \ge f\}$ is an infinite set of primes. By (2), no divisibility condition of the form $D(0,t)$ can hold in each of the rings $R(p_g,k_g) \quad (g \ge f)$. Consequently, if $m_\varepsilon = 0$ for a lattice identity ε then ε does not hold in U. Thus $m_\varepsilon > 0$ for all ε that hold in U, and these ε hold in $V(0)$ by (2). We have obtained that $V(0) \subseteq U$ and, of course, $0 \in G(m_\lambda, n_\lambda)$.

Now let us assume that $m_\lambda > 0$. First observe by Theorem B that for distinct primes p, q and any $0 \le k \le \infty$ the divisibility condition $D(q^{\ell+1}, q^\ell)$ holds in $R(p,k)$ for all $\ell \in \{0,1,2,\dots\}$. Hence, by (3), (2) and Theorem B, we conclude that, for every $g \ge f$, $\operatorname{expt}(m_\lambda, p_g) > \operatorname{expt}(n_\lambda, p_g)$ but $D(p_g^{\operatorname{expt}(n_\lambda,p_g)+1}, p_g^{\operatorname{expt}(n_\lambda,p_g)})$ fails in $R(p_g,k_g)$. Hence, by Theorem B, we conclude $i := \operatorname{expt}(p_g, n_\lambda) < k_g$ for all $g \ge f$. On the other hand, $\operatorname{expt}(m_\lambda, p) > \operatorname{expt}(n_\lambda, p)$ can hold for finitely many primes p only, whence there is a prime q such that J_q is infinite. I.e., U contains $L(q,k_g,g)^{u_g}$ for infinitely many g. Suppose ε holds in U and choose a $g \in J_q$ with $g \ge r(\varepsilon)$. From (2) we obtain $(m_\varepsilon, q^{k_g})' \mid n_\varepsilon$, whence $(m_\varepsilon, q^{i+1})' \mid n_\varepsilon$, implying that ε holds in $V(q^{i+1})$. We have obtained $V(q^{i+1}) \subseteq U$, and evidently q^{i+1} belongs to $G(m_\lambda, , n_\lambda)$. ☐

Proof of Theorem 2. Let us assume that $\Sigma \models^T \lambda$ and the conditions of the theorem are fulfilled. If $m_\lambda = 0$ but $m_\varepsilon > 0$ for all $\varepsilon \in \Sigma$ then, by Theorems A and B, Σ would hold but λ would fail in $V(0) \in T$. This is not the case and we conclude that $\{0\} \cap \{m_\lambda\} \subseteq \{m_\varepsilon\colon\ \varepsilon \in \Sigma\}$. If $\operatorname{expt}(m_\lambda, p) > \operatorname{expt}(n_\lambda, p) = i$ then, by Theorems A and B, λ and therefore Σ fails in $V(p^{i+1}) \in T$. Therefore, again by Theorems A and B, there exists an $\varepsilon \in \Sigma$ with $\operatorname{expt}(n_\lambda, p) = i \ge \operatorname{expt}(n_\varepsilon, p) <$

$\text{expt}(m_\varepsilon, p)$, proving (v).

Now assume that (v) holds but $\Sigma \models^T \lambda$ fails. Therefore there is a $U \in T$ such that λ fails in U but Σ holds in U. By Theorem 3, $V(h) \subseteq U$ for some $h \in G(m_\lambda, n_\lambda)$. Clearly, Σ holds in $V(h)$. If $h = 0 = m_\lambda$ then $m_\varepsilon = 0$ for some $\varepsilon \in \Sigma$ by (v). Hence, by Theorems A and B, ε cannot hold in $V(h)$. Therefore $h = p^{i+1}$ where $i = \text{expt}(n_\lambda, p) < \text{expt}(m_\lambda, p)$ for some p. By (v) there is an $\varepsilon \in \Sigma$ with $i \geq \text{expt}(n_\varepsilon, p) < \text{expt}(m_\varepsilon, p)$. Consequently, by Theorems A and B, ε cannot hold in $V(h)$; a contradiction again. □

4. The rest of the results

Most of the $\Sigma \models_c \lambda$ statements in the scope of Proposition 1 are settled by Theorem 1; there are only two exceptions, up to the author's present knowledge. It is shown in Freese and Jónsson [12] that mod $\models_c$ Arguesian law. In Freese, Herrmann and Huhn [11], some identities $\gamma_{n,m}(w_k)$ $(n$ odd, $n > 1$, $k > 1)$, even stronger than the Arguesian law, are constructed and it is shown that mod $\models_c$ $\gamma_{n,m}(w_k)$. Fortunately, the proof of these results is based on FJAP. Therefore Proposition 1 holds for these cases, too.

References

[1] Bloom, S. L., *Varieties of ordered algebras*, J. Comput. System Sci. **13** (1976), 200-212.

[2] Czédli, G., *Notes on congruence implications*, Archivum Mathematicum (Brno) **27** (1991), 149-153.

[3] Czédli, G., *How are diamond identities implied in congruence varieties*, Algebra Universalis **30** (1993), 291-293.

[4] Czédli, G., *Some nontrivial implications in congruence varieties*, Acta Sci. Math. (Szeged) **56** (1992), 15-18.

[5] Czédli, G., *On the lattice of congruence varieties of locally equational classes*, Acta Sci. Math. (Szeged) **41** (1979), 39-45.

[6] Czédli, G., Lenkehegyi, A., *On congruence n-distributivity of ordered algebras*, Acta Math. Hungar. **41** (1983), 17-26.

[7] Day, A., *p-modularity implies modularity in equational classes*, Algebra Universalis **3** (1973), 398-399.

[8] Day, A., Freese, R., *A characterization of identities implying congruence modularity I*, Canadian J. Math. **32** (1980), 1140-1167.

[9] Dziobiak, W., *Relative congruence-distributivity within quasivarieties of nearly associative φ-algebras*, Fund. Math. **135** (1990), 77-95.

[10] Freese, R., *The class of Arguesian lattices is not a congruence variety*, Notices Amer. Math. Soc. **23** (1976), #76T-A181.

[11] Freese, R., Herrmann, C., Huhn, A. P., *On some identities valid in modular congruence varieties*, Algebra Universalis **12** (1982), 322-334.

[12] Freese, R., Jónsson, B., *Congruence modularity implies the Arguesian identity*, Algebra Universalis **6** (1976), 225-228.

[13] Hagemann, J., Herrmann, C., *A concrete ideal multiplication for algebraic systems and its relation with congruence distributivity*, Arch. Math. **32** (1979), 234-245.

[14] Herrmann, C., *On the arithmetic of projective coordinate systems*, Trans. Amer. Math. Soc. **284** (1984), 759-785.

[15] Herrmann, C., Huhn, A. P., *Lattices of normal subgroups which are generated by frames*, Lattice Theory, Proc. Conf. Szeged 1974, Coll. Mat. Soc. J. Bolyai, vol. 14, North-Holland, Amsterdam, 1976, pp. 97-136.

[16] Huhn, A. P., *Schwach distributive Verbände I*, Acta Sci.Math. **33** (1972), 297-305.
[17] Huhn, A. P., *On Grätzer's problem concerning automorphisms of a finitely presented lattice*, Algebra Universalis **5** (1975), 65-71.
[18] Hutchinson, G., Czédli, G., *A test for identities satisfied in lattices of submodules*, Algebra Universalis **8** (1978), 269-309.
[19] Jónsson, B., *Congruence varieties*, Algebra Universalis **10** (1980), 355-394.
[20] Kearnes, K., McKenzie R., *Commutator theory for relatively modular quasivarieties*, Transactions of the AMS **331** (1992), 465-502.
[21] Mederly, P., *Three Mal'cev type theorems and their applications*, Math. Časopis Sloven. Akad. Vied. **25** (1975), 83-95.
[22] Nation, J. B., *Varieties whose congruences satisfy certain lattice identities*, Algebra Universalis **4** (1974), 78-88.
[23] Nurakunov, A. M., *Characterization of relatively distributive quasivarieties of algebras*, Algebra and Logic **29** (1990), 451-458.
[24] Pixley, A. F., *Local Mal'cev conditions*, Canadian Math. Bull. **15** (1972), 559-568.
[25] Weaver, N., *Generalized varieties*, Algebra Universalis **30** (1993), 27-52.
[26] Wille, R., *Kongruenzklassengeometrien*, Lecture Notes in Math., vol. 113, Springer-Verlag, Berlin — Heidelberg — New York, 1970.

GÁBOR CZÉDLI
JATE BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
H-6720 SZEGED, HUNGARY
E-mail: CZEDLI@MATH.U-SZEGED.HU