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TWO SORTS OF BOUNDARY–VALUE PROBLEMS OF NONLINEAR THIRD ORDER DIFFERENTIAL EQUATIONS

Michal Greguš

Abstract. Two sorts of nonlinear third order boundary-value problems are solved and the existence of eigenvalues and eigenfunctions is proved.

1. The aim of this paper is to study two sorts of boundary-value problems of the third order.

At the first we will study the boundary-value problem

$$u''' + q(t,\lambda)u' + p(t,\lambda)h(u) = 0\,, \tag{a}$$

$$u(-a,\lambda) = u'(-a,\lambda) = 0, \quad u(a,\lambda) = 0, a > 0\,, \tag{1}$$

or

$$u(-a,\lambda) = u''(-a,\lambda) = 0, \quad u(a,\lambda) = 0 \tag{2}$$

under certain suppositions on the functions q, p, h.

The problem $(a), (1), or (a), (2)$ is a generalization of the boundary-value problem for linear third order differential equation [2], where in par. 4, the so called generalized Sturm theory for linear third order boundary-value problems is developed.

At the second we will investigate the boundary-value problem of the form

$$u''' + [\mu f(t) + \lambda g(t)]u' + \lambda p(t)h(u) = 0\,, \tag{b}$$

$$u(-a,\lambda,\mu) = u(a,\lambda,\mu) = 0, a > 0\,, \tag{3}$$

$$\begin{aligned} \lambda \int_{-a}^{a} r(t,\mu)[g(t)u(t,\lambda,\mu) + \int_{-a}^{t}\{p(\tau)h(u(\tau,\lambda,\mu)) \\ - g'(\tau)u(\tau,\lambda,\mu)\}d\tau]dt = \mu \int_{-a}^{a} r(t,\mu)[f(t)u(t,\lambda,\mu) \\ - \int_{-a}^{t} f(\tau)u'(\tau,\lambda,\mu)d\tau]dt, \end{aligned} \tag{4}$$

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where λ, μ are parameters and f, g, p, h, r are suitable functions of their arguments. The boundary condition (4) is in the integral form. For the first time such a condition was formulated in [3] for a special linear third order boundary-value problem arising in physics. The problem was generalized for the linear third order differential equation in [1].

It will be shown that under certain conditions on the coefficients of (b) and on the function r and parameter μ the problem $(b), (3), (4)$ can be solved by means of the problem $(b), (2)$.

2. In this section we will investigate the nonlinear differential equation

$$u''' + q(t)u' + p(t)h(u) = 0\,, \tag{a$_1$}$$

where q, p are continuous functions of $t \in [-a, \infty), a > 0$, and h is a continuous function of $u \in (-\infty, \infty)$.

Under a solution of (a_1) we will understand a function u with continuous third derivative, defined on $[t_0, b), -a \leq t_0 < b$, that fulfils equation (a_1) on this interval. The solution u defined on $[t_0, b)$, nontrivial in a neigbourhood of b will be called oscillatory on $[t_0, b)$ if it has infinite number of zeros on this interval with the limit point at b. Otherwise the solution is called nonoscillatory. In this paper we will interested in the solutions defined on $[t_0, \infty), t_0 \geq -a$.

Lemma 1. *Let $|h(u)| < K, K > 0$ for all $u \in (-\infty, \infty)$.*
Then every solution u of (a_1) defined on $[t_0, b)$, $t_0 \geq -a$, $b > t_0$, is extendable to the interval $[t_0, \infty)$.

Proof. Let y_1, y_2, y_3 be a fundamental system of solutions of the linear differential equation

$$y''' + q(t)y' = 0$$

and let their wronskian $W(t) = 1$ for $t \in [-a, \infty)$.

Let u be a solution of (a_1) defined on $[t_0, b), b < \infty$.

Let $u(t_0) = u_0, u'(t_0) = u_0', u''(t_0) = u_0''$ and let at least one of the numbers u_0, u_0', u_0'' be different from zero.

Equation (a_1) can be writen in the form

$$u''' + q(t)u' = -p(t)h(u)\,,$$

where $u = u(t)$ for $t \in [t_0, b)$.

Then from the method of variation of constants there follows

$$u(t) = y(t) - \int_{t_0}^{t} p(\tau)h[u(\tau)]W(t, \tau)d\tau\,,$$

$$u'(t) = y'(t) - \int_{t_0}^{t} p(\tau)h[u(\tau)]W_t'(t, \tau)d\tau\,,$$

$$u''(t) = y''(t) - \int_{t_0}^{t} p(\tau)h[u(\tau)]W_t''(t, \tau)d\tau\,,$$

where $y(t_0) = u_0, y'(t_0) = u_0', y''(t_0) = u_0''$ and

$$W(t,\tau) = \begin{vmatrix} y_1(t), & y_2(t), & y_3(t) \\ y_1(\tau), & y_2(\tau), & y_3(\tau) \\ y_1'(\tau), & y_2'(\tau), & y_3'(\tau) \end{vmatrix} .$$

From the boundedness of $h[u(t)]$ there follows that for $b < \infty$ the functions u, u', u'' have at the point b the finite limits and therefore solution u is extendable to the right of b. □

Lemma 2. *Let p, q' be continuous functions of $t \in [-a, \infty)$ and let $p(t) > 0, q'(t) \le 0$ for all $t \in [-a, \infty)$.*
Let the differential equation

$$v'' + \frac{1}{4} q(t) v = 0$$

be oscillatory on $[-a, \infty)$. Let further h be continuous for every $u \in (-\infty, \infty)$ and let

(i) $$h(u)u > 0 \quad \text{for} \quad u \ne 0$$

and

(ii) $$\lim_{u \to 0} \frac{h(u)}{u} = \theta, 0 \le \theta < \infty .$$

If u_1 is a nontrivial solution of (a_1) defined on $[t_0, \infty), t_0 \ge -a$, with the property

(5) $$u_1(t_0)u_1''(t_0) - \frac{1}{2} u'^2_1(t_0) + \frac{1}{2} q(t_0) u_1^2(t_0) \le 0 ,$$

then u_1 is oscillatory on $[t_0, \infty)$.

Proof. Let u_1 be a solution of (a_1) defined on $[t_0, \infty)$ with property (5). u_1 fulfils at the same time the linear differential equation

(6) $$u''' + q(t)u' + p(t)H[u_1(t)]u = 0 ,$$

where

$$H[u_1(t)] = \begin{cases} \frac{h[u_1(t)]}{u_1(t)} & \text{for} \quad u_1(t) \ne 0 \\ \theta & \text{for} \quad u_1(t) = 0 \end{cases}$$

Equation (6) can be writen in the normal form [2]

$$u''' + q(t)u' + [\frac{1}{2} q'(t) + p(t)H[u_1(t)] - \frac{1}{2} q'(t)]u = 0 ,$$

where $p(t)H[u_1(t)] - \frac{1}{2} q'(t) \le 0$ for all $t \in [t_0, \infty)$. (It is so called Laguerre's invariant [2]). Equation (6) fulfils the supposition of Theorem 2 .4 [2] and therefore every its solution u with property (5) is oscillatory on $[t_0, \infty)$ and u_1 is a solution of (6). $\square$

3. In this section we will deal with the differential equation (a) where $p, q, q' = \frac{\partial q}{\partial t}$ are continuous functions of $t \in [-a, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and h is a function of $u \in (-\infty, \infty)$ with continuous first derivative h' on this interval. The aim of the section is the solution of the boundary-value problem $(a), (1), or (a), (2)$. From the general theorem on differential systems of the first order with the right saids continuously depending on parameter $\lambda \in (\Lambda_1, \Lambda_2)$ [4], it follows for every solution u of equation (a) defined on $[t_0, \infty)$, that u, u', u'' are continuous functions of t and λ in every closed twodimensional interval for t and λ which is a subset of the interval $[t_0, \infty)$ x (Λ_1, Λ_2).

Lemma 3. *Let the above suppositions on p, q, h be fulfilled and let $q'(t, \lambda) \le 0$ for all $t \in [-a, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$ and moreover let $(i), (ii)$ hold. If u_1 is a nontrivial solution of (a) defined on $[t_0, \infty), t_0 \ge -a$ with the property $u_1(t_0, \lambda) = 0$ for all $\lambda \in (\Lambda_1, \Lambda_2)$ then the zeros of u_1 on (t_0, ∞) (if exist) are continuous functions of the parameter $\lambda \in (\Lambda_1, \Lambda_2)$.*

Proof. Solution u_1 fulfils at the same time the linear differential equation

$$u''' + q(t, \lambda)u' + p(t, \lambda)H[u_1(t, \lambda)]u = 0\,. \tag{7}$$

Equation (7) fulfils the supposition of Lemma 4.2 [2] and the assertion of Lemma 3 follows from this Lemma 4.2. $\square$

Corollary 1. *Let the suppositions of Lemma 3 be fulfiled and let the differential equation*

$$v'' + \frac{1}{4}q(t, \lambda)v = 0$$

be oscillatory on $[-a, \infty)$ for every $\lambda \in [\bar{\Lambda}, \Lambda_1), \Lambda_1 < \bar{\Lambda} < \Lambda_2$.

If u_1 is a nontrivial solution of (a) with the property $u_1(t_0, \lambda) = 0, \lambda \in [\bar{\Lambda}, \Lambda_2)$ then u_1 is oscillatory on $[t_0, \infty)$ and its zeros are continuous functions of $\lambda \in [\bar{\Lambda}, \Lambda_2)$.

The proof follows from Lemma 2 and Lemma 3.

Lemma 4. *(Oscillation Lemma) Let the suppositions of Lemma 3 on p, q, h be satisfied and let further*

$$\lim_{\lambda \to \Lambda_2} q(t, \lambda) = +\infty$$

uniformly for all

$$t \in [-a, \infty)\,.$$

Let $-a \le t_0 < T < \infty$ and let u_1 be a nontrivial solution of (a) defined on $[t_0, \infty)$ with the property $u_1(t_0, \lambda) = 0$ for every $\lambda \in (\Lambda_1, \Lambda_2)$. With increasing $\lambda \to \Lambda_2$

the number of zeros of u_1 in $[t_0, T]$ increases to infinity and at the same time the distance between any two neighbouring zeros of u_1 converges to zero.

Proof. Solution u_1 of (a) is at the same time the solution of (7). The coefficients of (7) fulfil the suppositions of Theorem 4.5. b) in [2] (Oscillation Theorem) and therefore the assertion of Lemma 4 follows from this Theorem 4.5 b). □

Theorem 1. *Let p, q, h satisfy the suppositions of Lemma 4 on $[-a, \infty)$ and $\lambda \in (\Lambda_1, \Lambda_2)$. Let u be one of the nontrivial solutions of (a) with the property*

$$u(-a, \lambda) = u'(-a, \lambda) = 0 \tag{8}$$

defined on $[-a, \infty)$. Then there exists a natural number γ, or $\gamma = 0$ and a sequence of values of parameter λ, $\{\lambda_{\gamma+p}\}_{p=0}^{\infty}$ with a corresponding sequence of functions (eigenfunctions) $\{u_{\gamma+p}\}_{p=0}^{\infty}$ such that $u_{\gamma+p} = u(t, \lambda_{\gamma+p})$ is a solution of (a) satisfying the boundary conditions (1) and $u_{\gamma+p}$ has exactly $\gamma + p$ zeros in $(-a, a)$.

Proof. Let u be a solution of (a) with property (8) defined on $[-a, \infty)$. In vitue of Lemma 3 and Corollary 1 there exists such a $\lambda = \bar{\Lambda}$, that the solution u is oscillatory on $[-a, \infty)$ for every $\lambda \in [\bar{\Lambda}, \infty)$ and its zeros are continuous functions of $\lambda \in [\bar{\Lambda}, \Lambda_2)$. Denote by $t_n(\bar{\Lambda})$, $n = 1, 2, ...,$ the zeros of $u(t, \bar{\Lambda})$ to the right of $-a$. Let $u(t, \bar{\Lambda})$ have exactly γ zeros on $(-a, a)$. Then there is $t_\gamma(\bar{\Lambda}) < a \leq t_{\gamma+1}(\bar{\Lambda})$. According to Lemma 4 there exists $\bar{\lambda} > \bar{\Lambda}$ such that $t_{\gamma+1}(\bar{\lambda}) < a$ and according to Lemma 3 there exists such a $\lambda_\gamma, \bar{\Lambda} \leq \lambda_\gamma < \bar{\lambda}$ that $t_{\gamma+1}(\lambda_\gamma) = a$ and $u(t, \lambda_\gamma) = u_\gamma$ satisfies conditions (1) and has exactly γ zeros in $(-a, a)$. Proceeding in this way we prove the existence of sequences $\{\lambda_{\gamma+p}\}_{p=0}^{\infty}$ □

Remark 1. The boundary-value problem $(a), (2)$ can be solved by the same arguments as the problem $(a), (1)$, but it is necessary to take the condition

$$u(-a, \lambda) = u''(-a, \lambda) = 0$$

instead of condition (8).

4. Consider in this section the differential equation (b) and the boundary conditions (3), (4) and suppose that the functions f', g' and p are continuous functions of $t \in (-\infty, \infty)$. Then the following lemma is true.

Lemma 5. *Let μ^* be one of the eigenvalues and $r^*(t, \mu^*)$ be the corresponding eigenfunction of the second order eigenvalue problem*

$$r'' + \mu f(t) r = 0, r(-a, \mu) = r(a, \mu) = 0\,. \tag{9}$$

If $u = u(t, \lambda, \mu^*)$ is a solution of (b), which fulfils the boundary conditions for $\mu = \mu^*$

$$u(-a, \lambda, \mu^*) = u''(-a, \lambda, \mu^*) = u(a, \lambda, \mu^*) = 0\,, \tag{10}$$

then u is a solution of the boundary value problem (b), (3), (4), where $\mu = \mu^*$ and $r = r^*(t, \mu^*)$, too.

Proof. Integrating the differential equation (b), where $\mu = \mu^*$, writen in the form

$$u''' + \{[\mu^* f(t) + \lambda g(t)]u\}' + \{[-\mu^* f'(t) - \lambda g'(t)]u(t, \lambda, \mu^*) + \lambda p(t)h[u(t, \lambda, \mu^*]\} = 0$$

term by term from $-a$ to t, $t \le a$, and considering (10) we get

$$u'' + \mu^* f(t)u + \lambda g(t)u + \int_{-a}^{t} \{[-\mu^* f'(\tau) - \lambda g'(\tau)]u(\tau, \lambda, \mu^*) \\ + \lambda p(\tau)h[u(\tau, \lambda, \mu^*)]\}d\tau = 0\,.$$

Now multiply the last equality by $r^*(t, \mu^*)$ and integrate it from $-a$ to a. We come to the equality

$$-\int_{-a}^{a} r^*(t, \mu^*)[u''(t, \lambda, \mu^*) + \mu^* f(t)u(t, \lambda, \mu^*)]dt = \tag{11}$$
$$\lambda \int_{-a}^{a} r^*(t, \mu^*)\{g(t)u(t, \lambda, \mu^*) + \int_{-a}^{t} [p(\tau)h(u(\tau, \lambda, \mu^*)) - g'(\tau)u(\tau, \lambda, \mu^*)]d\tau\}dt \\ -\mu^* \int_{-a}^{a} r^*(t, \mu^*)\{f(t)u(t, \lambda, \mu^*) - \int_{-a}^{t} f(\tau)u'(\tau, \lambda, \mu^*)d\tau\}dt\,.$$

The right-hand side of (11) contains the expression which stands in the boundary condition (4). Therefore it is necessary to prove that the integral on the left-hand side of (11) is equal to zero. Calculate this integral and suppose (9) and (10). We obtain

$$\int_{-a}^{a} [u''(t, \lambda, \mu^*) + \mu^* f(t)u(t, \lambda, \mu^*)]r^*(t, \mu^*)]dt = u'(a, \lambda, \mu^*)r^*(a, \mu^*) - \\ u'(-a, \lambda, \mu^*)r^*(-a, \mu^*) + \int_{-a}^{a} [{r^*}''(t, \mu^*) + \mu^* f(t)r^*(t, \mu^*]dt = 0 \qquad \square$$

Corollary 2. *Let in equation (b) be $f(t) = 1, p(t) = 1, g(t) > 0$ and $g'(t) \le 0$ for $t \in [-a, \infty)$.*
Then every solution u of the boundary-value problem

$$u''' + \left[\left(\frac{k\Pi}{2a}\right)^2 + \lambda g(t)\right] u' + \lambda h(u) = 0 \tag{b$_1$}$$

$$u(-a, \lambda) = u''(-a, \lambda) = u(a, \lambda) = 0 \tag{12}$$

is also a solution of (b_1) which fulfils the boundary conditions

$$u(-a, \lambda) = u(a, \lambda) = 0$$

and

$$\int_{-a}^{a} \sin\frac{k\Pi}{2a}(a+t)[g(t)u(t,\lambda)+\int_{-a}^{t}\{h(u(\tau,\lambda))-g'(\tau)u(\tau,\lambda)\}d\tau]dt = 0 \tag{13}$$

Proof. It follows from Lemma 5, applied on the equation

$$u''' + [\mu + \lambda g(t)]\, u' + \lambda h(u) = 0 \tag{14}$$

The second order eigenvalue problem (9) for $f(t) = 1$ has the form

$$r'' + \mu r = 0, r(-a,\mu) = r(a,\mu) = 0$$

Its eigenvalues are $\mu_k^* = \left(\frac{k\Pi}{2a}\right)^2, k = 1, 2, ...$ and the corresponding eigenfunctions are

$$r_k^* = \sin\frac{k\Pi}{2a}(a+t), k = 1, 2, \dots.$$

It is necessary to prove that the boundary condition (4) has the form (13) in this case. It will be proved if the right-hand side of (4) is equal to zero. But it follows from the supposition $f(t) = 1$ and from the condition (3). □

At the end it is necessary to formulate the conditions on f, g, h for the solution of the problem (b), (10) and at the same time of the problem (b), (3), (4).

Theorem 2. *Let $f(t) > k > 0, g(t) > k > 0, p(t) > 0$ for $t \in [-a,\infty)$ and let $f'(t) \le 0, g'(t) \le 0$. Let further h have the properties (i), (ii) and h' be continuous on $(-\infty,\infty)$.*

Let μ be one of the positive eigenvalues of (9) and $r(t,\mu)$ its corresponding eigenfunction. Then there exists a natural number γ or $\gamma = 0$ and a sequence $\{\lambda_{\gamma+p}\}_{p=0}^{\infty}$ of the parameter λ and a corresponding sequence of functions $\{u_{\gamma+p}\}_{p=0}^{\infty}$ such that $u_{\gamma+p} = u(t,\lambda_{\gamma+p},\mu)$ is a solution of (b) which fulfils the conditions (10) for $\mu^* = \mu$ and $u[t,\lambda_{\gamma+p},\mu)$ has in $(-a,a)$ exactly $\gamma + p$ zeros.

Proof. At the first it is easy to see, that the coefficients of (b) fulfil the suppositions of Lemma 4 and Corollary 1, because equation (b) is of the form (a), where $q(t,\lambda) = \mu f(t) + \lambda g(t) > 0$ for $\mu > 0, \lambda > 0$ and μ is one of the positive eigenvalues of (9).

The equation

$$v'' + \frac{1}{4}[\mu f(t) + \lambda g(t)]\, v = 0$$

is oscillatory in $[-a,\infty)$ for $\lambda \ge \bar{\Lambda} > 0$ and therefore it follows from Lemma 2, that every solution $u(t,\lambda,\mu)$ of (b) with the property $u(-a,\lambda,\mu) = u''(-a,\lambda,\mu) = 0$, is oscillatory in $[-a,\infty)$ for $\lambda \ge \bar{\Lambda}$. Denote by u one of them and let $t_n\left(\bar{\Lambda}\right), n = 1, 2...$, be the zeros to the right of $-a$ of $u(t,\bar{\Lambda},\mu)$. Let for $n = \gamma$ be $t_\gamma(\bar{\Lambda}) < a$ and $t_{\gamma+1}(\lambda) \ge a$. According to Lemma 3, $t_{\gamma+1}(\lambda)$ is a continuous function of λ and hence in virtue of Lemma 4 there is $\bar{\lambda} > \bar{\Lambda}$ such that $t_{\gamma+1}(\bar{\lambda}) < a$ and from the continuity of $t_{\gamma+1}(\lambda)$ there is $\bar{\lambda} \ge \lambda_\gamma < \bar{\Lambda}$ such that $t_{\gamma+1}(\lambda_\gamma) = a$ and $u(t,\lambda_\gamma,\mu)$ satisfies the condition (10) and has exactly γ zeros in $(-a,a)$. Proceeding in this way we prove the existence of the sequences $\{\lambda_{\gamma+p}\}_{p=0}^{\infty}$ and $\{u_{\gamma+p}\}_{p=0}^{\infty}$ Thus the theorem is proved. □

Example. Have the boundary value problem

$$u''' + (k^2 + \lambda)u' + \lambda \; arctg \; u = 0 \tag{15}$$

$$u(-\frac{\Pi}{2}, \lambda) = u(\frac{\Pi}{2}, \lambda) = 0 \tag{16}$$

$$\int_{-\frac{\Pi}{2}}^{\frac{\Pi}{2}} \sin k \left(\frac{\Pi}{2} + t\right) [u(t, \lambda) + \int_{-\frac{\Pi}{2}}^{t} \; arctg \quad u(\tau, \lambda) d\tau] dt = 0 \tag{17}$$

From Theorem 2 there follows, that every solution u of equation (15) which fulfils the boundary conditions

$$u(-\frac{\Pi}{2}, \lambda) = u''(-\frac{\Pi}{2}, \lambda) = u(\frac{\Pi}{2}, \lambda) = 0$$

is the solution of (15), (16), (17), because it is easy to see, that the function $r(t, \mu) = \sin k(\frac{\Pi}{2} + t)$ is the eigenfunction of the problem $r'' + \mu r = 0, r(-\frac{\Pi}{2}, \mu) = r(\frac{\Pi}{2}, \mu) = 0$ with the eigenvalue $\mu = k^2$, where k is a natural number.

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Michal Greguš
Department of Mathematical Analysis
MFF UK, Mlynská Dolina
842 15 Bratislava, SLOVAKIA