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OSCILLATION THEOREMS FOR NEUTRAL DIFFERENTIAL EQUATIONS WITH THE QUASI — DERIVATIVES

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Abstract. The authors study the n-th order nonlinear neutral differential equations with the quasi – derivatives $L_n[x(t)+(-1)^r P(t)x(g(t))]+\delta Q(t)f(x(h(t)))=0$, where $n \geq 2$, $r \in \{1,2\}$, and $\delta = \pm 1$. There are given sufficient conditions for solutions to be either oscillatory or they converge to zero.

1. Introduction

We consider the neutral differential equation

$$(E_r) \qquad L_n[\, x(t) + (-1)^r P(t)\, x(g(t))\,] + \delta\, Q(t)\, f(x(h(t))) = 0,$$

$$\text{where} \quad n \geq 2, \quad r \in \{1,2\}, \quad \delta = \pm 1,$$

$$L_0 x(t) = x(t), \quad L_k x(t) = a_k(t)\, [L_{k-1}x(t)]', \quad k = 1,2,...,n, \quad a_n = 1,$$

$$a_i \in C[\,[t_0,\infty),(0,\infty)\,], \quad i = 1,2,...,n-1, \quad t_0 \geq 0,$$

$$P, Q, h, g \in C[\,[t_0,\infty),[0,\infty)\,], \quad P, Q \not\equiv 0 \text{ on any half line } [t,\infty),$$

$$g(t) \to \infty \text{ and } h(t) \to \infty \text{ as } t \to \infty, \quad f \in C[R,R], \; x\, f(x) > 0 \quad \text{for} \quad x \neq 0.$$

Every solution $x(t)$ of (E_r) considered here is nontrivial and defined on a half line $[T_x, \infty\,)$ $T_x \geq t_0$.

A solution of (E_r) is called *oscillatory* if it has arbitrarily large zeros; otherwise, it is called *nonoscillatory.*

We will use the following notation: $\gamma(t) = sup\{\, s \geq t_0,\, g(s) \leq t\,\}$, $g_1(t) = g(t)$, $g_k(t) = g(g_{k-1}(t))$, $k = 2,3,...,$ $g_{-1}(t) = g^{-1}(t)$, where $g^{-1}(t)$ is inverse function to $g(t)$, $g_{-k}(t) = g_{-1}(g_{-(k-1)}(t))$, $k = 2,3,...$

For any functions $a_i \in C[\,[t_0,\infty),(0,\infty)]$, $i = 1,2,...,n$, we define

$$I_0 = 1, \quad I_i(s,t;a_i,...,a_1) = \int_t^s \frac{1}{a_i(u)} I_{i-1}(u,t;a_{i-1},...,a_1)\, du, \quad t_0 \leq t \leq s.$$

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For each solution $x(t)$ of (E_r) we define

$$z(t) = x(t) + (-1)^r P(t)x(g(t)).$$

Sometimes we will require the following conditions to be satisfied:

$$\int^{\infty} \frac{1}{a_i(t)}dt = \infty, \quad i = 1, 2, ..., n-1; \tag{1}$$

There exist constants $\tau > 0$ and $b > 0$ such that

$$g(t) \le t - \tau, \quad \text{and } g(t) \text{ is increasing on } [t_0, \infty); \tag{2}$$

$$g(t) \le t, \quad \text{and} \quad g'(t) \ge b \text{ on } [t_0, \infty); \tag{2a}$$

$$h(t) \le t; \tag{3}$$

the functions g and h commute, i.e.,

$$g(h(t)) = h(g(t)); \tag{4}$$

$$\begin{aligned} f(u+v) &\le f(u) + f(v), \quad \text{if } u, v > 0,\\ f(u+v) &\ge f(u) + f(v), \quad \text{if } u, v < 0; \end{aligned} \tag{5}$$

$$\begin{aligned} f(ku) &\le k\,f(u), \quad \text{if } k \ge 0 \quad \text{and} \quad u > 0,\\ f(ku) &\ge kf(u), \quad \text{if } k \ge 0 \quad \text{and} \quad u < 0; \end{aligned} \tag{6}$$

(7) $f(u)$ is bounded away from zero if u is bounded away from zero,

$$\int^{\infty} Q(s)ds = \infty, \tag{8}$$

and there exists positive constant M such that

$$P(h(t))Q(t) \le MQ(g(t)). \tag{9}$$

The following two lemmas will be needed in the proofs of our results.

Lemma 1. ([4, Lemma 1]) *Let the condition (1) be satisfied and let* z *be either a positive or a negative function on the interval* $[t_x, \infty)$, $t_x \geq t_0$, *such that* $L_n z$ *exists on* $[t_x, \infty)$, $L_n z(t) \geq 0$ *or* $L_n z(t) \leq 0$ *for* $t \geq t_x$ *and is not identically zero on any interval of the form* $[t_2, \infty)$, $t_2 \geq t_x$. *Then there exists an integer* l, $0 \leq l \leq n$, *with* $n + l$ *even for* $z(t) L_n z(t) \geq 0$ *or* $n + l$ *odd for* $z(t) L_n z(t) \leq 0$, *such that for every* $t \geq t_x$

$$l > 1 \quad \textit{implies} \quad z(t) L_i z(t) > 0, \quad (i = 0, 1, ... l - 1)$$

and

$$l \leq n - 1 \quad \textit{implies} \quad (-1)^{l+i} z(t) L_i z(t) > 0, \quad (i = l, l+1, ..., n-1).$$

Further, for every $i = 0, 1, ..., n-1$, $\lim_{t \to \infty} L_i z(t)$ *exists in the extended real line* $R^* = R \cup \{-\infty, \infty\}$ *whereby*

$$\begin{aligned} &\textit{for} \quad l \leq n-1, && \lim_{t\to\infty} |L_l z(t)| = c_l \geq 0 && \textit{is finite,} \\ &\textit{for} \quad l \leq n-2, && \lim_{t\to\infty} L_i z(t) = 0 && (i = l+1, ..., n-1), \\ &\textit{for} \quad l \geq 2, && \lim_{t\to\infty} |L_i z(t)| = \infty && (i = 0, 1, ..., l-2). \end{aligned}$$

Lemma 2. ([5, Lemma 3]) *Let* $x, P, g : [t_0, \infty) \to R$, $z(t) = x(t) - P(t)\, x(g(t))$, $t \geq t_z = \gamma(t_0)$. *Suppose condition (2) holds and there exists a positive number* p_1 *such that* $0 \leq P(t) \leq p_1$. *Assume that* $x(t) > 0$ *for* $t \geq t_0$, $\liminf_{t\to\infty} x(t) = 0$ *and that* $\lim_{t\to\infty} z(t) = L \in R$ *exists. Then* $L = 0$.

2. Main results

In recent years there has been a growing interest in oscillation theory of functional differential equations of neutral type of the first and higher order; see, for example, the papers [1–5] and the references cited therein.

The purpose of this paper is to establish oscillation theorems for solutions of (E_r). The results from the papers [1] and [5] we extend for neutral differential equations with quasi-derivatives.

Theorem 1. *Let the conditions (1), (2) hold. Assume that there exist positive numbers* p_1 *and* p *such that* $P(t)$ *satisfies* $1 < p \leq P(t) \leq p_1 < \infty$. *If*

$$\int^{\infty} Q(s) I_{n-1}(s, t; a_{n-1}, ..., a_1)\, ds = \infty, \tag{10}$$

then

i) *every bounded solution* $x(t)$ *of* (E_1) *is oscillatory when* $(-1)^n \delta = -1$;
ii) *every bounded solution* $x(t)$ *of* (E_1) *is either oscillatory or* $\lim_{t\to\infty} x(t) = 0$ *when* $(-1)^n \delta = 1$.

Proof. Let $x(t)$ be a nonoscillatory bounded solution of (E_1). We may assume that $x(t)$ is eventually positive. Let $z(t) = x(t) - P(t)\,x(g(t))$. It is easy to see that $z(t)$ is bounded. We first claim that $z(t)$ is eventually negative; otherwise,

$$x(t) \geq P(t)\,x(g(t)) \geq p\,x(g(t)),$$

so by induction we would have

$$x(t) \geq p^m x(g_m(t)),$$

or

$$x(g_{-m}(t)) \geq p^m x(t),$$

for every positive integer m. But this last inequality implies that $x(t) \to \infty$ as $t \to \infty$, which contradicts to our assumption that $x(t)$ is bounded.

Now, from (E_1)

$$\delta L_n\, z(t) = -Q(t)\, f(x(h(t))) \leq 0.$$

Since $z(t)\,\delta L_n\, z(t) \geq 0$ and $z(t)$ is bounded, it follows from Lemma 1 that there exist a $t_2 \geq t_1$ and a number $l \in \{0,1\}$ with $(-1)^{n+l}\,\delta = 1$, such that for all $t \geq t_2$

$$(-1)^{i+l} L_i z(t) < 0, \quad i = l, l+1, ..., n-1. \tag{11}$$

Now, we integrate (E_1) from t to r ($r \geq t \geq t_2$) and see that

$$-\delta L_{n-1}\, z(t) + \int_t^r Q(s)\, f(x(h(s)))\, ds < 0. \tag{12}$$

Integrating (12) after dividing by $a_{n-1}(t)$ from t to r and interchanging the order of integration, we get

$$\delta L_{n-2}\, z(t) + \int_t^r Q(s)\, f(x(h(s))) \int_t^s \frac{1}{a_{n-1}(u)}\, du\, ds < 0.$$

Repeting this method $(n-2)$ times, and denoting by $z(\infty) = \lim\limits_{t\to\infty} z(t)$, we have

$$(-1)^n \delta[z(t) - z(\infty)] + \int_t^\infty Q(s)\, I_{n-1}(s,t; a_{n-1}, ..., a_1)\, f(x(h(s)))\, ds \leq 0. \tag{13}$$

In view of (10) and the fact that $z(t)$ is bounded, one can conclude from (13) that $\liminf\limits_{t\to\infty} f(x(t)) = 0$ or

$$\liminf_{t\to\infty} x(t) = 0. \tag{14}$$

Let $(-1)^n \delta = 1$, i.e. $l = 0$. We shall now proceed to show that $\lim\limits_{t\to\infty} x(t) = 0$. In view of (11) and Lemma 1, $z(t)$ approaches to a finite limit L as t tends to infinity.

Then by Lemma 2, L=0. Since $z(t) < 0$ and $z(t) \to 0$ as $t \to \infty$, for any $\varepsilon > 0$ there exists a T such that

$$z(t) > -\varepsilon, \quad \text{for all} \quad t \geq T.$$

So,

$$\begin{aligned} x(t) &> -\varepsilon + p\,x(g(t)) \\ p\,x(t) &< \varepsilon + x(g_{-1}(t)) \\ p^2 x(t) &< \varepsilon + p\,\varepsilon + x(g_{-2}(t)) \\ &\vdots \end{aligned}$$

$$p^m x(t) < \varepsilon + p\,\varepsilon + \cdots + p^{m-1}\varepsilon + x(g_{-m}(t)). \tag{15}$$

Because $x(t)$ is bounded, there exists a constant A such that $x(t) \leq A$. From (15) we obtain

$$x(t) < \varepsilon\,\frac{p^{-m}-1}{1-p} + A\,p^{-m}. \tag{16}$$

Because p^{-m} goes to zero as m tends to infinity, and ε is arbitrary, from (16) we have $x(t) \to 0$ as $t \to \infty$ as desired.

Suppose that $(-1)^n\delta = -1$. Because $z(t)$ is bounded and $l = 1$, $\lim_{t\to\infty} z(t)$ exists. In view of (14), it follows from Lemma 2 that $z(t) \to 0$, as $t \to \infty$. But this contradicts to the fact that $z(t)$ is negative and decreasing, and hence proves that $x(t)$ is oscillatory. The case when $x(t)$ is eventually negative is similar. The proof of Theorem 1 is complete. □

The following examples are ilustrative.

Example 1. Consider the neutral differential equation

$$(e^{-t}(e^{-t}(x(t) - (2 + e^{-t})\,x(t-1))')')' - (24\,e^{1-t} + 12\,e - 6)\,x(3t) = 0 \tag{17}$$

for $t \geq 1$. All conditions of Theorem 1 are satisfied, $\delta = -1$, $n = 3$ and hence every bounded solution $x(t)$ of (17) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$. One such solution is $x(t) = e^{-t}$.

Example 2. Consider the neutral differential equation

$$(e^{-t}(e^{-t}(x(t) - 2\,x(t - 2\,\pi))')')' + \left(\frac{(2\,e^{2\pi} - 1)\,10}{e^{\frac{3\pi}{2}}}\,e^{-2t}\right)\,x\left(t - \frac{3}{2}\,\pi\right) = 0 \tag{18}$$

for $t \geq 2\,\pi$. All conditions of Theorem 1 are satisfied, $\delta = 1$, $n = 3$ and hence every bounded solution $x(t)$ of (18) is oscillatory. One such solution is $x(t) = e^{-t}\sin t$.

Theorem 2. *Suppose $\delta = 1$ and conditions (1), (2a), (3)–(9) hold. Then*

i) *if n is even, every solution of (E_2) is oscillatory;*
ii) *if n is odd, any solution $x(t)$ of (E_2) is either oscillatory or satisfies $x(t) \to 0$ as $t \to \infty$.*

Proof. Suppose that (E_2) has an eventually positive solution $x(t)$, say $x(t) > 0$, $x(g(t)) > 0$, $x(h(t)) > 0$ and $x(g(h(t))) > 0$ for $t \geq t_1$, for some $t_1 \geq t_0$. It then follows from (5), (6) and (4) that

$$f(z(h(t))) = f(x(h(t))) + P(h(t))\, x(g(h(t))) \leq f(x(h(t))) + P(h(t))\, f(x(h(g(t)))).$$

Hence

$$L_n\, z(t) + Q(t)\, f(z(h(t))) \leq Q(t)\, P(h(t))\, f(x(h(g(t)))). \tag{19}$$

Since $z(t) > 0$, $L_n\, z(t) \leq 0$ for $t \geq t_1$, it follows from Lemma 1 that there exist a $t_2 \geq t_1$ and an integer l, $0 \leq l \leq n$ with $n + l$ odd such that for every $t \geq t_2$

$$\begin{aligned} L_i\, z(t) > 0 \quad &\text{for} \quad i = 0, 1, ..., l-1, \\ (-1)^{l+i} L_i\, z(t) > 0 \quad &\text{for} \quad i = l, l+1, ..., n-1. \end{aligned} \tag{20}$$

And hence $L_{n-1}\, z(t) > 0$ and

$$\lim_{t\to\infty} L_{n-1}\, z(t) \quad \text{is finite.}$$

From (E_2) we have

$$L_n\, z(g(t))\, g'(t) + Q(g(t))\, f(x(h(g(t))))\, g'(t) = 0,$$

and itegrating shows that

$$\int_{t_2}^{\infty} Q(g(s))\, f(x(h(g(s))))\, g'(s)\, ds < \infty.$$

This, together with (2a) and (9), implies that

$$\int_{t_2}^{\infty} Q(s)\, P(h(s))\, f(x(h(g(s))))\, ds < \infty.$$

An integration of (19) shows that

$$\int_{t_2}^{\infty} Q(s)\, f(z(h(s)))\, ds < \infty,$$

which, in view of (7) and (8), implies that

$$\liminf_{t\to\infty} z(t) = 0.$$

Therefore $z(t) \to 0$ as $t \to \infty$ since $z(t)$ is positive and monotonic. Clearly, $z'(t) < 0$, and from (20) we have $l = 0$ and n is odd. Because $P(t) \geq 0$,we get $x(t) \leq z(t) \to 0$ as $t \to \infty$. This completes the proof of the Theorem 2 in the case $x(t) > 0$. The proof when $x(t) < 0$ is similar and will be omitted. □

The following example is illustrative.

Example 3. Consider the neutral differential equation

$$\left[t\left(\frac{1}{t}[x(t)+2\,x(t-1)]'\right)'\right]' + \frac{1+2\,e}{e^2}\left(1+\frac{1}{t}+\frac{1}{t^2}\right)x(t-2) = 0, \quad t \geq 2.$$

All conditions of Theorem 2 are satisfied and any solution of this equation is either oscillatory or $\lim\limits_{t\to\infty} x(t) = 0$. One such solution is $x(t) = e^{-t}$.

Theorem 3. *Suppose* $\delta = -1$, *conditions (1), (2a), (3)–(9) hold and P(t) is bounded. Then*

i) *if n is even, any bounded solution* $x(t)$ *of* (E_2) *is either oscillatory or satisfies* $x(t) \to 0$ *as* $t \to \infty$;

ii) *if n is odd, every bounded solution of* (E_2) *is oscillatory.*

Proof. Let $x(t)$ be a bounded and eventually positive solution of (E_2), say $x(t) > 0$, $x(g(t)) > 0$, $x(h(t)) > 0$, $x(g(h(t))) > 0$ for $t \geq t_1$. Also, note that $z(t)$ is positive and bounded since $P(t)$ is bounded. Since $L_n\, z(t) \geq 0$ for $t \geq t_1$, it follows from Lemma 1 that there exist a $t_2 \geq t_1$ and an integer l, $l \in \{0, 1\}$ with $n + l$ even such that for every $t \geq t_2$

$$(-1)^{l+i} L_i\, z(t) > 0 \qquad \text{for } i = 1, 2, ..., n-1. \tag{21}$$

Conditions (5), (6), (4) and (9) yield

$$L_n\, z(t) + M\, Q(g(t))\, f(x(h(g(t)))) \geq Q(t)\, f(z(h(t))). \tag{22}$$

As in the proof of Theorem 2, it follows from (E_2) that

$$\int_{t_2}^{\infty} Q(g(s))\, f(x(h(g(s))))\, g'(s)\, ds < \infty.$$

Hence by (2a) and an integration of (22) we see that

$$\int_{t_2}^{\infty} Q(s)\, f(z(h(s)))\, ds < \infty.$$

Conditions (7) and (8) then imply

$$\liminf_{t\to\infty} z(t) = 0,$$

and in view of the monotonicity of z we get $z(t) \to 0$ as $t \to \infty$. From (21) we get $l = 0$ and n is even. Then $x(t) \le z(t) \to 0$ as $t \to \infty$.

If $x(t)$ is eventually negative, the proof can be done in a similar way. The proof of Theorem 3 is complete. □

Example 4. Consider the neutral differential equation

$$\left[\frac{1}{t}\,(\,x(t) + 3\,x(t-1))'\right]' - e^{-4}\,(1 + 3\,e)\,\left(\frac{1}{t} + \frac{1}{t^2}\right)\,x(t-4) = 0, \quad t \ge 4.$$

All conditions of Theorem 3 are satisfied and any solution of this equation is either oscillatory or $\lim\limits_{t\to\infty} x(t) = 0$. One such solution is $x(t) = e^{-t}$.

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