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A STRONG RELAXATION THEOREM FOR MAXIMAL MONOTONE DIFFERENTIAL INCLUSIONS WITH MEMORY

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Abstract. We consider maximal monotone differential inclusions with memory. We establish the existence of extremal strong and then we show that they are dense in the solution set of the original equation. As an application, we derive a "bang-bang" principle for nonlinear control systems monitored by maximal monotone differential equations.

1. Introduction

In a recent paper [13], we studied maximal monotone differential inclusions with memory defined on $\mathbb{R}^N$ (with N being a positive integer) of the form

$$\left\{ \begin{array}{c} -\dot{x}(t) \in Ax(t) + F(t, x_t) \text{ a.e. on } \quad T = [0, b] \\ x(v) = \varphi(v) \quad \text{for} \quad v \in T_0 = [-r, 0]. \end{array} \right\} \tag{1}$$

Here $b \in \mathbb{R}_+$, $A(\cdot)$ is a maximal monotone operator on $\mathbb{R}^N$, $F(t, x_t)$ is a multivalued vector field (orientor field) and $x_t \in C(T_0, \mathbb{R}^N)$ is defined by $x_t(v) = x(t + v)$. Hence $x_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t. Among the results proved in [13], was a relaxation theorem, which says that the solution set of the above multivalued Cauchy problem is dense for the $C(\widehat{T}, \mathbb{R}^N)$-topology ($\widehat{T} = [-r, b]$), in the solution set of the Cauchy problem in which the orientor field $F(t, x)$ is replaced by its convexification $\overline{conv}\, F(t, x)$ (see theorem 5.1 in [13]).

In this paper, we prove a stronger version of the relaxation theorem, which is closely related to the "bang-bang" principle for control systems. So instead of problem (1), we consider the following multivalued Cauchy problem:

$$\left\{ \begin{array}{c} -\dot{x}(t) \in Ax(t) + ext\, F(t, x_t) \text{ a.e. on } \quad T \\ x(v) = \varphi(v), \quad v \in T_0\, . \end{array} \right\} \tag{2}$$

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Here $ext\, F(t, x_t)$ stands for the extreme points of the compact, convex set $F(t, x_t)$. First we answer the question of existence of solutions for problem (2). The nonconvex existence theorem proved in [13] (see theorem 3.2), is not applicable here, because the multifunction $(t, y) \to ext\, F(t, y)$ is not in general closed valued and $y \to ext\, F(t, y)$ is not necessarily lower semicontinuous ($l.s.c.$). The nonemptiness of the solution set $S_e \subseteq C(\widehat{T}, \mathbb{R}^n)$ ($\widehat{T} = [-r, b]$) of (2) is established in theorem 3.1. Then in section 4, in theorem 4.1, we show that S_e is dense in $S \subseteq C(\widehat{T}, \mathbb{R}^N)$ the solution set of (1), for the $C(\widehat{T}, \mathbb{R}^N)$-topology. This way we obtain a genuine new approximation (relaxation) result. Note that in the relaxation theorem of [13] (see theorem 5.1), the nonconvex valued orientor field $F(t, y)$ was assumed to be closed valued and Hausdorff-Lipschitz in the y-variable, conditions that in general are not true for the multifunction $(t, y) \to ext\, F(t, y)$, even if $(t, y) \to F(t, y)$ is very regular. Finally in section 5, we consider an application to nonlinear control systems, monitored by maximal monotone differential equations.

2. Preliminaries

In this section we fix our notation and we briefly recall some basic definitions and facts that we will need in the sequel.

So let (Ω, Σ) be a measurable space and X a separable Banach space. We will be using the following notations:

$$P_f(X) = \{A \subseteq X : \text{ nonempty, closed } \}$$
$$\text{and} \quad P_{kc}(X) = \{A \subseteq X : \text{ nonempty, compact and convex}\}\,.$$

A multifunction $F : \Omega \to P_f(X)$ is said to be measurable, if for all $x \in X$, the $\mathbb{R}_+$-valued function $\omega \to d(x, F(\omega)) = \inf\{\|x - z\| : z \in F(\omega)\}$ is measurable. Other equivalent definitions of the measurability of a $P_f(X)$-valued multifunction, can be found in Wagner [16]. Let $\mu(\cdot)$ be a finite measure defined by Σ. By S_F^1 we will denote the set of all selectors of F, that belong in the space $L^1(\Omega, X)$; i.e. $S_F^1 = \{f \in L^1(\Omega, X) : f(\omega) \in F(\omega)\mu-\text{ a.e.}\}$. For a measurable multifunction this set is nonempty if and only if $\omega \to \inf\{\|z\| : z \in F(\omega)\} \in L_+^1$ (for details, we refer to Papageorgiou [14]).

On the set $P_f(X)$, we can define a generalized metric, better known as the Hausdorff metric, by setting

$$h(A, B) = \max[\sup_{a \in A} d(a, B),\ \sup_{b \in B} d(b, A)]$$

for every $A, B \in P_f(X)$. It is well known (see for example Klein-Thompson [8]), that $(P_f(X), h)$ is a complete generalized metric space. A multifunction $F : X \to P_f(X)$ is said to be Hausdorff continuous (h-continuous), if it is continuous from X into the metric space $(P_f(X), h)$.

If Y, Z are Hausdorff topological spaces, a multifunction $G : Y \to 2^Z \setminus \{\emptyset\}$ is said to be lower semicontinuous ($l.s.c.$), if for all $C \subseteq Z$ closed, $G^+(C) = \{y \in Y : G(y) \subseteq C\}$ is closed in Y.

Finally, if H is a Hilbert space, an operator $A : D(A) \subseteq H \to 2^H$ is said to be "monotone" if $(x - x', y - y') \geq 0$ for all $[x, y], [x', y'] \in GrA$ (here $(\cdot, \cdot)$ denotes the inner product in H). The operator A is said to be "maximal monotone" if and only if $(x - v, y - w) \geq 0$ for all $[x, y] \in GrA$, implies $[v, w] \in GrA$ (i.e. the graph of $A(\cdot)$ is not properly included in any other monotone subset of $H \times H$). From a result of Minty, we know that the operator $A(\cdot)$ is maximal monotone if and only if for some $\lambda > 0$ (equivalently for every $\lambda > 0$), $R(I + \lambda A) = H$.

3. Existence of extremal solutions

In this section we establish the nonemptiness of the solution set $S_e \subseteq C(\widehat{T}, \mathbb{R}^N)$ of (2). For this we will need the following hypotheses:

$H(A)$: $A : D(A) \subseteq \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone operator.

$H(F)$: $F : T \times C(T_0, \mathbb{R}^N) \to P_{kc}(\mathbb{R}^N)$ is a multifunction *s.t.*

(1) $t \to F(t, y)$ is measurable,

(2) $y \to F(t, y)$ is h-continuous,

(3) $|F(t, y)| = \sup\{\|v\| : v \in F(t, y)\} \leq \alpha(t) + \beta(t)\|y\|_\infty$ a.e. with $\alpha(\cdot), \beta(\cdot) \in L^p_+$, $1 < p < \infty$.

$H(\varphi)$: $\varphi \in C(T_0, \mathbb{R}^N)$ and $\varphi(0) \in \overline{D(A)}$.

First we need an auxiliary result. Let $L_w(T, \mathbb{R}^N)$ be the space of equivalence classes of Lebesgue integrable functions $x : T \to \mathbb{R}^N$, equipped with the "weak" norm $\|x\|_w = \sup\{\|\int_{t_1}^{t_2} x(s)\, ds\| : 0 \leq t_1 \leq t_2 \leq b\}$. The notation $\xrightarrow{\|\cdot\|_w}$ stands for convergence in $L_w(T, \mathbb{R}^N)$.

Lemma 3.1. *If $\{f_n\}_{n\geq1} \subseteq L^p(T, \mathbb{R}^N)$ are such that $\sup\limits_{n\geq1}\|f_n\|_p < \infty$ and $f_n \xrightarrow{\|\cdot\|_w} 0$ as $n \to \infty$, then $f_n \xrightarrow{w} 0$ in $L^p(T, \mathbb{R}^N)$.*

Proof. Since by hypothesis $\{f_n\}_{n\geq1}$ is bounded in $L^p(T, \mathbb{R}^N)$ and step functions are dense in $L^q(T, \mathbb{R}^N)$ with $\frac{1}{p} + \frac{1}{q} = 1$ (see Dunford-Schwartz [6]), we only need to show that $((f_n, s)) \to 0$ as $n \to \infty$, for each $s : T \to \mathbb{R}^N$ of the form $s(t) = \sum\limits_{k=1}^m \chi_{(t_{k-1}), t_k}(t) v_k^*$, $v_k^* \in \mathbb{R}^N$ and with $((\cdot, \cdot))$ being the duality brackets for the pair $(L^p(T, \mathbb{R}^N), L^q(T, \mathbb{R}^N))$. We have:

$$|((f_n, s))| = |\sum_{k=1}^m \int_{t_{k-1}}^{t_k} (f_n(s), v_k^*)\, ds| \leq \sum_{k=1}^m \|\int_{t_{k-1}}^{t_k} f_n(s)\, ds\| \cdot \|v_k^*\|$$

$$\Rightarrow |((f_n, s))| \leq \|f_n\|_w \cdot \sum_{k=1}^m \|v_k^*\| \to 0 \quad \text{as} \quad n \to \infty$$

□

Now we are ready for our existence theorem, concerning problem (2).

Theorem 3.1. *If hypotheses $H(A)$, $H(F)$ and $H(\varphi)$ hold, then $S_e \subseteq C(\widehat{T}, \mathbb{R}^N)$ is nonempty.*

Proof. From the proof of theorem 3.1 in [13], we know that there exists $M_1 > 0$ such that for all $t \in T$ and all $x \in S$ ($S \subseteq C(\widehat{T}, \mathbb{R}^N)$ being the solution set of (1)), we have

$$\|x_t\|_\infty \leq M_1 .$$

Hence because of hypothesis $H(F)$ (3), we may assume, without any loss of generality that

$$|F(t,y)| \leq \alpha(t) + \beta(t) M_1 = \psi(t) \quad \text{a.e.}$$

with $\psi(\cdot) \in L^p_+$ (otherwise in what follows replace $F(t,y)$ by $F(t, p_{M_1}(y))$ with $p_{M_1}(\cdot)$ being the M_1-radial retraction map).

Next let $K = \{v \in L^1(T, \mathbb{R}^N) : \|v(t)\| \leq \psi(t) \quad a.e.\}$ and let $\eta : L^1(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$ be the map which assigns to each $q \in L^1(T, \mathbb{R}^N)$, the unique strong solution of the Cauchy problem $-\dot{x}(t) \in Ax(t) + q(t)$ a.e., $x(0) = \varphi(0)$ (see Brezis [5], theorem 3.4, p. 65 and proposition 3.8, p. 82). Since K is bounded in $L^1(T, \mathbb{R}^N)$ and since the semigroup of nonlinear contractions generated by $-A$ on $\overline{D(A)}$, is compact (because $\mathbb{R}^N$ is finite dimensional), we may invoke theorem 1 of Baras [2] and get that $W = \overline{\eta(K)}^{C(T,\mathbb{R}^N)}$ is compact. Extend the elements of W on $\widehat{T} = [-r, b]$, by simply setting $x(v) = \varphi(v)$ for $v \in T_0$, when $x \in W$ (recall $x(0) = \varphi(0) \in \overline{D(A)}$). Denote the set of these extensions by $\widehat{W}_0$. Clearly $\widehat{W}_0 \subseteq C(\widehat{T}, \mathbb{R}^N)$ is compact. Let $\widehat{W} = \overline{conv}\widehat{W}_0$. Then $\widehat{W} \subseteq C(\widehat{T}, \mathbb{R}^N)$ is compact and convex (Mazur's theorem; see Dunford-Schwartz [6], theorem 6, p. 414). On $\widehat{W}$, we consider the $C(\widehat{T}, \mathbb{R}^N)$-norm topology and on $L^1(T, \mathbb{R}^N)$ the norm topology. Then define $R : \widehat{W} \to 2^{L^1(T,\mathbb{R}^N)} \setminus \{\emptyset\}$ by $R(x) = S^1_{F(\cdot, x.)}$. Note that $R(\cdot)$ has closed, decomposable values (i.e. if $f_1, f_2 \in R(x)$ and $B \subseteq T$ is measurable, then $\chi_B f_1 + \chi_{B^c} f_2 \in R(x)$) and is *l.s.c.* (in fact h-continuous; see theorem 4.5 of Papageorgiou [10]). So applying theorem 1.1 of Tolstonogov [15], we can find a continuous map $\theta : \widehat{W} \to L_w(T, \mathbb{R}^N)$ such that $\theta(x) \in ext\, R(x)$, for all $x \in \widehat{W}$. But from Benamara [3], we know that $ext\, R(x) = ext\, S^1_{F(\cdot, x.)} = S^1_{ext\, F(\cdot, x.)}$. Then let $u = \eta \circ \theta : \widehat{W} \to W$ and let $\widehat{u}(x)$ be the extension of $u(x)$ on $\widehat{T}$ by $\widehat{u}(x)(v) = \varphi(v)$, $v \in T_0$. Clearly, $\widehat{u} : \widehat{W} \to \widehat{W}$ and because $\psi \in L^q_+$ by virtue of lemma 3.1, we can easily check using the $\|\cdot\|_w$-continuity of $\theta(\cdot)$, that $\widehat{u}(\cdot)$ is continuous. Apply Schauder's fixed point theorem to get $x = \widehat{u}(x)$. Obviously $x \in S_e \neq \emptyset$. $\square$

4. Strong relaxation

In this section, we show that S_e is dense in S (the solution set of (1) for the $C(\widehat{T}, \mathbb{R}^N)$-topology).

For this we will need the following stronger hypothesis on the orientor field $F(t,y)$:

$H(F)_1 : F : T \times C(T_0, \mathbb{R}^N) \to P_{kc}(\mathbb{R}^N)$ is a multifunction *s.t.*

(1) $t \to F(t,y)$ is measurable,

(2) $h(F(t,y), F(t,z)) \leq k(t)\|y-z\|_\infty$ a.e. with $k(\cdot) \in L^1_+$,

(3) $|F(t,y)| = \sup\{\|v\| : v \in F(t,y)\} \leq \alpha(t) + \beta(t)\|y\|_\infty$ a.e. with $\alpha, \beta \in L^p_+$, $1 < p < \infty$.

Theorem 4.1. *If hypotheses $H(A)$, $H(F)_1$ and $H(\varphi)$ hold, then $S = \overline{S_e}^{C(\widehat{T},\mathbb{R}^N)}$.*

Proof. Let $x(\cdot) \in S$. Then $x(\cdot) \in C(\widehat{T}, \mathbb{R}^N)$ is a strong solution of

$$\left\{\begin{array}{c} -\dot{x}(t) \in Ax(t) + f(t) \text{ a.e. on } \quad T = [0,b] \\ x(v) = \varphi(v), \quad v \in T_0 = [-r,0], \varphi(0) \in \overline{D(A)} \end{array}\right\}$$

with $f \in L^p(T, \mathbb{R}^N)$, $f(t) \in F(t, x_t)$ a.e. Let $\widehat{W} \subseteq C(\widehat{T}, \mathbb{R}^N)$ be as in the proof of theorem 3.1. Given $y \in \widehat{W}$ an $\epsilon > 0$, let $\Gamma : T \to 2^{\mathbb{R}^N} \setminus \{\emptyset\}$ be defined by

$$\Gamma(t) = \left\{u \in \mathbb{R}^N : \|f(t) - u\| < \frac{\epsilon}{2M_1 b} + d(f(t), F(t, y_t)), u \in F(t, y_t)\right\}$$

where $M_1 > 0$ is the a priori bound for the elements in S (see the proof of theorem 3.1). Then

$$\begin{aligned} Gr\Gamma &= \left\{(t,u) \in T \times \mathbb{R}^N : u \in F(t, y_t), \|f(t) - u\| < \frac{\epsilon}{2M_1 b} + d(f(t), F(t, y_t))\right\} \\ &= \left\{(t,u) \in GrF(\cdot, y_\cdot) : \|f(t) - u\| < \frac{\epsilon}{2M_1 b} + d(f(t), F(t, y_t))\right\}. \end{aligned}$$

Using hypotheses $H(F)_1$ (1) and (2) and theorem 3.3 of Papageorgiou [11], we get that $GrF(\cdot, y_\cdot) \in B(T) \times B(\mathbb{R}^N)$ where $B(T)$ (resp. $B(\mathbb{R}^N)$) is the Borel σ-field of T (resp. of $\mathbb{R}^N$). Furthermore, $(t,u) \to \|f(t) - u\| - d(f(t), F(t, y_t))$ is clearly jointly measurable. Hence $Gr\Gamma \in B(T) \times B(\mathbb{R}^N)$. Apply Aumann's selection theorem (see for example, Wagner [16], theorem 5.10), to get $u : T \to \mathbb{R}^N$ a measurable map *s.t.* $u(t) \in \Gamma(t)$ a.e. Therefore, if we define $L : \widehat{W} \to 2^{L^1(T,\mathbb{R}^N)}$ by

$$L(y) = \left\{u \in S^1_{F(\cdot, y_\cdot)} : \|f(t) - u(t)\| < \frac{\epsilon}{2M_1 b} + d(f(t), F(t, y_t)) \text{ a.e.}\right\},$$

it follows that $L(\cdot)$ has nonempty, decomposable values. In addition, proposition 4 of Bressan-Colombo [4] tells us that $L(\cdot)$ is *l.s.c.* Therefore, $y \to \overline{L(y)}$ is *l.s.c.* with nonempty, closed and decomposable values. Apply theorem 3 of Bressan-Colombo [4], to get a continuous map $u_\epsilon : \widehat{W} \to L^1(T, \mathbb{R}^N)$ *s.t.* $u_\epsilon(y) \in \overline{L(y)}$ for all $y \in \widehat{W}$. Then we have:

$$\begin{aligned} \|f(t) - u_\epsilon(y)(t)\| &\leq \frac{\epsilon}{2M_1 b} + d(f(t), F(t, y_t)) \\ &\leq \frac{\epsilon}{2M_1 b} + k(t)\|x - y\|_\infty \text{ a.e. on } \quad T. \end{aligned}$$

Use theorem 1.1 of Tolstonogov [15] to get $v_\epsilon : \widehat{W} \to L_w(T,\mathbb{R}^N)$ a continuous map *s.t.* $v_\epsilon(y) \in ext\, S^1_{F(\cdot,y_\cdot)} = S^1_{ext\, F(\cdot,y_\cdot)}$ and $\|u_\epsilon(y) - v_\epsilon(y)\|_w < \epsilon$ for all $y \in \widehat{W}$.

Next let $\epsilon_n \downarrow 0$ and set $u_n = u_{\epsilon_n}$ and $v_n = v_{\epsilon_n}$. Let $\widehat{x}_n \in S_e \subseteq \widehat{W}$ be such that $\widehat{x}_n = \widehat{v}_n(\widehat{x}_n)$, where $\widehat{v}_n(\widehat{x}_n)$ is the extension by φ on T_0, of $(\eta \circ \theta_n)\,(\widehat{x}_n)$; see the proof of theorem 3.1 (the existence of x_n follows from Schauder's fixed point theorem). Since $\widehat{W} \subseteq C(\widehat{T},\mathbb{R}^N)$ is compact, we may assume that $\widehat{x}_n \to \bar{x}$ in $C(\widehat{T},\mathbb{R}^N)$. Then exploiting the monotonicity of the operator $A(\cdot)$, we have

$$(-\dot{x}(t) + \dot{\widehat{x}}_n(t), \widehat{x}_n(t) - x(t)) \le (f(t) - v_n(\widehat{x}_n)(t), \widehat{x}_n(t) - x(t)) \quad \text{a.e. on} \quad T$$

$$\Rightarrow \frac{1}{2}\|\widehat{x}_n(t) - x(t)\|^2 \le \int_0^t (f(s) - u_n(\widehat{x}_n)(s), \widehat{x}_n(s) - x(s))\,ds \tag{3}$$
$$+ \int_0^t (u_n(\widehat{x}_n)(s) - v_n(\widehat{x}_n)(s), \widehat{x}_n(s) - x(s))\,ds\,.$$

Note that by construction $u_n(\widehat{x}_n) - v_n(\widehat{x}_n) \xrightarrow{\|\cdot\|_w} 0$ and $\{u_n(\widehat{x}_n) - v_n(\widehat{x}_n)\}_{n\ge 1}$ is bounded in $L^p(T,\mathbb{R}^N)$. So from lemma 3.1, we get that $u_n(\widehat{x}_n) - v_n\widehat{x}_n \xrightarrow{w} 0$ in $L^p(T,\mathbb{R}^N)$. Since $\widehat{x}_n - x \to \bar{x} - x$ in $C(\widehat{T},\mathbb{R}^N)$, we get

$$\int_0^t (u_n(\widehat{x}_n)(s) - v_n(\widehat{x}_n)(s), \widehat{x}_n(s) - x(s))\,ds \to 0 \quad \text{as} \quad n \to \infty\,.$$

Also we have:

$$\begin{aligned} &\int_0^t (f(s) - u_n(\widehat{x}_n)(s), \widehat{x}_n(s) - x(s))\,ds \\ &\le \int_0^t \|f(s) - u_n(\widehat{x}_n)(s)\| \cdot \|\widehat{x}_n(s) - x(s)\|\,ds \\ &\le \int_0^t \Big(\frac{\epsilon_n}{2M_1 b} + d(f(s), F(s,(\widehat{x}_n)_s))\Big) \cdot \|\widehat{x}_n(s) - x(s)\|\,ds \\ &\le \epsilon_n + \int_0^t k(s) \cdot \|(\widehat{x}_n)_s - x_s\|^2_\infty\,ds\,. \end{aligned}$$

So by passing to the limit as $n \to \infty$ in (3), we get

$$\begin{aligned} &\|x_t - \bar{x}_t\|^2_\infty \le 2\int_0^t k(s) \cdot \|x_s - \bar{x}_s\|^2_\infty\,ds \\ &\Rightarrow x = \bar{x} \quad \text{(Gronwall's inequality)} \\ &\Rightarrow \widehat{x}_n \to x \quad \text{in} \quad C(\widehat{T},\mathbb{R}^N)\,. \end{aligned}$$

Since $\widehat{x}_n \in S_e$ and $S \subseteq C(\widehat{T},\mathbb{R}^N)$ is compact (see [13]), we conclude that $S = \overline{S_e}^{C(T,\mathbb{R}^N)}$. □

5. Control systems

In this section, we use theorem 4.1 to derive a "bang-bang" principle for nonlinear control systems monitored by maximal monotone differential equations. Specifically, we consider the following two systems:

$$(4)\qquad \left\{\begin{array}{c} -\dot{x}(t) \in Ax(t) + f(t, x_t)u(t) \text{ a.e. on } \quad T \\ x(v) = \varphi(v), \quad v \in T_0 \\ u(t) \in U(t) \text{ a.e., } u(\cdot) - \text{measurable} \end{array}\right\}$$

and

$$(5)\qquad \left\{\begin{array}{c} -\dot{x}(t) \in Ax(t) + f(t, x_t)u(t) \text{ a.e. on } \quad T \\ x(v) = \varphi(v), \quad v \in T_0 \\ u(t) \in ext\, U(t) \text{ a.e., } u(\cdot) - \text{measurable}\,. \end{array}\right\}$$

We will need the following hypotheses on the data:

$H(f)$: $\quad f : T \times C(T_0, \mathbb{R}^N) \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^N) = \mathbb{R}^{N\times m}$ is a map *s.t.*

(1) $t \to f(t, y)u$ is a measurable for all $(y, u) \in C(T_0, \mathbb{R}^N) \times \mathbb{R}^m$,

(2) $\|f(t, y) - f(t, y')\|_{\mathcal{L}} \leq k(t) \cdot \|y - y'\|_\infty$ a.e. with $k(\cdot) \in L^1_+$,

(3) $\|f(t, y)\|_{\mathcal{L}} \leq \alpha(t) + \beta(t)\|y\|_\infty$ a.e. with $\alpha, \beta \in L^p_+$, $1 < p < \infty$.

$H(U)$: $\quad U : T \to P_{kc}(\mathbb{R}^m)$ is a measurable multifunction *s.t.*

$|U(t)| = \sup\{\|u\| : u \in U(t)\} \leq M,\ M > 0$.

By $S, S_e \subseteq C(\widehat{T}, \mathbb{R}^N)$ we will denote that the sets of trajectories of (4) and (5) respectively and by $R(t)$ and $R_e(t)$, the corresponding reachable sets at time $t \in T$; i.e. $R(t) = \{x(t) : x \in S\}$ and $R_e(t) = \{x(t) : x \in S_e\}$. The nonemptiness of these sets follows from theorem 3.1, if in addition to $H(f)$ and $H(U)$, we also assume that $H(A)$ and $H(\varphi)$ hold.

Now we are ready to state and prove our nonlinear "bang-bang" principle.

Theorem 5.1. *If hypotheses $H(A)$, $H(f)$, $H(U)$ and $H(\varphi)$ hold, then $S = \overline{S_e}^{C(T_0,\mathbb{R}^N)}$ and for every $t \in T$, $R(t) = \overline{R_e(t)}^{\mathbb{R}^N}$.*

Proof. Let $F : T \times C(T_0, \mathbb{R}^N) \to P_{kc}(\mathbb{R}^N)$ be defined by

$$F(t, y) = f(t, y)U(t)\,.$$

Let $u_n : T \to \mathbb{R}^m, n \geq 1$, be measurable function *s.t.* $U(t) = \overline{\{u_n(t)\}}_{n\geq 1}$, $t \in T$. They exist since by hypothesis $H(U)$, $U(\cdot)$ is measurable (see Wagner [16], theorem 4.2). Then for $v \in \mathbb{R}^N$, we have:

$$\begin{aligned} d(v, F(t, y)) &= \inf_{n\geq 1} \|v - f(t, y)u_n(t)\| \\ &\Rightarrow t \to d(v, F(t, y)) \quad \text{is measurable} \\ &\Rightarrow t \to F(t, y) \quad \text{is measurable}\,. \end{aligned}$$

Next let $y, y' \in C(T_0, \mathbb{R}^N)$ and $v \in F(t,y)$. Then by definition $v = f(t,y)u$, $u \in U(t)$. Because of hypotheses $H(f)$ (2) and $H(U)$, we have

$$\begin{aligned} d(v, F(t,y')) &\leq \|f(t,y)u - f(t,y')u\| \leq Mk(t) \cdot \|y - y'\|_\infty \\ &\Rightarrow h(F(t,y), F(t,y')) \leq \widehat{k}(t) \cdot \|y - y'\|_\infty (\widehat{k} = Mk) . \end{aligned}$$

Finally because of hypothesis $H(f)$ (3), we have

$$|F(t,x)| \leq \widehat{\alpha}(t) + \widehat{\beta}(t)\|x\| \quad \text{a.e.}$$

with $\widehat{\alpha} = M\alpha$, $\widehat{\beta} = M\beta \in L^p_+$. So through an easy application of Aumann's selection theorem, we see that system (4) (resp. (5)) can be equivalently rewritten in the "deparametrized" (i.e. control free) form (1) (resp. (2)), with $F(t,y)$ as above. An application of theorem 4.1 leads to the desired conclusions. □

Our formulation incorporates gradient and more generally, subdifferential systems which are important in nonsmooth optimal control. In particular, if $A = \partial\delta_K$, where δ_K is the indicator function of a nonempty, closed and convex set $K \subseteq \mathbb{R}^N$ (i.e. $\delta_K(x) = 0$ if $x \in K$ and $+\infty$ otherwise), then the resulting system is known as "Differential Variational Inequality" and arises in mathematical economics (see Aubin-Cellina [1] and Henry [7]) in the study of resource allocation mechanisms and in theoretical mechanics (see Moreau [9]), in the study of unilateral problems. Recall that $\partial\delta_K = N_K$ (the normal cone to K). So system (1) has the following particular form:

$$\left\{ \begin{array}{c} -\dot{x}(t) \in N_K(x(t)) + F(t, x_t) \ \ a.e. \text{ on } \quad T = [0,b] \\ x(v) = \varphi(v) \quad v \in T_0 = [-r, 0]. \end{array} \right\}$$

Such systems, with no memory (i.e. $r = 0$) and with a time-dependent set K, were studied by the author in [12].

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