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SOME NOTES ON THE COMPOSITE G–VALUATIONS

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Abstract. In analogy with the notion of the composite semi-valuations, we define the composite G-valuation v from two other G-valuations w and u. We consider a lexicographically exact sequence $(a, \beta) : A_u \to B_v \to C_w$ and the composite G-valuation v of a field K with value group B_v. If the assigned to v set $R_v = \{x \in K / v(x) \geq 0$ or $v(x)$ non comparable to $0\}$ is a local ring, then a G-valuation w of K into C_w is defined with its assigned set R_w a local ring, as well as another G-valuation u of a residue field is defined with G-value group A_u.

1. Preliminaries

It is our main aim to show that under some differentiations and some adjustments it is possible to transfer the theory of the composite semi-valuations as it is exposed by Ohm in [2], to the case of the G-valuations. So an appropriate homomorphism is introduced, the composite G-valuations are defined by analogy to the former ones and similar conditions are stated under which an ordered exact sequence splits.

1.1. As it is known (e.g.[1]) a *G-valuation* is a function v of the multiplicative group K^* of a field K, in an ordered group G such that for all x, y in K^*:

(i) $v(xy) = v(x) + v(y)$
(ii) if $v(x) > \gamma$ and $v(y) > \gamma$, then $v(x+y) > \gamma$, for each $\gamma \in G$
(iii) $v(-1) = 0$

We can extend v on K by specifying that $v(0) = \infty$, where ∞ is a symbol such that $a < \infty$ and $a + \infty = \infty$ for all $a \in G$.

Relation (ii) may be written as
(ii)′ $v(x+y) \geq inf_{\tilde{G}}\{v(x), v(y)\}$.

In fact, the $inf_{\tilde{G}}$ means the infimum in a concrete order- completion, where the relation $a \geq inf_{\tilde{G}}(a_1, a_2)$ gives that a is larger than or equal to the smaller of a_1, a_2, but it would be parallel to the smaller or to both of them.

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1.2. As usual a short exact sequence of ordered groups

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0 \tag{1}$$

is called *lexicographically exact* if $B^+ = \{b \in B : \beta(b) > 0 \text{ or } b \in \alpha(A^+)\}$, A^+ and B^+ are the positive cones of A and B, respectively.

The notation $(\alpha, \beta) : A \to B \to C$ will also be used for the short exact sequence (1).

1.3. The G-homomorphism. If B and C are ordered groups and β is a homomorphism of B into C, then β is said to be a *G-homomorphism* if for every b_1, b_2, ..., b_n in B the relation $b_0 \geq inf_{\tilde{B}}\{b_1, ..., b_n\}$ implies $\beta(b_0) \geq inf_{\tilde{C}}\{\beta(b_1), ..., \beta(b_n)\}$.

It is not difficult for one to prove the following:

Propositions.

(1) *If v is a G-valuation defined on a field K, ranging over an ordered group B and if $\beta : B \to C$ is a G-homomorphism, then $\beta \circ v$ is a G-valuation.*

(2) *If, in the short exact sequence $(\alpha, \beta) : A \to B \to C, \alpha$ and β are G-homomorphisms, then $\beta \circ \alpha$ is also a G-homomorphism.*

(3) *If the sequence $(\alpha, \beta) : A \to B \to C$ is lexicographically exact, then α is a G-homomorphism.*

(4) *If B and C are lattice groups, the homomorphism $\beta : B \to C$ is a G-homomorphism iff β preserves the positiveness of the positive elements and moreover $inf_B\{b_1, ..., b_n\} = inf_C\{\beta(b_1), ..., \beta(b_n)\}$ for every subset $\{b_1, ..., b_n\}$ of B.*

1.4. The rings of a G-valuation. Let K be a field and v a G-valuation of it. The set $R = \{x \in K : v(x) \geq 0\}$ is not in general a ring, but as long as it is a ring, the set $M = \{x \in k : w(x) > 0\}$ is a maximal ideal.

It is possible to be defined some rings of K via a G-valuation, for instance the set

$$R_1 = \{x \in K : w(x) \text{ is larger than all the negative elements of } G\}$$

is a ring.

On the other hand there holds the following (the non-comparable elements are called parallel):

Proposition. *Given a G-valuation w of a field K, if the positive elements of the value group are larger than the parallel to zero elements, then the set $R = \{x \in K : w(x) \geq 0$ or $w(x)$ parallel to zero$\}$ is a ring and the set $M = \{x \in K : w(x) > 0\}$ is a maximal ideal.*

In the sequel, given a G-valuation w of a field K we symbolize by R_w the set

$$R_w = \{x \in K : w(x) \geq 0 \text{ or } w(x) \text{ parallel to zero}\} \tag{2}$$

2. The composite G-valuations

Throughout the text we fix the following notation: K is always a field, w is a G-valuation of K and assume that the set R_w is a quasi- local ring with maximal ideal m_w and residue field $k = R_w/m_w$. We note by h the canonical homomorphism of R_w onto k. Let u be a G-valuation of k, and let v be a G-valuation of K assigned to the subset $R_v = h^{-1}(R_u)$.

If R_u and R_v are rings, then v is said to be *composite with w and u.*

Let, furthermore, A_u, B_v and C_w denote the respective G-value groups of u, v and w and let U_u, U_v and U_w be the respective multiplicative groups of units of R_u, R_v and R_w.

2.1. Proposition. *Suppose that R_v and R_w are rings; then there exist G-homomorphisms α and β which complete commutatively the diagram below and make the bottom row lexicographically exact (i the identity, h' the restriction of h to U_w).*

$$\begin{tikzcd}
 & U_w \arrow[r, "i"] \arrow[d, "uh'"'] & K^* \arrow[d, "v"'] \arrow[dr, "w"] & & \\
0 \arrow[r] & A_u \arrow[r, "a"] & B_v \arrow[r, "\beta"] & C_w \arrow[r] & 0
\end{tikzcd}$$

The proof follows as in [2]. The definition of α and β becomes as follows:

$$Ker\beta = Imv|_{\{x\in R_v: w(x)=0\}} \text{and } Ker\alpha = Imuh'|_{U_v}\,.$$

2.2. The case of C_w being a totally ordered group. In such a case R_w is a ring and given w and v we define $v : K^* \to A_u \oplus C_w$ by

$$v(x) = (uh(x), w(x))\,. \tag{4}$$

Then it is true the following:

Proposition. *If A_u is a G-value group and C_w a totally ordered group, then $A_u \oplus C_w$ is a G-value group.*

Proof. It follows from a well-known statement of Krull (cited in [3], p.31). We define a G-valuation w with value group C_w, while (by the definition of A_u) a G-valuation u is defined on the set k.

In that case the short exact sequence $(\alpha, \beta) : A_u \to B_v \to C_w$ splits, that is

$$\begin{tikzcd}
0 \arrow[r] & A_u \arrow[r, "a"] \arrow[d, "i_1"] & B_v \arrow[r, "\beta"] \arrow[d, "i_3"] & C_w \arrow[r] \arrow[d, "i_2"] & 0 \\
0 \arrow[r] & A_u \arrow[r, "a"] & A_u \oplus C_w \arrow[r, "\beta"] & C_w \arrow[r] & 0
\end{tikzcd}$$

where i_1, i_3 are the identity maps and i_2 is an order-isomorphism. $\square$

2.3. Theorem. *Let $(\alpha, \beta) : A_u \to B_v \to C_w, A_u \neq \{0\}$ be a lexicographically exact sequence and v a G-valuation of a field K with G-value group B_v and its assigned set R_v a local ring. Then, (1) a G-valuation w of K into C_w is defined with R_w a local ring, (2) the ideal m_w is maximal and (3) a G-valuation u of the residue field R_w/m_w is defined with G-value group A_u and for which the known commutative diagram (3) is valid.*

Proof (1). Put $w(x) = \beta v(x)$. Then $\beta v(xy) = \beta v(x) + \beta v(y)$, or $w(xy) = w(x) + w(y)$.

Let now be $w(x_1) > \gamma, w(x_2) > \gamma$ or $\beta v(x_1) > \beta v(b)$ (where $(\beta v(b) = \gamma)$ and $\beta v(x_2) > \beta v(\gamma)$ or $\beta(v(x_1) - v(b)) > 0$, $\beta(v(x_2) - v(b)) > 0$, that is $v(x_1) > v(b), v(x_2) > v(b)$, hence $v(x_1+x_2) > v(b)$. We examine whether $v(x_1+x_2) - v(b)$ belongs also to $\alpha(A_u^+)$. Since $A_u \neq \{0\}$, there exists an element a in A_u neither zero nor smaller than zero; thus $v(x_1+x_2) - v(b) + \alpha(a) \in \alpha(A_u)$ and $v(x_1) - v(b) > v(x_1+x_2) - v(b) + \alpha(a)$ (since $\beta(v(x_1) - v(b) - v(x_1+x_2) - v(b) + \alpha(a)) = \beta(v(x_1) - v(b)) - \beta(v(x_1+x_2) - v(b) + \alpha(a)) = \beta(v(x_1) - v(b)) > 0)$.

Similarly, $v(x_2) - v(b) > v(x_1+x_2) - v(b) + \alpha(\alpha)$, hence $v(x_1+x_2) - v(b) > v(x_1+x_2) - v(b) + \alpha(a)$ or $\alpha(a) < 0$, which is absurd, and thus $v(x_1+x_2) - v(b) \notin \alpha(A_u)$ and $v(x_1+x_2) - v(b) > 0$, that is $w(x_1+x_2) > \beta(v(b)) = \gamma$.

If R_v is a ring, m_v is a maximal ideal. Let be $w(x), w(y) \in R_w$; if $w(x+y) \notin R_w$, then $w(x+y) < 0$, $\beta v(x+y) < 0$. But then $v(x+y) < \alpha(A_u)$, which is absurd (because, if $w(x), w(y) \in \alpha(A_u)$, then $v(x+y)$ would be smaller than both of them, if $v(x) \notin \alpha(A_u), v(y) \notin \alpha(A_u)$, then $v(x+y) \geq 0$ or parallel to zero, that is $\beta v(x+y) = 0$ if it belonged to $\alpha(A_u)$ or ≥ 0 or parallel to zero if it didn' t belong to $\alpha(A_u)$). It remains the case $v(x+y) \notin \alpha(A_u)$ and one of $v(x), v(y)$ belongs to $\alpha(A_u)$. But then, one of $v(x), v(y)$, say $v(x) \notin A_u$, is parallel to zero, thus it is not possible $v(x+y) < 0$.

(2) As usual $m_w \subset R_v$. If $x \in R_v$, then, either $v(x) \in \alpha(A_u)$ or not, it is $\beta v(x) = w(x) \in R_w$. Since $A_u \neq \{0\}$ contains positive elements, then there is an $a \in A_u$ with $\alpha(a) < 0$, $\beta\alpha(a) = 0$, that is $R_v \neq R_w$. Besides, there holds $U_v + m_w \subset U_v$.

(3) Definition of u: let h denote the canonical homomorphism of R_w onto $k = R_w/m_w$ and h' the restriction of h into U_w. The homomorphism (uh') is defined by $\alpha^{-1}vi$. It is $kerh' = 1 + m_w \subset U_v = ker(\alpha^{-1}vi)$. So, u is well defined.

It is a G-valuation because h' preserves the addition and the (uh') is a G-valuation. If x and y are elements of $U_w(modm_w)$, then $v(x) = v(y)$. It means that the equivalent elements have equal values $\alpha^{-1}vi(x), \alpha^{-1}vi(y)$, hence correspond to an element of A_u and so u can be defined. There holds: let be $u(x) > \gamma, u(y) > \gamma$ and $\gamma = \alpha^{-1}(\gamma')$. Then, $\alpha(u(x) - \gamma) = v(x) - \gamma' > 0$, $v(y) > \gamma'$, hence $v(x+y) > \gamma'$ and thus $\alpha^{-1}(v(x+y) - \gamma') > 0 \Rightarrow \alpha^{-1}v(x+y) > \alpha^{-1}(\gamma') = \gamma \Rightarrow uh'(x+y) = \alpha^{-1}v(x+y) > \gamma$.
We also have $uh'(xy) = \alpha^{-1}vi(xy) = \alpha^{-1}(vi(x) + vi(y))$.

2.4. The non-archimedean character of B_v.

Suppose there exists an element $a \in A_u$ neither parallel nor equal to zero. Let $\alpha(a) = a^*$. Observe that na^*, for every $n \in N$, must not be larger than any positive or parallel to zero element of $B_v - \alpha(A_u)$. In fact, at that case we will have for some $b^* \in B_v - \alpha(A_u)$ that $\beta(a^* - b^*) > 0$ or $\beta(a^*) > \beta(b^*)$ or $\beta(b^*) < 0$, which is absurd.

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