ARCHIVUM MATHEMATICUM (BRNO)
Tomus 31 (1995), 37 – 42

ON SOME ITERATION SEMIGROUPS

Janusz Brzdęk

Abstract. Let F be a disjoint iteration semigroup of C^n diffeomorphisms mapping a real open interval $I \neq \varnothing$ onto I. It is proved that if F has a dense orbit possesing a subset of the second category with the Baire property, then $F = \{f_t\colon f_t(x) = f^{-1}(f(x)+t)$ for every $x \in I, t \in \mathbb{R}\}$ for some C^n diffeomorphism f of I onto the set of all reals $\mathbb{R}$. The paper generalizes some results of J.A.Baker and G.Blanton [3].

Throughout this paper $I \neq \varnothing$ denotes an open interval. $\mathbb{R}$ and $\mathbb{N}$ are the sets of all reals and positive integers, respectively. In connection with a problem raised by O.Borůvka and F.Neumann (cf. e.g. [8]), J.A.Baker and J.Blanton [3](Theorem 1) (cf. also [2], [4] and [5]) have proved that every complete and disjoint group F of C^n bijections from I to I has the form $F = F[f] := \{f_t : f_t(x) = f^{-1}(f(x)+t)$ for $x \in I$, $t \in \mathbb{R}\}$ for some C^n diffeomorphism f of I onto $\mathbb{R}$. For $n = 0$ this follows also from some earlier results of J.Aczél [1].

Let us remind (cf. [2]–[5]) that a family of functions $F \subset I^I$ is said to be disjoint provided the graphs of any two distinct members of F are disjoint (i.e. if $f, g \in F$ and $f(a) = g(a)$ for some $a \in I$, then $f = g$) and F is complete if $\bigcup F = I \times I$, where $\bigcup F := \{(a, f(a)) : a \in I, f \in F\}$. Further, we say that F has a dense orbit provided there is $b \in I$ such that the set $F(b) := \{f(b) : f \in F\}$ is dense in I.

Clearly, if $f\colon I \to \mathbb{R}$ is a C^n diffeomorphism (i.e. f is a C^n bijection and $f'(x) \neq 0$ for all $x \in I$), then $F[f]$ is a complete disjoint group of C^n functions (cf. [2]–[5]).

We generalize the outcome from [3]. Namely we will prove the given below theorem.

Theorem 1. *Let n be a positive integer and F be a semigroup of C^n bijection from I onto I. Suppose that F has a dense orbit $F(b)$ possesing a subset $D \subset F(b)$ of the second category with the Baire property (cf. e.g. [7], p.599) and $F \cup \{i\}$ is*

1991 *Mathematics Subject Classification*: 26A18, 39B12.
Key words and phrases: iteration semigroup, diffeomorphism, ordered semigroup, Baire property.
Received January 3, 1994.

disjoint, where $i\colon I \to I$ *and* $i(a) = a$ *for every* $a \in I$. *Then* $F = F[f]$ *for some* C^n *diffeomorphism* f *of* I *onto* $\mathbb{R}$.

We will as well show the following

Theorem 2. *Let* $F = \{g_t : t \in T\}$ *be a semigroup of homeomorphisms from* I *onto* I. *Suppose that* F *has a dense orbit and* $F \cup \{i\}$ *is disjoint. Then* F *is a subsemigroup of the group* $F[f]$ *for some homeomorphism* f *of* I *onto* $\mathbb{R}$. *Furhermore, if the dense orbit has a subset of the second category with the Baire property, then* $F = F[f]$.

Theorem 2 is a generalization of Theorem 6 from [3] and, to some extend, of the results of J.Aczél [1]. Actually it is not supposed in [3] that the members of F are homeomorphisms, nevertheless this easily follows from the assumption that F is a group (see e.g. [3], p.121).

Since Theorem 1 is an immediate consequence of Theorem 1 in [3] and our Theorem 2, it suffices to prove only Theorem 2. For the proof we need a theorem of Alimov. Let us recall it.

Theorem A (see e.g. [6], Theorem 4, ch.XI). *Suppose that* B *is a cancellative fully ordered semigroup and for every* $a, c \in B$ *neither*

$$a^n < c^{n+1} \quad \text{and} \quad c^n < a^{n+1} \quad \text{for all} \quad n \in \mathbb{N}$$

nor

$$a^n > c^{n+1} \quad \text{and} \quad c^n > a^{n+1} \quad \text{for all} \quad n \in \mathbb{N}.$$

Then there is an order preserving isomorphism of semigroups mapping B *onto an additive subsemigroup of* $\mathbb{R}$.

Proof of Theorem 2. Fix $b \in I$ such that the set $B = \{g_t(b) : t \in T\}$ is dense in I and define a binary operation $\centerdot : B \times B \to B$ by the formula:

$$g_s(b) \centerdot g_t(b) = g_t(g_s(b)) \quad \text{for} \;\; s, t \in T.$$

It is easily seen that $(B, \centerdot)$ is a cancellative semigroup. We want to show that B satisfies the assumptions of Theorem A with the natural order from I.

According to the hypotheses, for every $s, t \in T$, $s \neq t$,

(1) either $g_t(a) < g_s(a)$ for all $a \in I$ or $g_t(a) > g_s(a)$ for all $a \in I$.

Moreover, since g_t has no fixed points (of course if $g_t \neq i$),

(2) g_t is strictly increasing for every $t \in T$.

Thus from (1) and (2) we derive

(3) $x \centerdot z < x \centerdot y$ and $z \centerdot x < y \centerdot x$ for every $x, y, z \in B$ with $z < y$,

which means that B is a fully ordered semigroup. By induction we as well get

(4) $(g_t(b))^n < (g_t(b))^{n+1} < (g_s(b))^{n+1}$ for $n \in \mathbb{N}$, $s,t \in T$ with $b < g_t(b) < g_s(b)$

and

(5) $(g_s(b))^{n+1} < (g_t(b))^{n+1} < (g_t(b))^n$ for $n \in \mathbb{N}$, $s,t \in T$ with $g_s(b) < g_t(b) < b$,

where $x^1 = x$ and $x^{n+1} = x^n \cdot x$ for $n \in \mathbb{N}$ and $x \in B$. Finally we have

(6) for every $x, z \in B$ with $b < x < z$ $\quad (z < x < b$, respectively$)$
there is $n \in \mathbb{N}$ such that $x^n > z$ $\quad (x^n < z$, respectively$)$.

The proof of (6) is analogous to the proof of Proposition 3 in [3]. However for the sake of completeness we present it.

Take $s,t \in T$ with $b < g_t(b) < g_s(b)$ (the case $g_s(b) < g_t(b)$ is similar) and suppose that $(g_t(b))^n < g_s(b)$ for every $n \in \mathbb{N}$. By virtue of (4), $(g_t(b))^n < (g_t(b))^{n+1} < g_s(b)$ for $n \in \mathbb{N}$. Thus there is $y = \lim_{n\to\infty}(g_t(b))^n \in I$. Hence $g_t(y) = g_t(\lim_{n\to\infty}(g_t(b))^n) = \lim_{n\to\infty} g_t((g_t(b))^n) = \lim_{n\to\infty}(g_t(b))^{n+1} = y$. This is a contradiction, because g_t has no fixed points.

For the proof of the remaining assumption of Theorem A fix $s,t \in T$. We consider only the case $b < g_s(b) < g_t(b)$, for the case $g_t(b) < g_s(b) < b$ is analogous and the case $g_s(b) < b < g_t(b)$, in view of (4) and (5), is trivial. On account of (2) and the definition of B, there exist $x \in B$ and $v \in T$ with $x < g_s(b) < g_s(x) < g_t(b)$ and $b < g_v(b) < x$. Further, since g_v is an increasing homeomorphism, there is $u \in T$ such that $b < g_u(b) < g_v(b)$ and $g_v(b) < g_v(g_u(b)) < x$. Note that by (1) $(g_u(b))^2 = g_u(g_u(b)) < g_v(g_u(b)) < x$. Thus (3) yields

$$(g_u(b))^2 \cdot g_s(b) < x \cdot g_s(b) = g_s(x) < g_t(b). \tag{7}$$

According to (4) and (6) there is $n \in \mathbb{N}$, $n > 1$, such that

$$(g_u(b))^{n-1} \le g_s(b) < (g_u(b))^n.$$

Hence (3), (4) and (7) imply

$$g_s(b) < (g_u(b))^n < (g_u(b))^{n+1} \le (g_u(b))^2 \cdot g_s(b) < g_t(b).$$

Consequently

$$(g_s(b))^{n+1} < (g_u(b))^{(n+1)n} < (g_t(b))^n.$$

On the other hand, by (4),

$$(g_s(b))^n < (g_s(b))^{n+1} < (g_t(b))^{n+1}.$$

In this way we have proved that B fulfils the assumptions of Theorem A. So there exists an order preserving isomorphism g of B onto an additive subsemigroup A of $\mathbb{R}$. We will show that A is dense in $\mathbb{R}$.

Put $A^- = \{a \in A : a < 0\}$ and $A^+ = \{a \in A : a > 0\}$. It results from (6) that $A^- = \{g(x) : x \in B, x < b\}$ and $A^+ = \{g(x) : x \in B, x > b\}$. Let $i = infA^+$ and $s = supA^-$. By the density of B in I we have $i \notin A^+$ or $s \notin A^-$. Consequently $i = 0 = s$, because $A^+ + A^- \subset A$, $s \leq s + i \leq i$, and $sup(A^- + i) = s + i = inf(A^+ + s)$. This means that A is dense in $\mathbb{R}$.

Define a function $f : I \to \mathbb{R}$ by: $f(a) = \lim_{n\to\infty} g(x_n)$ for $a \in I$, where $(x_n : n \in \mathbb{N}) \subset B$ is any sequence with $a = \lim_{n\to\infty} x_n$. It is easily seen that the definition is correct and f is an increasing homeomorphism onto $\mathbb{R}$ such that $f(x) = g(x)$ for $x \in B$ (cf. e.g. [3], the proof of Corollary 5). Fix $a \in I, t \in T$, and a sequence $(x_n : n \in \mathbb{N}) \subset B$ with $a = \lim_{n\to\infty} x_n$. Then

$$g_t(a) = g_t(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} g_t(x_n) = \lim_{n\to\infty} x_n \cdot g_t(b).$$

Consequently

$$\begin{aligned} f(g_t(a)) &= \lim_{n\to\infty} f(x_n \cdot g_t(b)) = \lim_{n\to\infty} g(x_n \cdot g_t(b)) \\ &= \lim_{n\to\infty} (g(x_n) + g(g_t(b))) = \lim_{n\to\infty} f(x_n) + g(g_t(b)) \\ &= f(a) + g(g_t(b)). \end{aligned}$$

This completes the first part of the proof.

Now suppose that B has a subset of the second category with the Baire property. Then A has a subset of the second category with the Baire property, because $A = g(B) = f(B)$ and f is a homeomorphism. Thus on account of the theorem of S.Piccard (see e.g. [7], Theorem 2) $intA \neq \varnothing$. This, in view of the density of A in $\mathbb{R}$, gives $A = \mathbb{R}$. Hence $F = F[f]$, which ends the proof. □

From Theorem 1 we get the following partial answer to the problem of O.Borůvka and F.Neumann (see [3], p.122; cf. also [2], [4], [5] and [8]).

Corollary 1. *Let F be a disjoint group of C^n functions from I into I such that the set $\bigcup F$ is dense in $I \times I$ and has a subset of the second category with the Baire property. Then $F = F[f]$ for some C^n diffeomorphism f of I onto $\mathbb{R}$.*

Proof. Since $\bigcup F$ has a subset of the second category with the Baire property, there is $b \in I$ such that $F(b)$ has a subset of the second category with the Baire property in I (see e.g. [9], p.56–57). Further, according to Proposition 7 in [3], $F(b)$ is dense in I. (Actually Proposition 7 in [3] is proved for $I = (0,1)$, however since every open interval is homeomorphic to $(0,1)$, we do not lose generality (see the proof of Theorem 8 in [3])). Thus Theorem 1 yields the statement. □

Finally we have the subsequent

Proposition. *Let $f_1, f_2 : I \to \mathbb{R}$ be homeomorphisms and F be a subsemigroup of $F[f_1]$. Suppose that F has a dense orbit. Then F is a subsemigroup of $F[f_2]$ iff $f_1(x) = cf_2(x) + d$ for $x \in \mathbb{R}$ with some $c, d \in \mathbb{R}, c \neq 0$.*

Proof. First suppose that $f_1(x) = cf_2(x) + d$ for $x \in \mathbb{R}$ with some $c, d \in \mathbb{R}, c \neq 0$. Fix $g_t \in F$. There is $a \in \mathbb{R}$ such that

$$g_t(x) = f_1^{-1}(f_1(x) + a) \qquad \text{for } x \in I.$$

Thus, for every $x \in \mathbb{R}$,

$$g_t(x) = f_1^{-1}(cf_2(x) + d + a) = f_1^{-1}(cf_2(f_2^{-1}(f_2(x) + \frac{a}{c})) + d) = f_2^{-1}(f_2(x) + \frac{a}{c}).$$

Now assume that F is a subsemigroup of $F[f_2]$. Let B and $\centerdot : B \times B \to B$ be defined as in the proof of Theorem 2. For $j = 1, 2$ define function $h_j : I \to \mathbb{R}$ by the formula:

$$h_j(f_j^{-1}(f_j(b) + x)) = x \qquad \text{for } x \in \mathbb{R}.$$

Since f_1 and f_2 are homeomorphisms, h_1 and h_2 are strictly monotonic.

Take $g_t, g_s \in F$. There are $x_s^1, x_s^2, x_t^1, x_t^2 \in \mathbb{R}$ with

$$g_t(x) = f_j^{-1}(f_j(x) + x_t^j) \qquad \text{for } x \in I,\ j = 1, 2,$$
$$g_s(x) = f_j^{-1}(f_j(x) + x_s^j) \qquad \text{for } x \in I,\ j = 1, 2.$$

Thus, for $j = 1, 2$,

$$g_t(b) \centerdot g_s(b) = g_s(g_t(b)) = f_j^{-1}(f_j(g_t(b)) + x_s^j) = f_j^{-1}(f_j(b) + x_t^j + x_s^j)$$

and consequently

$$h_j(g_t(b) \centerdot g_s(b)) = x_t^j + x_s^j = h_j(g_t(b)) + h_j(g_s(b)).$$

In this way we have proved that $h_j(x \centerdot y) = h_j(x) + h_j(y)$ for $x, y \in B, j = 1, 2$. Put $A_j = h_j(B)$ for $j = 1, 2$. Then A_1 and A_2 are additive subsemigroups of $\mathbb{R}$ and the function $h = h_1 \circ h_2^{-1}|_{A_2}$ is additive and strictly monotonic. Further, A_1 and A_2 are dense in $\mathbb{R}$, because $A_j = f_j(B) - f_j(b)$ for $j = 1, 2$. Define a function $\bar{h} : \mathbb{R} \to \mathbb{R}$ by: $\bar{h}(a) = \lim_{n\to\infty} h(x_n^a)$ for $a \in \mathbb{R}$, where $(x_n^a : n \in \mathbb{N}) \subset A_2$ is any sequence with $a = \lim_{n\to\infty} x_n^a$. It is easy to see that $\bar{h}$ is monotonic and additive. Thus there is $c \in \mathbb{R}$, $c \neq 0$, such that $\bar{h}(x) = cx$ for $x \in \mathbb{R}$. Hence, for every $x \in B$, $h_1(x) = h_1 \circ h_2^{-1}(h_2(x)) = ch_2(x)$.

Fix $a \in I$ and a sequence $(x_n : n \in \mathbb{N}) \subset B$ with $a = \lim_{n\to\infty} x_n$. Then $f_1(a) = \lim_{n\to\infty} f_1(x_n) = f_1(b) + \lim_{n\to\infty} h_1(x_n) = f_1(b) - cf_2(b) + c(f_2(b) + \lim_{n\to\infty} \frac{1}{c} h_1(x_n)) = f_1(b) - cf_2(b) + cf_2(a)$. This ends the proof. $\square$

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Janusz Brzdęk
Department of Mathematics
Pedagogical University
Rejtana 16 A
PL-35-310 Rzeszów, POLAND