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EMBEDDING OF HILBERT MANIFOLDS WITH SMOOTH BOUNDARY INTO SEMISPACES OF HILBERT SPACES

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Abstract. In this paper we prove the existence of a closed neat embedding of a Hausdorff paracompact Hilbert manifold with smooth boundary into $H \times [0, +\infty)$, where H is a Hilbert space, such that the normal space in each point of a certain neighbourhood of the boundary is contained in $H \times \{0\}$. Then, we give a neccesary and sufficient condition that a Hausdorff paracompact topological space could admit a differentiable structure of class ∞ with smooth boundary.

0. Introduction

A generalization of Whitney's embedding theorem was given by J. Mc Alpin on 1965 [1] and [8]: "Every separable C^r–manifold without boundary modeled on a separable Hilbert space can be C^r–embedded as a closed submanifold of a separable Hilbert space".

On 1970 J. Eells and K.D. Elworthy [4] proved the following immersion theorem: "Let E be a C^∞–smooth Banach space of infinite dimension, with a Shauder base. Suppose that X is a separable metrizable C^∞–manifold without boundary modeled on E. If X is parallelizable, then there is a C^∞–embedding of X onto an open subset of E".

The purpose of this paper is to study embeddings in case that the infinite dimensional manifolds have boundary. We shall prove the following two theorems:

Theorem A

Let X be a Hausdorff paracompact differentiable manifold of class $p+1, p \geq 1$. Assume that X is a Hilbert manifold such that $\partial(X) \neq \phi$ and $\partial^2(X) = \phi$. Then there are a real Hilbert space H, a closed embedding $g : H \to H \times [0, +\infty)$ of class p with $g^{-1}(X \times \{0\}) = \partial(X)$, a collar neighbourhood (f, A) of $\partial(X)$ in X of class p and an open set G of $\partial(X) \times [0, +\infty)$ such that $\partial(X) \times \{0\} \subset G$, $gf(x, t) = (p_1 g(x), t)$ for every $(x, t) \in G$, $f(G) = G_1$ is an open set in X with

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$\partial(X) \subset G_1$ and $N_x^g(X) \subset H \times \{0\}$ for every $x \in G_1$, where $N_x^g(X)$ is the normal space of g at x.

Theorem B

Let X be a Hausdorff paracompact topological space. The following statements are equivalent:

a) X admits a Hilbert differentiable structure of class ∞ with $\partial(X) \neq \phi$ and $\partial^2(X) = \phi$.

b) There are a real Hilbert space H, an open set U of $H \times [0, +\infty)$ with $U \cap (H \times \{0\}) \neq \phi$ and a map $r : U \to U$ of class ∞ such that $r \cdot r = r$, $r(\partial(U)) \subset \partial(U)$, $\ker(D(r)(x)) \subset H \times \{0\}$ for every $x \in r(U) \cap \partial(U)$ and $r(U)$ is homeomorphic to X.

1. Prerequisites

Along this paper manifolds may have boundary if otherwise is not specified. Terminology and notation can be found in [6] but we explain here some of them.

Let E be a real Banach space and Λ a finite linearly independent system of elements of $\mathcal{L}(E, R)$. Then the quadrant $\{x \in E / \lambda(x) \geq 0$ for all $\lambda \in \Lambda\}$ will be denoted by E_Λ^+ and the closed linear subspace $\{x \in E/\lambda(x) = 0$ for all $\lambda \in \Lambda\}$ by E_Λ^0.

If X is a manifold, a chart of X will be denoted by $(U, \varphi, (E, \Lambda))$, where U is the domain of the chart, φ is the morphism, E is the model space, $\varphi : U \to E_\Lambda^+$ is injective and $\varphi(U)$ is an open set of E_Λ^+. For instance (E, Λ_E, E) is the natural chart of E and $(E_\Lambda^+, j, (E, \Lambda))$ is the natural chart of E_Λ^+, where j is the inclusion map.

Let E_Λ^+ be a quadrant, U an open set of E_Λ^+ and $x \in E_\Lambda^+$. Then we call index of x and denote $\mathrm{ind}(x)$, the cardinal of the set $\{i/\lambda_i(x) = 0, \lambda_i \in \Lambda\}$. The set $\{y \in U |\ \mathrm{ind}(y) \geq 1\}$ will be called boundary of U and denoted $\partial(U)$. The set $\{y \in U\ /\ \mathrm{ind}(y) = 0\}$ will be called interior of U and denoted by $\mathrm{int}(U)$, the set $\{x \in U/\ \mathrm{ind}(x) = k\}$ will be denoted by $B_k(U)$ and the set $\{x \in U/\ \mathrm{ind}(x) \geq k\}$ will be denoted by $\partial^k(U)$, where $k \in N \cup \{0\}$. From the local boundary invariance theorem we can define, in a natrual way, the index and the boundary of manifolds.

If X is a manifold and $a \in X$, we take the set $\{(c, v)/c = (U, \varphi, (E, \Lambda))$ is a chart of X with $a \in U$ and $v \in E\}$ and we consider the binary relation, $\sim$, on this set defined by:

$$(c, v) \sim (c', v') \Leftrightarrow D(\varphi' \varphi^{-1})(\varphi(a))(v) = v'.$$

Then this relation is an equivalence relation and the quotient set will be denoted by $T_a(X)$.

Let $c = (U, \varphi, (E, \Lambda))$ be a chart of X and $a \in U$. It is clear that the map $\theta_c^a : E \to T_a(X)$ defined by $\theta_c^a(v) = \sim ((c, v))$ is a bijective map. The class of equivalence $\sim ((c, v))$ will be also denoted by $[(c, v)]$. Via the map θ_c^a the space $T_a(X)$ becomes a real Banach space that will be called tangent space of X at a and θ_c^a becomes a linear homeomorphism. Moreover if $c = (U, \varphi, (E, \Lambda))$,

$c' = (U'.\varphi', (E', \Lambda'))$ are charts of X with $a \in U \cap U'$, then $(\theta^a_{c'})^{-1}\ \theta^a_c = D(\varphi'\varphi^{-1})$ $(\varphi(a))$.

If $f : X \to X'$ is a map of class p and $a \in X$, it is clear that there is a unique continuous linear map $T_a(f) : T_a(X) \to T_{f(a)}(X')$ such that for every chart $c = (U, \varphi, (E, \Lambda))$ of X at a and every chart $c' = (U', \varphi', (E', \Lambda'))$ of X' at $f(a)$, it holds $T_a(f) = \theta^{f(a)}_{c'} D(\varphi' f \varphi^{-1})\ (\varphi(a))(\theta^a_c)^{-1}$.

If X is a manifold of class p we denote by $T(X)$ the set $\{(x,v)/x \in X,\ v \in T_x(X)\}$ and by τ_X the map $\tau_X : T(X) \to X$ defined by $\tau_X(x,v) = x$. Then for every chart $c = (U, \varphi, (E, \Lambda))$ of X, the triplet $d_c = (\tau_X^{-1}(U),\ \varphi_c, (E \times E, \Lambda p_1))$ is a chart of $T(X)$ where the map $\varphi_c : \tau_X^{-1}(U) \to E \times E$ is defined by $\varphi_c((x,v)) = (\varphi(x), (\theta^x_c)^{-1}(v))$. In this way we obtain an atlas for $T(X)$ and $T(X)$ with this differentiable structure will be called tangent bundle manifold of X.

Let X be a manifold of class p and $x \in X$. A curve of class r on X with origin $x, 0 \leq r \leq p$, is a map $\alpha : [0,a) \to X$ of class r such that $\alpha(0) = x$.

If α is a curve of class r on X $(1 \leq r \leq p)$ with origin x defined on $[0,a)$, then the element of $T_x(X)$ defined by $T_0(\alpha)\theta^0_{c_0}(1)$, where $c_0 = ([0,a), i, (R, 1_R))$ is the natural chart of $[0,a)$ is called tangent vector to α at 0 and denoted $\dot\alpha(0)$. We note that if $c = (U, \varphi, (E, \Lambda))$ is a chart of X at x, then $\dot\alpha(0) = T_0(\alpha)\theta^0_{c_0}(1) = \theta^x_c D(\varphi\alpha)(0)(1) = \theta^x_c \lim\limits_{t \to 0^+} \dfrac{\varphi\alpha(t) - \varphi\alpha(0)}{t} = \theta^x_c(\varphi\alpha)'(0)$, where $\theta^0_{c_0} : R \to T_0([0,a))$ and $\theta^x_c : E \to T_x(X)$ are the natural linear homeomorphism.

If v is a tangent vector of X at x given by a curve $\alpha : [0,a) \to X$ of class 1 on X with origin x, i.e. $\dot\alpha(0) = v$, then we shall say that v is an inner tangent vector at x. The set of the inner tangent vectors at x will be denoted by $(T_xX)^i$. It holds that $T_xX = L((T_xX)^i)$, where L is the linear operator.

If $c = (U, \varphi, (E, \Lambda))$ is a chart of X such that $x \in U$ and $\varphi(x) \in E^0_\Lambda$, then $\theta^x_c(E^+_\Lambda) = (T_xX)^i = (T_xX)^+_{\Lambda'}$, where $\Lambda' = \Lambda(\theta^x_c)^{-1}$.

Let X be a manifold of class p and X' a subset of X. We say that X' is a submanifold of class p of X if for every $x' \in X'$ there are a chart $c = (U, \varphi, (E, \Lambda))$ of X with $x' \in U$ and $\varphi(x') = 0$, a closed linear subspace E' of E that admits a topological supplement in E and a finite linearly independent system Λ' of elements of $\mathcal{L}(E', R)$ such that $\varphi(U \cap X') = \varphi(U) \cap E'^+_{\Lambda'}$ and this set is open in $E'^+_{\Lambda'}$.

We say that the submanifold X' is a totally neat submanifold if $\mathrm{ind}_{X'}(x') = \mathrm{ind}_X(x')$ for every $x' \in X'$.

If only $\partial(X') = \partial(X) \cap X'$ we say that X' is a neat submanifold.

Let $(E, <,>_E)$, $(F, <,>_F)$ be real Hilbert spaces and $u : E \to F$ a linear continuous map. Then there is a unique map $u^* : F \to E$ such that $< u(x), y >_F = < x, u^*(y) >_E$ for every $x \in E$ and $y \in F$. The map u^* will be called adjoint operator of u. This operator has the following properties:

1) $u^* : F \to E$ is a linear continuous map and $\|u^*\| = \|u\|$.

2) The map $\alpha : \mathcal{L}(E,F) \to \mathcal{L}(F,E)$ defined by $\alpha(u) = u^*$ is a linear homeomorphism which is also an isometry.

3) $u^{**} = u$ for all $u \in \mathcal{L}(E,F)$.

4) If G is a real Hilbert space and $v : F \to G$ is a linear continuous map, then

$(v.u)^* = u^*.v^*$. If $E = F$, then $1_E^* = 1_E$. Therefore if $u \in \mathcal{L}(E, F)$ is an invertible operator, then u^* is also an invertible operator and $(u^*)^{-1} = (u^{-1})^*$.

5) If $(E, <,>)$ is a real Hilbert space, F is a closed linear subspace of E and $u : E \to E$ is a linear homeomorphism, then $(u(F))^\perp = (u^*)^{-1}(F^\perp)$.

Lemma 1.1 (R. Godement)

Let U, M be Hausdorff topological spaces, $g : U \to M$ a local homeomorphism, X a closed set of M and $s : X \to U$ a continuous section of g, i.e. $gs = 1_X$. Suppose that $g(U)$ is a Hausdorff paracompact space. Then, there exists an open neighbourhood W of X in M and there exists a prolongation of s to a continuous section, $\bar{s} : W \to U$, of g such that $\bar{s}(W) = U_0$ is an open set of U. □

Corollary 1.2

Let Y and Y' be Hausdorff differentiable manifolds, $f : Y \to Y'$ a differentiable map of class p and X a closed subset of Y'. Suppose that:

1) Y' is a Hausdorff paracompact space.

2) There exists a continuous map, $s : X \to Y$, such that $fs = 1_X$

3) For every $x \in X$, f is a local diffeomorphism of class p at $s(x)$.

Then there exists an open set U_0 of Y and there exists an open set W of Y' with $X \subset W$ such that $f_{|U_0} : U_0 \to W$ is a diffeomorphism of class p and $s = (f_{|U_0})^{-1}_{|X}$. □

2. The normal bundle manifold of an immersion which ranges over a Hilbert space.

Proposition 2.1

Let $(H, <,>)$ be a real Hilbert space, Y a differentiable manifold of class $p+1$, $(p \geq 1)$, and $f : Y \to H$ an immersion of class $p+1$. For every $y \in Y$ let us consider the sets $T_y^f(Y) = \left(\theta_c^{f(y)}\right)^{-1} T_y(f)\, T_y(Y) \subset H$, where $c = (H, 1_H, H)$ is the natural chart of H and $\theta_c^{f(y)} : H \to T_{f(y)}(H)$ is the natural linear homeomorphism, and $N_y^f(Y) = \{z \in H / < z, u >= 0 \text{ for every } u \in T_y^f(Y)\} = [T_y^f(Y)]^\perp \subset H$. Now we take the sets $T^f(Y) = \sum_{y \in Y} T_y^f(Y) = \{(y, v) \in Y \times H / v \in T_y^f(Y)\} \subset Y \times H$ and $N^f(Y) = \sum_{y \in Y} N_y^f(Y) = \{(y, v) \in Y \times H / v \in N_y^f(Y)\} \subset Y \times H$.

Then we have that:

a) $T^f(Y)$ and $N^f(Y)$ are closed totally neat submanifolds of class p of $Y \times H$, ($N^f(Y)$ will be called normal bundle manifold of f). In particular $\partial(T^f(Y)) = T^f(Y) \cap [\partial Y \times H]$ and $\partial(N^f(Y)) = N^f(Y) \cap [\partial Y \times H]$.

Moreover the map $\ell : T(Y) \to T^f(Y)$, defined by

$$\ell(y, v) = (y, \left(\theta_c^{f(y)}\right)^{-1} T_y(f)(v)),$$

is a diffeomorphism of class p from $T(Y)$ onto $T^f(Y)$.

b) The maps $\tau_1 : T^f(Y) \to Y$, $\tau_2 : N^f(Y) \to Y$ defined by $\tau_1(y,u) = y$, $\tau_2(y,u) = y$ are submersions of class p.

c) The maps $P : Y \times H \to T^f(Y)$ and $Q : Y \times H \to N^f(Y)$ defined by $P(y,v) = (y, p_{T^f_y(Y)}(v))$, $Q(y,v) = (y, p_{N^f_y(Y)}(v))$, where $p_{T^f_y(Y)}$ is the orthogonal projection of H onto $T^f_y(Y)$ and $p_{N^f_y(Y)}$ is the orthogonal projection of H onto $N^f_y(Y)$, (we note that $H = T^f_y(Y) \oplus_T N^f_y(Y)$ and $v = p_{T^f_y(Y)}(v) + p_{N^f_y(Y)}(v)$), are maps of class p such that $P.P = P$, $Q.Q = Q$ and $p_2 Q = p_2 - p_2 P$, where $p_2 : Y \times H \to H$ is the 2–projection.

d) $T^f(Y) \times_Y N^f(Y) = \{((y,u),(y,v))/(y,u) \in T^f(Y),\ \ (y,v) \in N^f(Y)\}$ is a submanifold of class p of $T^f(Y) \times N^f(Y)$ and it is also a submanifold of class p of $(Y \times H) \times (Y \times H)$.

e) The map $\alpha : T^f(Y) \times_Y N^f(Y) \to Y \times H$ defined by $\alpha((y,u),(y,v)) = (y, u+v)$ is a diffeomorphism of class p whose inverse is $\alpha^{-1}(y,v) = (P(y,v), Q(y,v))$. Therefore, $T^f(Y)$ and $N^f(Y)$ are closed submanifolds of $Y \times H$.

f) If $\partial(Y) = \phi$, then the map $e : N^f(Y) \to H$ of class p defined by $e(y,v) = f(y) + v$ is a local diffeomorphism of class p at $(y,0) \in N^f(Y)$ for every $y \in Y$.

Proof

a) Let y_0 be an element of Y. Since f is an immersion of class $p+1$ at y_0, there is a chart $c_1 = (U, \phi, (E, \Lambda))$ of Y with $y_0 \in U$ and $\phi(y_0) = 0$ and there is a chart $\bar{c} = (V, \Psi, H)$ of class $p+1$ of $(H, <,>)$ with $\Psi f(y_0) = 0$ and $f(U) \subset V$ such that E is a closed linear subspace of H (hence it admits a topological supplement in H), $\phi(U) \subset \Psi(V)$ and $\Psi f_{|U} \phi^{-1} = j : \phi(U) \hookrightarrow \Psi(V)$ is the inclusion map.

Then we have that

$$\begin{aligned} T^f_y(Y) &= \left(\theta^{f(y)}_c\right)^{-1} T_y(f) T_y(Y) = \left(\theta^{f(y)}_c\right)^{-1} \theta^{f(y)}_{\bar{c}} D(\Psi f \phi^{-1})(\phi(y)) \left(\theta^{y}_{c_1}\right)^{-1} T_y(Y) \\ &= \left(\theta^{f(y)}_c\right)^{-1} \theta^{f(y)}_{\bar{c}}(E) = D(\Psi^{-1})\,(\Psi f(y))(E) \end{aligned}$$

for every $y \in U$ and therefore $D\Psi(f(y))(T^f_y(Y)) = E$ for every $y \in U$. We note that $T_y(f)$ is an injective map and $im(T_y(f))$ admits a topological supplement in $T_{f(y)}(H)$.

Let $\beta : U \to GL(H) \subset \mathcal{L}(H,H)$ be the map of class p defined by $\beta(y) = D\Psi^{-1}(\Psi f(y))$ and let G be the orthogonal space of E in $(H, <,>)$, $(G = E^{\perp})$.

Since the map $\nu : GL(H) \to GL(H)$ defined by $\nu(u) = u^{-1}$ is a map of class ∞, then the map $\beta^{-1} : U \to GL(H)$ defined by $\beta^{-1}(y) = (\beta(y))^{-1} = D\Psi(f(y))$ is a map of class p. On the other hand the map $* : \mathcal{L}(H,H) \to \mathcal{L}(H,H)$ defined by $*(u) = u^*$ is a linear continuous map and therefore is a map of class ∞. Moreover $*(GL(H)) = GL(H)$ and $(u^*)^{-1} = (u^{-1})^*$ for every $u \in GL(H)$. Thus the maps $\beta^* : U \to GL(H))$ and $(\beta^*)^{-1} : U \to GL(H)$ defined by $\beta^*(y) = (\beta(y))^*$ and $(\beta^*)^{-1}(y) = (\beta^*(y))^{-1} = ((\beta(y))^{-1})^*$ are of class p.

Let us consider the map of class p

$$\Phi : \phi(U) \times H \to \phi(U) \times H$$

defined by $\Phi(z,v) = (z, \beta(\phi^{-1}(z))(p_E(v)) + (\beta^*)^{-1}(\phi^{-1}(z))(p_G(v)))$, where p_E, p_G are the orthogonal projections of H over E and G respectively.

Then for every $z \in \phi(U)$ the induced map $\Phi_z : H \to H$ is a linear homeomorphism. Since

$$\left[\beta(\phi^{-1}(z))(E)\right]^{\perp} = \left[T^f_{\phi^{-1}(z)}(Y)\right]^{\perp} = (\beta^*)^{-1}(\phi^{-1}(z))(G) = N^f_{\phi^{-1}(z)}(Y).$$

It is clear that Φ is a bijective map of class p,

$$D\Phi(z,v)(w,u) = (w, D^1(p_2\Phi)(z,v)(w) + D^2(p_2\Phi)(z,v)(u)),$$

$D^2(p_2\Phi)(z,v)(u) = \Phi_z(u)$, $D\Phi(z,v)$ is a linear homeomorphism for every $(z,v) \in \phi(U) \times H$ and $\Phi(\partial(\phi(U) \times H)) = \partial(\phi(U) \times H)$. Hence Φ is a diffeomorphism of class p and

$$\begin{aligned}\Phi^{-1}(z,u) &= (z, (\beta(\phi^{-1}(z)))^{-1} p_{\beta(\phi^{-1}(z))(E)}(u) + \beta^*(\phi^{-1}(z)) p_{(\beta^*)^{-1}(\phi^{-1}(z))(G)}(u)) \\ &= (z, (\beta(\phi^{-1}(z)))^{-1} p_{T^f_{\phi^{-1}(z)}(Y)}(u) + \beta^*(\phi^{-1}(z)) p_{|_{N^f_{\phi^{-1}(z)}(Y)}}(u)).\end{aligned}$$

Then we can take the chart $c^* = (U \times H, \Phi^{-1}(\phi \times 1_H) = \phi^*, (E \times H, \Lambda p_1))$ of class p of $Y \times H$ and we have that $\phi^*((U \times H) \cap T^f(Y)) = \phi^*(U \times H) \cap (E^+_\Lambda \times E) = \phi(U) \times E$, $\phi^*((U \times H) \cap N^f(Y)) = \phi^*(U \times H) \cap (E^+_\Lambda \times G) = \phi(U) \times G$ and $\phi^*(U \times H) = \phi(U) \times H$.

Thus we have that $T^f(Y)$ and $N^f(Y)$ are submanifolds of class p of $Y \times H$ and $c_1^* = ((U \times H) \cap T^f(Y),\ \phi_1^* = \phi^*_{|_{(U \times H) \cap T^f(Y)}}, (E \times E, \Lambda p_1))$ is a chart of $T^f(Y)$ and $c_2^* = ((U \times H) \cap N^f(Y),\ \phi_2^* = \phi^*_{|_{(U \times H) \cap N^f(Y)}}, (E \times G, \Lambda p_1'))$ is a chart of $N^f(Y)$. It is clear, using these charts, that $T^f(Y)$ and $N^f(Y)$ are totally neat submanifolds of $Y \times H$.

b) and c) are easily checked by localization.

d) We take the charts c_1^* and c_2^* constructed in the statement a). Then $c_1^* \times c_2^* = (S = ((U \times H) \cap T^f(Y)) \times ((U \times H) \cap N^f(Y)), \phi_1^* \times \phi_2^*, ((E \times E) \times (E \times G), \Lambda p_1^* \cup \Lambda p_3^*))$ is a chart of $T^f(Y) \times N^f(Y)$, $H' = \{(u,v),(u,w))/u \in E,\ v \in E,\ w \in G\}$ is a closed linear subspace of $(E \times E) \times (E \times G)$ that admits topological supplement in $(E \times E) \times (E \times G)$ and $\Lambda(p^*_{1|H'})$ is a finite linearly independent system of elements of $\mathcal{L}(H', R)$. Since $(\phi_1^* \times \phi_2^*)(S \cap (T^f(Y) \times_Y N^f(Y))) = (\phi_1^* \times \phi_2^*)(S) \cap H'^{+}_{\Lambda p^*_{1|_{H'}}}$ and $H'^{+}_{\Lambda p^*_{1|_{H'}}} \subset [(E \times E) \times (E \times G)]^+_{\Lambda p_1^* \cup \Lambda p_3^*}$, it happens that $T^f(Y) \times_Y N^f(Y)$ is a submanifold of class p of $T^f(Y) \times N^f(Y)$ and it is also a submanifold of $(Y \times H) \times (Y \times H)$.

e) It is clear that α is a bijective map of class p and $\alpha^{-1} = (P, Q)$. Moreover, from c) and d), α^{-1} is a map of class p, hence α is a diffeomorphism of class p.

f) We have that $(y_0, 0) \in (U \times H) \cap N^f(Y)$ and $(e|_{(U\times H)\cap N^f(y)})\ (\phi_2^{*-1}|_{\varphi(U)\times G}) = \gamma$, where $\gamma(z,u) = (\beta^*(\phi^{-1}(z)))^{-1}(u) + \Psi^{-1}(z)$.

Since

$$D\gamma(0,0)(u_1,u_2) = D(\Psi^{-1})(0)(u_1) + (\beta^*(y_0))^{-1}(u_2) = \beta(y_0)(u_1) + (\beta^*(y_0))^{-1}(u_2)\,,$$

$D\gamma(0,0) : E \times G \to H$ is a linear homeomorphism and therefore e is a local diffeomorphism of class p at $(y_0, 0) \in N^f(Y)$, because of $\partial(N^f(Y)) = \phi$.

In fact we have the more general situation:

Proposition 2.2

Let $(H, <,>)$ be a real Hilbert space, Λ_H a finite linearly independent system of elements of $\mathcal{L}(H,R)$, Y a differentiable manifold of class $p+1$, $(p \geq 1)$, $f : Y \to H^+_{\Lambda_H}$ an immersion of class $p+1$, $c = (H^+_{\Lambda_H},\ 1_{H^+_{\Lambda_H}},\ (H, \Lambda_H))$ the natural chart of $H^+_{\Lambda_H}$ and $c' = (H, 1_H, H)$ the natural chart of H, (We note that $jf : Y \to H$ is also an immersion of class $p+1$, where $j : H^+_{\Lambda_H} \hookrightarrow H$ is the inclusion map, $\left(\theta_c^{f(y)}\right)^{-1} T_y(f) = \left(\theta_{c'}^{f(y)}\right)^{-1} T_y(jf)$ for every $y \in Y$ and $T_y(f)(T_yY)^i \subset \left[T_{f(y)}\left(H^+_{\Lambda_H}\right)\right]^i$ for every $y \in Y$). For every $y \in Y$ let us consider the sets $T^f_y(Y) = \left(\theta_c^{f(y)}\right)^{-1} T_y(f) T_y(Y) \subset H$ and $N^f_y(Y) = [T^f_y(Y)]^\perp \subset H$, (We note that $\theta_c^{f(y)} : H \to T_{f(y)}\left(H^+_{\Lambda_H}\right)$ is the natural isomorphism, $T_y(f)$ is an injective map, $N^f_y(Y) \oplus_T T^f_y(Y) = H$, $\left(\theta_c^{f(y)}\right)^{-1} T_y(f)[T_y(Y)]^i = [T^f_y(Y)]^+_{M_y}$ for every $y \in Y$, where $[T^f_y(Y)]^+_{M_y}$ is a quadrant of $T^f_y(Y)$, $\left(\theta_c^{f(y)}\right)^{-1} \left(T_{f(y)} H^+_{\Lambda_H}\right)^i = H^+_{\Lambda_H}$ for every y such that $f(y) \in H^0_{\Lambda_H}$ and $T^f_y(Y) \subset H^0_{\Lambda_H}$ for every $y \in$ *int* (Y) such that $f(y) \in H^0_{\Lambda_H}$).

Now we take the sets $T^f(Y) = \{(y,v) \in Y \times H / v \in T^f_y(Y)\} \subset Y \times H$ and $N^f(Y) = \{(y,v) \in Y \times H / v \in N^f_y(Y)\} \subset Y \times H$, (Of course we have that $T^f_y(Y) = T^{jf}_y(Y)$, $N^f_y(Y) = N^{jf}_y(Y)$, $T^f(Y) = T^{jf}(Y)$ and $N^f(Y) = N^{jf}(Y)$).

Then we have that:

a) $T^f(Y)$ and $N^f(Y)$ are closed totally neat submanifolds of class p of $Y \times H$. Moreover the map $\ell : T(Y) \to T^f(Y)$ defined by $\ell(y,v) = (y,\ \left(\theta_c^{f(y)}\right)^{-1} T_y(f)(v))$ is a diffeomorphism of class p from $T(Y)$ over $T^f(Y)$.

b) The maps $\tau_1 : T^f(Y) \to Y$, $\tau_2 : N^f(Y) \to Y$ defined by $\tau_1(y,u) = y$, $\tau_2(y,u) = y$ are submersions of class p.

c) The maps $P : Y \times H \to T^f(Y)$ and $Q : Y \times H \to N^f(Y)$ defined by $P(y,v) = (y, p_{T^f_y(Y)}(v))$, $Q(y,v) = (y, p_{N^f_y(Y)}(v))$ are maps of class p such that $P.P = P$, $Q.Q = Q$ and $p_2Q = p_2 - p_2P$, where $p_2 : Y \times H \to H$ is the 2–projection.

d) $T^f(Y) \times_Y N^f(Y) = \{((y,u),(y,v)) / (y,u) \in T^f(Y),\ (y,v) \in N^f(Y)\}$ is a submanifold of class p of $T^f(Y) \times N^f(Y)$ and it is also a submanifold of class p of $(Y \times H) \times (Y \times H)$.

e) The map $\alpha : T^f(Y) \times_Y N^f(Y) \to Y \times H$ defined by $\alpha((y,u),(y,v)) = (y, u+v)$ is a diffeomorphism of class p whose inverse is $\alpha^{-1}(y,v) = (P(y,v), Q(y,v))$. Therefore $T^f(Y)$ and $N^f(Y)$ are closed submanifolds of $Y \times H$.

f) Suppose that $\partial(Y) = f^{-1}(\partial\left(H^+_{\Lambda_H}\right))$ and that there is an open neighbourhood G of $\partial(Y)$ in Y and there is an open neighbourhood V^0 of 0 in H, such that $[V^0 \cap N^f_y(Y)] + f(y) \subset H^+_{\Lambda_H}$ for every $y \in G$ and $[V^0 \cap N^f_y(Y)] + f(y) \subset \partial H^+_{\Lambda_H}$ for every $y \in \partial Y$.

Then there is an open neighbourhood A of $\{(y,0)/y \in Y\}$ in $N^f(Y)$ such that the map $e : A \to H^+_{\Lambda_H}$ of class p defined by $e(y,v) = f(y) + v$ is a local diffeomorphism of class p at $(y,0) \in A$ for every $y \in Y$.

g) If $\partial(Y) = f^{-1}(\partial(H^+_{\Lambda_H}))$, then

$$T_y(f)(T_yY)^i \subset \left[T_{f(y)}\left(H^+_{\Lambda_H}\right)\right]^i = K;\ T_y(f)(\partial(T_y(Y))^i) \subset \partial(K)$$

and $T_y(f)\ (int\ ((T_y(Y))^i)) \subset\ int\ (K)$ for every $y \in Y$. □

3. Closed embeddings into Hilbert spaces. Tubular neighbourhoods.

Proposition 3.1

Let $(H, <,>)$ be a real Hilbert space, Λ_H a finite linearly independent system of elements of $\mathcal{L}(H,R)$, Y a differentiable manifold of class $p+1$ $(p \geq 1)$, and $f : Y \to H^+_{\Lambda_H}$ a closed embedding of class $p+1$. Suppose that $\partial(Y) = f^{-1}(\partial\left(H^+_{\Lambda_H}\right))$ and that there is an open neighbourhood G of $\partial(Y)$ in Y and there is an open neighbourhood V^0 of 0 in H such that $[V^0 \cap N^f_y(Y)] + f(y) \subset H^+_{\Lambda_H}$ for every $y \in G$ and $[V^0 \cap N^f_y(Y)] + f(y) \subset \partial\left(H^+_{\Lambda_H}\right)$ for every $y \in \partial Y$.

Consider the totally neat submanifolds $T^f(Y)$ and $N^f(Y)$ of class p of $Y \times H$ and the map $e : A \subset N^f(Y) \to H^+_{\Lambda_H}$ of class p defined by $e(y,v) = f(y) + v$, (we know, from Proposition 2.2 that e is a local diffeomorphism of class p at $(y,0) \in A$ for every $y \in Y$. Moreover $[T^f_y(Y)]^+_{M_y} = T^f_y(Y) \cap H^+_{\Lambda_H}$ for every y with $f(y) \in H^0_{\Lambda_h}$).

Then we have that:

a) There is an open set Ω_A of $A \subset N^f(Y)$ with $Y \times \{0\} \subset \Omega_A$ and there is an open set W of $H^+_{\Lambda_H}$ with $f(Y) \subset W$ such that $e_{|\Omega_A} : \Omega_A \to W$ is a diffeomorphism of class p and $e\xi = f$, where $\xi : Y \to N^f(Y)$ is defined by $\xi(y) = (y,0)$.

b) $f(Y)$ is a neat submanifold of $H^+_{\Lambda_H}$ and the map $\pi : W \to W$ defined by

$$\pi = e.\xi.p_{1|\Omega_A}\left(e_{|\Omega_A}\right)^{-1}$$

is a map of class p such that $\pi(W) \subset f(Y)$ and $\pi f(y) = f(y)$ for every $y \in Y$. Hence $\pi : W \to f(Y)$ is a submersion of class p at every $f(y) \in f(Y)$. Lastly $\pi(\partial(W)) \subset \partial(f(Y))$ and there is an open set W_1 of W such that $f(Y) \subset W_1$ and $\pi_{|W_1} : W_1 \to f(Y)$ is a submersion of class p.

c) Suppose that for every $y \in \partial(Y)$ there is an open neighbourhood W^0_y of 0 in H such that

$$W_y^0 \cap [T_y^f(Y)]^+_{M_y} + f(y) \subset H^+_{\Lambda_H}$$

and $\partial\left(W_y^0 \cap [T_y^f(Y)]^+_{M_y}\right) + f(y) \subset \partial\left(H^+_{\Lambda_H}\right)$. Then for every $y \in Y$, there is an open neighbourhood U_y^0 of 0 in $(T_y(Y))^i$ and there is an open neighbourhood V^y of y in Y such that $f(y) + \left(\theta_c^{f(y)}\right)^{-1} T_y(f)(u) \in W_1$ for every $u \in U_y^0$ and the map $e_y : U_y^0 \to V^y$ defined by $e_y(u) = f^{-1}\pi[f(y) + \left(\theta_c^{f(y)}\right)^{-1} T_y(f)(u)]$, is a diffeomorphism of class p, where $c = (H^+_{\Lambda_H}, i, (H, \Lambda_H))$ is the natural chart.

d) Suppose that there is an open neighbourhood W^0 of 0 in H such that for all $y \in \partial(Y)$,

$$W^0 \cap [T_y^f(Y)]^+_{M_y} + f(y) \subset H^+_{\Lambda_H}$$

and $\partial(W^0 \cap [T_y^f(Y)]^+_{M_y}) + f(y) \subset \partial\left(H^+_{\Lambda_H}\right)$. Then there exists an open set A_k of $\sum_{y \in B_k(Y)} (T_yY)^i$ and there exists an open set A_k^* of $B_k(Y) \times Y$ such that $\Delta_{B_k(Y)} \subset A_k^*$, $\{(y,0)/y \in B_k(Y)\} \subset A_k$ and the map $E_k(y,v) = (y, f^{-1}\pi(f(y) + \left(\theta_c^{f(y)}\right)^{-1} T_y(f)(v))$ is a diffeomorphism of class p from A_k onto A_k^*, $k \geq 0$.

e) If $H = R^q$, then there exists an open set W_2 of W_1, such that $H^+_{\Lambda_H} \supset W \supset W_1 \supset \bar{W}_2 \supset W_2 \supset f(Y)$ and $\pi_{|_{\bar{W}_2}} : \bar{W}_2 \to f(Y)$ is a proper map.

Proof

a) For every $y \in Y$, there is an open neighbourhood $V^{(y,0)}$ of $(y,0)$ in $A \subset N^f(Y)$ and there is an open neighbourhood $V^{f(y)}$ of $e(y,0) = f(y)$ in $H^+_{\Lambda_H}$ such that $e : V^{(y,0)} \to V^{f(y)}$ is a diffeomorphism of class p. Let us consider the open sets $M = \bigcup_{y \in Y} V^{(y,0)} \subset A \subset N^f(Y)$ and $U = \bigcup_{y \in Y} V^{f(y)} \subset H^+_{\Lambda_H}$. Then the map $e|_M : M \to U$ is a local diffeomorphism of class p and therefore a local homeomorphism. On the other hand $f(Y)$ is a closed set in $U, \xi : Y \to A$ defined by $\xi(y) = (y,0)$ is a map of class p and the map $s : f(Y) \to M$ defined by $s(z) = \xi.f^{-1}(z)$ is a section of class p of $e_{|M}$. Then using Godement's Lemma there is an open neighbourhood W of $f(Y)$ in U and there is a prolongation of s to a continuous section, $\bar{s} : W \to M$, of $e_{|M}$ such that $\bar{s}(W) = \Omega_A$ is an open set of $M \subset A$. Thus $e_{|\Omega_A} : \Omega_A \to W$ is a bijective local diffeomorphism of class p and therefore a diffeomorphism of class p, which fulfils that $e.\xi = f$ and $\xi(Y) \subset \Omega_A$.

b) Let $z \in W$. Then there is a unique $(y_z, v_z) \in \Omega_A$ such that $e(y_z, v_z) = z = f(y_z) + v_z$. Hence $\pi(z) = f(y_z) \in f(Y)$.

On the other hand if $z = f(y)$, then $(y_z, v_z) = (y, 0)$ and $\pi(z) = f(y_z) = f(y) = z$.

c) Let y be an element of Y and let $\alpha_y : T_y(Y) \to H$ be the map defined by $\alpha_y(u) = f(y) + \left(\theta_c^{f(y)}\right)^{-1} T_y(f)(u)$. It is clear that α_y is a continuous map and $\alpha_y(0) = f(y) \in W_1$. We note that if $y \in int(Y)$ then $f(y) \in int(W_1)$ and if $y \in \partial(Y)$ then $f(y) \in \partial(W_1)$ and there is an open neighbourhood $\tilde{W}_y^0$ of 0 in $(T_yY)^i$ such that $\alpha_y(\tilde{W}_y^0) \subset H^+_{\Lambda_H}$ and $\left(\theta_c^{f(y)}\right)^{-1} T_y(f)(\tilde{W}_y^0) \subset W_y^0 \cap [T_y^f(Y)]^+_{M_y}$.

Then there is an open neighbourhood $V_y^0 \subset \tilde{W}_y^0$ of 0 in $(T_y(Y))^i$ such that $\alpha_y(V_y^0) \subset W_1 \subset W \subset H^+_{\Lambda_H}$. Moreover if $y \in int(Y)$, then $\alpha_y(V_y^0) \subset int(W_1)$. Thus we have the map $e_y : V_y^0 \subset (T_y(Y))^i \to Y$ of class p defined by $e_y(u) = f^{-1}\pi[f(y) + \left(\theta_c^{f(y)}\right)^{-1} T_y(f)(u)]$.

Let us consider the map of class p defined by:

$$\mu : V_y^0 \xrightarrow{T_y(f)|V_y^0} T_y(f)\ (V_y^0) \xrightarrow{\left(\theta_c^{f(y)}\right)^{-1}} \left(\theta_c^{f(y)}\right)^{-1} T_y(f)(V_y^0) \xrightarrow{\tau_{f(y)}} W_1$$

Then we have that $e_y = f^{-1}\pi\mu$, $T_0(e_y) = T_{f(y)}(f^{-1}).\, T_{f(y)}(\pi).T_0(\mu)$ and

$$\mu|V_y^0 = \tau_{f(y)}((\theta_c^{f(y)})^{-1} T_y(f)|V_y^0).$$

Therefore $T_0(\mu) = \theta_c^{f(y)}\left(0_{c_1}^0\right)^{-1}$, where $c_1 = (V_y^0, \left(\theta_c^{f(y)}\right)^{-1} T_y(f), (T_y^f(Y), M_y))$.

On the other hand $f^{-1}\pi f = 1_Y$, $T_{f(y)}(f^{-1})T_{f(y)}(\pi)T_y(f) = 1_{T_y(Y)}$ and $T_0(e_y) = T_{f(y)}(f^{-1})T_{f(y)}(\pi)\ \theta_c^{f(y)}\left(\theta_{c_1}^0\right)^{-1}$. Hence if $w \in T_0(V_y^0)$, then

$$\begin{aligned} T_0(e_y)(w) &= T_{f(y)}(f^{-1})T_{f(y)}(\pi)\theta_c^{f(y)}\left(\theta_{c_1}^0\right)^{-1}(w) \\ &= T_{f(y)}(f^{-1})T_{f(y)}(\pi)\theta_c^{f(y)}\left(\theta_c^{f(y)}\right)^{-1} T_y(f)(u) = u\,, \end{aligned}$$

where $\left(\theta_c^{f(y)}\right)^{-1} T_y(f)(u) = \left(\theta_{c_1}^0\right)^{-1}(w)$, and $T_0(e_y) =$
$T_y(f))^{-1}\left(\theta_c^{f(y)}\right)_{|T_y^f(Y)}.\left(\theta_{c_1}^0\right)^{-1}$ is a linear homeomorphism.

Finally one has that $e_y(\partial(V_y^0)) \subset \partial(Y)$ for all $y \in \partial Y$ and $e_y(0) \in int(Y)$ for all $y \in int(Y)$ which, using the inverse mapping theorem, ends the proof of c).

d) The set $H_k = \sum_{y \in B_k(Y)} (T_yY)^i$ is a submanifold of $T(Y)$ and a submanifold of $\sum_{y \in B_k(Y)} T_y(Y)$.

Let us consider the continuous map $\bar{e} : T(Y) \to H$ defined by $\bar{e}(y,v) = f(y) + \left(\theta_c^{f(y)}\right)^{-1} T_y(f)(v)$. For all $k > 0$ we take an open neighbourhood G^k of $\{(y,0)/y \in B_k(Y)\}$ in H_k, such that $\bar{e}(G^k) \subset H^+_{\Lambda_H}$.

Then there exists an open neighbourhood G_1^k of $\{(y,0)/y \in B_k(Y)\}$ in G^k such that $\bar{e}(G_1^k) \subset W_1 \subset H^+_{\Lambda_H}$, $k > 0$. If $k = 0$ there exists an open neighbourhood G_1^0 of $\{(y,0)/y \in B_0(Y)\}$ in H_0 such that $\bar{e}(G_1^0) \subset intW_1$. Now we take the map $E_k : G_1^k \to B_k(Y) \times Y$ of class p defined by $E_k(y,v) = (y, f^{-1}\pi(f(y) + \left(\theta_c^{f(y)}\right)^{-1} T_y(f)(v)), k \geq 0$.

From the statement c) one has that E_k is a local diffeomorphism of class p at $(y,0)$ for all $y \in B_k(Y)$, $k \geq 0$.

Since the map $\tau : \Delta_{B_k(Y)} \to G_1^k$ defined by $\tau(y,y) = (y,0)$ is a continuous section of E_k, using Godement's Lemma we have that there exists an open neighbourhood G_2^k of $\Delta_{B_k(Y)}$ in $B_k(Y) \times Y$ and there exists a prolongation of τ to a continuous section, $\bar{\tau} : G_2^k \to G_1^k$, of E_k such that $\bar{\tau}(G_2^k) = B^k$ is an open set of G_1^k. Thus $E_k : B^k \to G_2^k$ is a bijective local diffeomorphism of class p at $(y,0)$ for all $y \in B_k(Y)$, $k \geq 0$.

e) We have that $p_1 : Y \times B_1^{=}(0) \to Y$ is a proper map. Hence the map $\gamma = p_1|_{(Y \times B_1^{=}(0)) \cap N^f(Y)} : (Y \times B_1^{=}(0)) \cap N^f(Y) \to Y$ is also a proper map, because of $N^f(Y)$ is a closed set in $Y \times R^q$.

On the other hand $N^f(Y) \supset (Y \times B_1^{=}(0)) \cap \Omega_A \supset (Y \times B_1(0)) \cap \Omega_A \cap (e_{|\Omega_A})^{-1}(W_1) \supset Y \times \{0\}$ and from the normality of $N^f(Y)$ we have that there exists an open set V of $N^f(Y)$, such that $(Y \times B_1(0)) \cap \Omega_A \cap (e_{|\Omega_A})^{-1}(W_1) \supset \bar{V} \supset V \supset Y \times \{0\}$. Then $e_{|\bar{V}} : \bar{V} \to e(\bar{V}) \subset W_1$ is a homeomorphism and $H^+_{\Lambda_H} \supset W \supset W_1 \supset e(\bar{V}) \supset e(V) \supset f(Y)$. Again by the normality of $H^+_{\Lambda_H}$, there exists an open set W_2 of $H^+_{\Lambda_H}$, such that

$$e(V) \supset \overline{W}_2 \supset W_2 \supset f(Y).$$

Now it is clear that $\pi_{|\overline{W}_2} = e_{|Y \times \{0\}} \xi \gamma j (e_{|\overline{V}})^{-1} : \overline{W}_2 \to f(Y)$, where $j : (e_{|\overline{V}})^{-1}(\overline{W}_2) \hookrightarrow (Y \times B_1^{=}(0)) \cap N^f(Y)$ is the inclusion map, is a proper map. $\square$

4. Collar neighbourhood of $\partial(X)$ in X. Embedded and collared manifolds

Proposition 4.1

Let X be a Hausdorff paracompact Hilbert differentiable manifold of class p. Then there is a real Hilbert space, $(H, <,>)$ and there is a closed embedding f of class p from X into H. Therefore the manifold X is diffeomorphic of class p to a closed submanifold of H. Moreover, for every $x \in X$ there is an open neighbourhood W^x of x in X, there is a closed vector subspace H_1 of H and there is a quadrant $(H_1)^+_{\Lambda_1}$ of H_1 such that $f_{|W^x} : W^x \to (H_1)^+_{\Lambda_1}$ is an embedding of class p which fulfils that $f(W^x)$ is a totally neat submanifold of $(H_1)^+_{\Lambda_1}$, (see [2]). $\square$

Lemma 4.2

Let X be a Hausdorff paracompact Hilbert differentiable manifold of class p with $\partial(X) \neq \phi$. Then there exists a function $g : X \to R$ of class p such that

1) $g(x) \geq 0$ for all $x \in X$.

2) $g^{-1}(0) = \partial(X)$.

3) $\partial^2(X) = g^{-1}(0) \cap \{x \in X / T_x g = 0\}$. Hence $T_x(g) = 0$ for all $x \in \partial^2(X)$.

4) If $\partial^2(X) = \phi$, then $T_x(g) \neq 0$ for all $x \in \partial(X)$ and $T_x j T_x B_1 X = \ker T_x(g) = \partial((T_x X)^i)$ for all $x \in B_1(X)$, where $j : B_1 X \to X$ is the inclusion map.

Proof

The manifold X admits partitions of unity of class p. Let us consider the atlas $\mathcal{A} = \{c_i = (U_i, \Psi_i = (\Psi_i^0, \Psi_i^1, \ldots, \Psi_i^{n_i}), (E_i \times R^{n_i}, (p_j^i)_{j \in J_i = \{1, \ldots, n_i\}}))/i \in I\}$ of class p of X, where $p_j^i(x^0, x^1, \ldots, x^{n_i}) = x^j$. Then there exists a partition of unity $\{\theta_i\}_{i \in I}$ of class p in X which is subordinated to the open covering $\{U_i\}_{i \in I}$.

For every $i \in I$ let us consider the function $g_i : X \to R$ of class p defined by

$$g_i(x) = \begin{cases} \theta_i(x) \prod_{j \in J_i} \Psi_i^j(x) & if\ x \in U_i \\ 0 & if\ x \notin U_i \end{cases}$$

Then $g = \sum_{i \in I} g_i : X \to R$ is a function of class p that fulfils the following properties:

1) $g(x) \geq 0$ for all $x \in X$.

2) $g^{-1}(0) = \partial(X)$. Indeed, if $x \in \partial(X)$ and $x \in U_i$, there is $j \in J_i$ such that $p_j^i \Psi_i(x) = \Psi_i^j(x) = 0$ and $g_i(x) = 0$ and therefore $g(x) = 0$. If $x \in g^{-1}(0)$, then there is $i_0 \in I$ such that $\theta_{i_0}(x) \neq 0$ and there is $j_0 \in J_{i_0}$ such that $\Psi_{i_0}^{j_0}(x) = 0$. Hence $x \in \partial X$.

3) Let x be an element of $\partial(X)$ and let $c = (U, \Psi = (\Psi^0, \Psi^1, \ldots, \Psi^n),\ (E \times R^n, (p_j)_{j \in J = \{1,2,\ldots,n\}}))$ be a chart of X with $x \in U$ and $\Psi(x) = 0$, where $p_j(x^0, x^1, \ldots, x^n) = x^j$. Then for all $j_0 \in \{0, 1, 2, \ldots, n\} = J \cup \{0\}$, one has that $\frac{\partial(g\Psi^{-1}(0))}{\partial x_{j_0}} = \sum_{i \in I} \frac{\partial(g_i \Psi^{-1})(0)}{\partial x_{j_0}}$ and for all $i \in I$, $\frac{\partial(g_i \Psi^{-1}(0))}{\partial x_{j_0}} = 0$ if $x \in X - U_i \subset X - \sup(\theta_i)$ and

$$\frac{\partial(g\Psi^{-1}(0))}{\partial x_{j_0}} = \frac{\partial((\theta_i . \prod_{j \in J_i} \Psi_i^j)\Psi^{-1})(0)}{\partial x_{j_0}} =$$

$$= \theta_i(x) \left(\sum_{k \in J_i} \left(\prod_{j \in J_i - \{k\}} \Psi_i^j(x) \right) . \frac{\partial(\Psi_i^k\ \Psi^{-1})(0)}{\partial x_{j_0}} \right) \ if\ x \in U_i.$$

Thus if ind $(x) \geq 2$, then $\frac{\partial(g_i \Psi^{-1}(0))}{\partial x_{j_0}} = 0$ and $\partial^2 X \subset g^{-1}(0) \cap C(g)$, where $C(g) = \{x \in X / T_x g = 0\}$.

If $x \in B_1(X)$ and $x \in U_i$, there is a unique $j_0 \in J_i$ such that $\Psi_i^{j_0}(x) = 0$. Moreover $J = \{1\}$, $p_1(z, t) = t$ and

$$\frac{\partial(g_i \Psi^{-1})(0)}{\partial t} = \theta_i(x) \left(\prod_{j \in J_i - \{j_0\}} \Psi_i^j(x) \right) \frac{\partial(\Psi_i^{j_0} \Psi^{-1})}{\partial t}(0).$$

Since $D(\Psi_i\Psi^{-1})(0)$ is a linear homeomorphism, it holds that $D(\Psi_i^{j_0}\Psi^{-1})(0) \neq 0$. On the other hand $\Psi_i^{j_0}\Psi^{-1}(y) \geq 0$ for all $y \in \Psi(U \cap U_i)$ and $\Psi_i^{j_0}\Psi^{-1}(0) = 0$ and therefore $\frac{\partial(\Psi_i^{j_0}\Psi^{-1})(0)}{\partial z} = 0$ and $\frac{\partial(\Psi_i^{j_0}\Psi^{-1})(0)}{\partial t} > 0$. Then we have that $T_x g \neq 0$, that is $x \notin C(g)$, and $\partial^2(X) = g^{-1}(0) \cap C(g)$.

4) If $\partial^2(X) = \phi$, then $g^{-1}(0) \cap C(g) = \phi$ or $\partial(X) \cap C(g) = \phi$. □

We note that the Lemma 4.2 is also true if X is a differentiable manifold which admits partition of unity of class p.

Corollary 4.3

Let X be a Hausdorff paracompact differentiable manifold of class p whose charts are modelled over real Banach spaces which satisfy the Urysohn condition of class p (In particular, X can be a Hausdorff paracompact Hilbert differentiable manifold). Suppose that $\partial(X) \neq \phi$ and $\partial^2(X) = \phi$. Then there exists a real Banach space $(E, \| \, \|)$ (or there exists a real Hilbert space $(E, <,>)$, if X is a Hausdorff paracompact Hilbert manifold) and there exists a closed embedding g of class p from X into the quadrant $E \times (R^+ \cup \{0\})$ of $E \times R$ such that $g(\partial(X)) = g(X) \cap (E \times \{0\})$ and $g(X)$ is a closed totally neat submanifold of $(E \times R)^+_{p_2}$, where $p_2(x,t) = t$. Moreover for all $x \in X$ there is an open neighbourhood W^x of x in X, there is a closed vector subspace E_1 of E and there is a quadrant $E^{1+}_{\Lambda_1}$ of E_1 such that $p_1 g : W^x \to E^{1+}_{\Lambda_1}$ is an embedding of class p and $p_1 g(W^x)$ is a totally neat submanifold of $E^{1+}_{\Lambda_1}$.

Proof

There exists a real Banach space, $(E, \| \, \|)$, and there exists a closed embedding f of class p from X into E such that for every $x \in X$ there is an open neighbourhood W^x of x in X, there is a closed vector subspace E_1 of E and there is a quadrant $(E_1)^+_{\Lambda_1}$ of E_1 which fulfil that $f_{|W^x} : W^x \to (E_1)^+_{\Lambda_1}$ is an embedding of class p and $f(W^x)$ is a totally neat submanifold of $(E_1)^+_{\Lambda_1}$ (see [2]). In particular, there exists a real Hilbert space, $(E, <,>)$, and there exists a closed embedding f of class p from X into E with the same local property if X is a Hausdorff paracompact Hilbert differentiable manifold).

From Lemma 4.2 it follows that there exists a function $\lambda : X \to R$ of class p such that $\lambda(x) \geq 0$ for all $x \in X$, $\lambda^{-1}(0) = \partial(X)$ and $T_x(\lambda) \neq 0$ for all $x \in \partial(X) = \lambda^{-1}(0)$ and $T_x(j) T_x B_1(X) = \ker(T_x(\lambda)) = \partial((T_x(X))^i)$ for all $x \in B_1(X)$, where $j : B_1(X) \to X$ is the inclusion map. Let us consider the map, $g = (f, \lambda) : X \to E \times (R^+ \cup \{0\})$. Obviously g is an injective map. Moreover g is a closed map. Indeed, if C is a closed subset of X and $\{g(x_n) = (f(x_n), \lambda(x_n))\}_{n \in N}$ is a sequence in $g(C)$ which converges to (u_0, t_0) in $E \times (R^+ \cup \{0\})$, then $x_n \in C$ for all $n \in N$ and the sequence $\{f(x_n)\}_{n \in N}$ converges to u_0 in E. Hence the sequence $\{x_n\}_{n \in N}$ converges to $x_0 \in C$ in X. Thus $(u_0, t_0) = g(x_0) \in g(C)$ and $g(C)$ is a closed set in $E \times (R^+ \cup \{0\})$.

Then it occurs that $g : X \to g(X)$ is a homeomorphism, $g(X)$ is a closed set in

$E \times (R^+ \cup \{0\})$ and $g(\partial(X)) = g(X) \cap (E \times \{0\})$.

Let x_0 be an element of X. Since $T_{x_0}(f)$ is a linear injective map and $im\ (T_{x_0}(f))$ admits a topological supplement in $T_{f(x_0)}(E)$, we have that $T_{x_0}(g) \equiv (T_{x_0}(f), T_{x_0}(\lambda))$ is a linear injective map and $im\ (T_{x_0}(g))$ admits a topological supplement in $T_{(f(x_0),\lambda(x_0))}(E \times R^+ \cup \{0\}))$.

On the other hand $g^{-1}(\partial((E \times R)^+_{p_2})) = \partial(X)$, and $ind\ (v) = ind\ (T_{x_0}(g)(v))$ for all $v \in (T_{x_0}(X))^i$, because of $\ker(T_x(\lambda)) = \partial((T_x X)^i)$ for all $x \in B_1(X)$. Then, by Theorem 2.3 of [2], the map g is an immersion at x_0 and therefore g is a closed embedding of class p of X into $E \times (R^+ \cup \{0\})$ which fulfils that $g^{-1}(E \times \{0\}) = \partial(X)$. $\square$

Definition 4.4

Let X be a differentiable manifold of class p with $\partial X \neq \phi$ and $\partial^2 X = \phi$. We say that (f, A) is a collar neighbourhood of ∂X in X of class p if A is an open neighbourhood of $\partial(X)$ in X and $f : \partial X \times (R^+ \cup \{0\}) \to A$ is a diffeomorphism of class p such that $f(x, 0) = x$ for all $x \in \partial(X)$.

Lemma 4.5

Let X be a differentiable manifold of class p which admits partitions of unity of class p (In particular X could be a Hausdorff paracompact Hilbert manifold), let $M = \{M_i / i \in I\}$ be a locally finite family of subsets of X such that $X = \cup_{i\in I} M_i$ and $\varepsilon = \{\varepsilon_i\}_{i\in I}$ a family of positive real numbers. Then there exists a map $\delta : X \to R^+$ of class p such that $\delta(x) < \varepsilon_i$ for all $i \in I$ and all $x \in M_i$. $\square$

Theorem 4.6

Let X be a Hilbert Hausdorff paracompact differentiable manifold of class $p+1$, $p \geq 1$, such that $\partial(X) \neq \phi$ and $\partial^2(X) = \phi$. Then there exists a collar neighbourhood (f, A) of $\partial(X)$ in X of class p. Moreover there are a real Hilbert space H, a closed embedding $\beta : \partial(X) \to H$ of class $p+1$ and an open set A^* in $N^\beta(\partial(X))$ such that $\{(y, 0)/y \in \partial(X)\} \subset A^*$ and $e : A^* \to H$, defined by $e(y, v) = f(y) + v$ is a local diffeomorphism of class p at $(y, 0) \in A^*$ for every $y \in \partial X$.

Proof

By Proposition 4.1, there is a real Hilbert space $(H, <,>)$, and there is a closed embedding g of class $p{+}1$, from X into H. Obviously, $g_{|\partial(X)} : \partial(X) = B_1(X) \to H$ is also a closed embedding of class $p+1$.

Moreover, for every $x \in X$ there is an open neighbourhood W^x of x in X, there is a closed vector subspace H_1 of H, and there is a quadrant $(H_1)^+_{\Lambda_1}$ of H_1, such that $g_{|W^x} : W^x \to (H_1)^+_{\Lambda_1}$ is an embedding of class $p+1$ which fulfils that $g(W^x)$ is a totally neat submanifold of $(H_1)^+_{\Lambda_1}$.

Then, by Proposition 3.1, we have that $g(\partial(X))$ is a submanifold without boundary of H and there are an open set W of H and a map $\pi : W \to W$ of class

p such that $g(\partial(X)) \subset W$, $\pi(W) \subset g(\partial(X))$, $\pi\, g(y) = g(y)$ for every $y \in \partial(X)$ and $\pi : W \to g(\partial(X))$ is a submersion of class p at every $g(y) \in g(\partial(X))$. Thus $U = g^{-1}(W)$ is an open set in X such that $\partial(X) \subset U$ and $r = (g_{|\partial(X)})^{-1}.\pi.g_{|U} : U \to \partial(X)$ is a retraction of class p.

On the other hand, from Lemma 4.2, there exists a function $\alpha : X \to R^+ \cup \{0\}$ of class $p+1$ such that $\alpha^{-1}(0) = \partial(X)$ and $T_x(\alpha) \neq 0$ for all $x \in \partial(X)$ and $T_x(j)(T_x(B_1(X))) = \ker(T_x(\alpha)) = \partial((T_x(X))^i)$ for all $x \in B_1(X)$, where $j : B_1(X) \to X$ is the inclusion map.

Let us consider the map $h = (r, \alpha_{|U}) : U \to \partial(X) \times (R^+ \cup \{0\})$ of class p. Then it is clear that $h(\partial(U)) = h(\partial(X)) = [\partial(X)] \times \{0\} = \partial[\partial(X) \times (R^+ \cup \{0\})]$ and $T_{x_0}(h)$ is a linear homeomorphism for every $x_0 \in \partial(X)$. Indeed, we have that $T_{x_0}(h) \equiv (T_{x_0}(r), T_{x_0}(\alpha_{|U})) : T_{x_0}(U) \to (T_{x_0}(\partial(X))) \times (T_0(R^+ \cup \{0\}))$ and $T_{x_0}(X) = T_{x_0}(j)(T_{x_0}(B_1(X))) \oplus_T L\{v_1\}$, where $T_{x_0}(\alpha)(v_1) \neq 0$. Then for every $u \in T_{x_0}(X)$, there exists $u_1 \in T_{x_0}(j)(T_{x_0}(\partial(X))$ and there exists $u_2 \in L\{v_1\}$ such that $u = u_1 + u_2$ and $T_{x_0}(h) = (u_1 + T_{x_0}(r)(u_2), T_{x_0}(\alpha)(u_2))$ which proves that $T_{x_0}(h)$ is a linear homeomorphism. Thus there exists an open neighbourhood V^{x_0} of x_0 in U there exists $\varepsilon_{x_0} > 0$ and there exists an open neighbourhood W^{x_0} of x_0 in X, such that $h_{|V^{x_0}} : V^{x_0} \to (W^{x_0} \cap \partial(X)) \times [0, \varepsilon_{x_0})$ is a diffeomorphism of class p for all $x_0 \in \partial(X)$. Clearly the map $s : \partial(X) \times \{0\} \to U$, defined by $s(x,0) = x$, is a continuous section of h and using the Godement's lemma we have that there are an open neighbourhood G_1 of $\partial(X) \times \{0\}$ in $\partial(X) \times (R^+ \cup \{0\})$, an open set U_1 in U with $\partial(X) \subset U_1$ and a prolongation of s to a continuous section, $\bar{s} : G_1 \to U_1$ of $h_{|U_1}$ such that $\bar{s}(G_1) = U_0$ is an open set of U_1 with $\partial(X) \subset U_0$. Hence $h_{|U_0} : U_0 \to G_1$ is a diffeomorphism of class p.

By the Lemma 4.5 there exists a function $\gamma : \partial(X) \to R^+$ of class $p+1$ such that $\{x\} \times [0, \gamma(x)) \subset G_1$ for every $x \in \partial(X)$. Then the set $G_2 = \bigcup_{x \in \partial(X)} \{x\} \times [0, \gamma(x)) \subset G_1$ is an open set in $\partial(X) \times (R^+ \cup \{0\})$ and the map $\mu : G_2 \to \partial(X) \times (R^+ \cup \{0\})$, defined by $\mu(x,t) = \left(x, \frac{t}{\gamma(x)-t}\right)$, is a diffeomorphism of class $p+1$ whose inverse map is $\mu^{-1}(y,u) = \left(y, \frac{u.\gamma(y)}{u+1}\right)$. Thus the set $A = \left(h_{|U_0}\right)^{-1}(G_2)$ is an open set of $U_0 \subset U_1$ such that $\partial(X) \subset A$ and $f^* = \mu h_{|A}$ is a diffeomorphism of class p of A onto $\partial(X) \times (R^+ \cup \{0\})$, which verifies that $f^*(x) = (x, 0)$ for all $x \in \partial(X)$. Finally we take $f = (f^*)^{-1}$. $\square$

Proof of Theorem A

By Theorem 4.6, there exists a collar neighbourhood $(\bar{f}, U)$ of $\partial(X)$ in X of class p. Then $\bar{f}(\partial(X) \times [0, +\infty)) = U$ is an open set in X with $\partial(X) \subset U$ and $\bar{f} : \partial(X) \times [0, +\infty) \to U$ is a diffeomorphism of class p such that $\bar{f}(x, 0) = x$ for every $x \in \partial(X)$.

Since X is a normal topological space and $\partial(X)$ is a closed set in X, there is an open set V in X, such that $X \supset U \supset \bar{V} \supset V \supset \partial(X)$. Therefore $(\bar{f})^{-1}(V)$ is an open set in $\partial(X) \times [0, +\infty)$ and, from Lemma 4.5, there exists a map $\gamma : \partial(X) \to R^+$ of class $p+1$, such that $\{x\} \times [0, \gamma(x)) \subset (\bar{f})^{-1}(V)$ for every $x \in \partial(X)$.

We consider the maps of class p

$$\alpha : U \xrightarrow{(\bar{f})^{-1}} \partial(X) \times [0,+\infty) \xrightarrow{p_1} \partial(X),$$

$$\beta : U \xrightarrow{(\bar{f})^{-1}} \partial(X) \times [0,+\infty) \xrightarrow{p_2} [0,+\infty)$$

and
$$\gamma_1 = \gamma\alpha : U \to R^+.$$

Then it is clear that α, β are surjective maps, $(\bar{f})^{-1}(x) = (\alpha(x), \beta(x))$ for all $x \in U$, $U_1 = \{x \in U/\beta(x) < \frac{3}{4}\gamma\alpha(x)\}$ is an open set of U, $U_2 = \{x \in U/\beta(x) < \frac{5}{8}\gamma\alpha(x)\}$ is an open set of U, $U_1^* = \{x \in U/\beta(x) \leq \frac{3}{4}\gamma\alpha(x)\}$ is a closed set of U, $U_2^* = \{x \in U/\beta(x) \leq \frac{5}{8}\gamma\alpha(x)\}$ is a closed set of U, $\partial(X) \subset U_1 \subset U_1^*, \partial(X) \subset U_2 \subset U_2^* \subset U_1 \subset U_1^*$ and $\partial(X) \subset U_1^* \subset V \subset \bar{V} \subset U \subset X$. Hence U_1^*, U_2^* are closed sets of X and there exists a map of class $p{+}1$, $\mu : X \to [0,1]$, such that $\mu(\bar{U}_1) = \{1\}$ and $\mu(X - V) = \{0\}$ and there exists a map of class $p+1$, $\nu : X \to [0,1]$ such that $\nu(\bar{U}_2) = \{0\}$ and $\nu(X - U_1) = \{1\}$.

Let us consider the map $\tau : X \to [0,+\infty)$ of class p, defined by

$$\tau(x) = \begin{cases} \mu(x).\beta(x) + \nu(x) & if \quad x \in U \\ \nu(x) & if \quad x \in X - U, \end{cases}$$

the open set of U, $U' = \{x \in U/\beta(x) < \frac{1}{2}\gamma\alpha(x)\}$ and the open set of $\partial(X) \times [0,+\infty)$, $W_1 = \{(x,t) \in \partial(X) \times [0,+\infty)/t < \frac{1}{2}\gamma(x)\}$. Then we have that $\partial(X) \subset U' \subset U_2 \subset U_1 \subset U$, $\partial(X) \times \{0\} \subset W_1$, $\tau(\partial(X)) = \{0\}$, $\tau_{|U'} = \beta_{|U'}$, $\tau_{|X-\bar{V}=\nu|_{X-\bar{V}}}$ and τ is a submersion at x for every $x \in \partial(X)$.

Hence $T_x(j)T_x(\partial(X)) = \ker(T_x(\tau))$ for every $x \in \partial(X)$, where $j : \partial(X) \to X$ is the inclusion map. Moreover $T_x(j)T_x(\partial X) = (T_xX)^0 = \partial(T_xX)^i) = \ker(T_x(\tau))$ for every $x \in \partial(X)$.

On the other hand $C_1 = \{(x,t) \in \partial(X) \times [0,+\infty)/t \leq \frac{1}{4}\gamma(x)\}$ and $C_2 = \{(x,t) \in \partial(X) \times [0,+\infty) / \frac{3}{8}\gamma(x) \leq t \leq \frac{1}{2}\gamma(x)\}$ are disjoint closed sets of $\partial(X) \times [0,+\infty)$ such that $\partial(X) \times \{0\} \subset C_1$, $C_1 \subset W_1$ and $\mathring{C}_1 = \{(x,t) \in \partial(X) \times [0,+\infty)/t < \frac{1}{4}\gamma(x)\}$.

There is a map $r : [0,\frac{1}{2}] \to [0,1]$ of class ∞ such that $r(t) = 0$ for every $t \in [0,\frac{1}{4}]$, $r(t) = 1$ for all $t \in [\frac{3}{8},\frac{1}{2}]$ and $r(t_1) < r(t_2)$ for all $t_1, t_2 \in (1/4, 3/8)$ with $t_1 < t_2$.

Let us consider the map $\Phi : W_1 \to [0,1]$ defined by $\Phi(x,t) = r\left(\frac{t}{\gamma(x)}\right)$. Then we have that Φ is a map of class $p+1$, $\Phi^{-1}(0) = C_1$, $\Phi^{-1}(1) = C_2 \cap W_1$ and $\Phi(x,t_1) < \Phi(x,t_2)$ for every $(x,t_1), (x,t_2) \in W_1$ with $t_1, t_2 \in (\frac{1}{4}\gamma(x), \frac{3}{8}\gamma(x))$ and $t_1 < t_2$. Hence for every $x \in \partial(X)$, $\Phi_x : (\frac{1}{4}\gamma(x), \frac{3}{8}\gamma(x)) \to (0,1)$ is a bijective map.

Now we consider the map $u : W_1 \to \partial(X) \times [0,+\infty)$ of class $p+1$, defined by $u(x,t) = (x, t.\Phi(x,t))$ and the map, $q : X \to X$, defined by

$$q(x) = \begin{cases} \bar{f}u(\bar{f})^{-1}(x) & if \quad x \in U' \subset U_2 \subset U_1 \\ x & if \quad x \notin U' \end{cases}$$

(Note that for every $x \in U'$, $(\bar{f})^{-1}(x) = (\alpha(x), \beta(x)) \in W_1$). Since the set $C = \{x \in U / \beta(x) \leq \frac{3}{8}\gamma\alpha(x)\} \subset U_1^* \subset V$ is a closed set of X, $C \subset U'$, $X = U' \cup (X - C)$ and $\bar{f}(\alpha(x), \beta(x)\Phi(\alpha(x), \beta(x))) = \bar{f}(\alpha(x), \beta(x)) = x$ for every $x \in U' \cap (X - C)$, we have that q is a map of class p.

By Proposition 4.1, there exists a real Hilbert space $(H, <,>)$ and there exists a closed embedding h of class $p+1$, from X into H. Let g be the map of class p, $g = (hq, \tau) : X \to H \times [0, +\infty)$.

Then the map g has the following properties:

a) $g(\partial X) = g(X) \cap (H \times \{0\}), \partial(X) = \tau^{-1}(0)$.

b) $g \circ (\bar{\rho}|_{C_1}) = ((p_1 \circ g|_{\partial(x)}) \times 1_{[0,+\infty)})|_{C_1}$

and $h(x) = p_1 g(x)$ for all $x \in \partial(X)$. Indeed, for all $(x,t) \in C_1$, we have that $\beta\bar{f}(x,t) = t \leq \frac{1}{4}\gamma(x) < \frac{1}{2}\gamma\alpha(\bar{f}(x,t)), \bar{f}(x,t) \in U'$ and

$$\begin{aligned} g\bar{f}(x,t) &= (hq\bar{f}(x,t), \tau\bar{f}(x,t)) = (h\bar{f}u(x,t), \beta\bar{f}(x,t)) \\ &= (h\bar{f}(x,0), t) = (h(x), t) = (p_1 g(x), t)\,. \end{aligned}$$

In particular $g\bar{f}(x,0) = g(x) = (h(x), 0)$ for every $x \in \partial(X)$.

c) The map g is an injective map. Indeed,

1) For all $y \in X - \{x \in U/\beta(x) < \frac{3}{8}\gamma\alpha(x)\} = M_1$, we have that $g(y) = (hq(y), \tau(y)) = (h(y), \tau(y))$.

2) For every $y \in \{x \in U/\frac{1}{4}\gamma\alpha(x) < \beta(x) < \frac{3}{8}\gamma\alpha(x)\} = M_2$, we have that $g(y) = (h\bar{f}u(\alpha(y), \beta(y)), \beta(y))$.

3) For every $y \in \{x \in U/\beta(x) \leq \frac{1}{4}\gamma\alpha(x)\} = M_3 \subset U'$ it occurs that $g(y) = (hq(y), \tau(y)) = (h\bar{f}u(\bar{f})^{-1}(y), \tau(y)) = (h\alpha(y), \beta(y))$.

Obviously $X = M_1 \cup M_2 \cup M_3$ and M_1, M_3 are closed sets of X. Let x, y be elements of X with $x \neq y$ such that $g(x) = g(y)$.

c_1) If $x, y \in M_1$, then $g(x) = (h(x), \tau(x)) = (h(y), \tau(y)) = g(y)$ and $h(x) = h(y)$ which is a contradiction.

c_2) If $x \in M_1$ and $y \in \{z \in U/\beta(z) < \frac{3}{8}\gamma\alpha(z)\}$, then $g(x) = (h(x), \tau(x))$ and $g(y) = (h\alpha(y), \beta(y))$ if $y \in M_3$ or $g(y) = (h\bar{f}u(\alpha(y), \beta(y)), \beta(y))$ if $y \in M_2$.

In the first case $x = \alpha(y) \in \partial(X)$ which is a contradiction. In the second case we have that $x = \bar{f}(\alpha(y), \beta(y)\Phi(\alpha(y), \beta(y))$ which implies that $\alpha(x) = \alpha(y)$ and $\frac{3}{8}\gamma\alpha(x) \leq \beta(x) = \beta(y)\Phi(\alpha(y), \beta(y)) \leq \beta(y) < \frac{3}{8}\gamma\alpha(y)$ which is a contradiction.

c_3) If $x, y \in \{z \in U/\beta(z) < \frac{3}{8}\gamma\alpha(z)\}$, then $\beta(x) = \beta(y)$ and it happens that $h\alpha(x) = h\alpha(y)$ if $x, y \in M_3$, $\bar{f}u(\alpha(x), \beta(x)) = \bar{f}u(\alpha(y), \beta(y))$ if $x, y \in M_2$ and $\alpha(x) = \bar{f}u(\alpha(y), \beta(y)) = \bar{f}(\alpha(y), \beta(y)\Phi(\alpha(y), \beta(y)))$ if $x \in M_3$ and $y \in M_2$. All these cases give us that $\alpha(x) = \alpha(y)$, which is a contradiction.

d) The map $g : X \to g(X)$ is a homeomorphism and $g(X)$ is a closed set of $H \times [0, +\infty)$.

Indeed,

d_1) The map $g_{|M_1} : M_1 \to g(M_1)$ is a homeomorphism whose inverse map is $\alpha_1 = h^{-1}p_{1|g(M_1)}$. Indeed, we note that $p_1 g(M_1) = h(M_1), p_1 g(x) = h(x)$ for all $x \in M_1$ and $\alpha_1 g(x) = \alpha_1(h(x), \tau(x)) = h^{-1}h(x) = x$ for every $x \in M_1$ and $g\alpha_1(y,t) = g\alpha_1 g(z) = g\alpha_1(h(z), \tau(z)) = g(z) = (y,t)$ for all $(y,t) \in g(M_1)$.

d_2) The map $g_{|M_3} : M_3 \to g(M_3)$ is a homeomorphism whose inverse map is $\alpha_3 = \bar{f}(\theta^{-1})_{|g(M_3)} : g(M_3) \to M_3$, where $\theta = \left(h_{|\partial(X)\times^1[0,+\infty)}\right)_{|C_1} : C_1 \to \left(h_{|\partial(X)\times^1[0,+\infty)}\right)(C_1)$. Indeed, we note that $g(M_3) \subset \theta(C_1)$ $\bar{f}\theta^{-1}g(M_3) = M_3$, $\bar{f}\theta^{-1}g(x) = x$ for every $x \in M_3$, $g\bar{f}\theta^{-1}g(x) = g(x)$ for all $x \in M_3$ and $\bar{f}\theta^{-1}g(x) = x$ for every $x \in M_3$.

d_3) Let M_2 be the closed subset of X, $\{x \in U / \frac{1}{4}\gamma\alpha(x) \leq \beta(x) \leq \frac{3}{8}\gamma\alpha(x)\} \subset U'$. Then the map $g_{|M_2^*} : M_2^* \to g(M_2^*)$ is a homeomorphism.

Indeed, we have that the map $u_x : [\frac{1}{4}\gamma(x), \frac{3}{8}\gamma(x)] \to [0, \frac{3}{8}\gamma(x)]$ is a bijective map for every $x \in \partial(X)$, the map $u_{|c_3} : C_3 = \{(y,t) \in \partial(X) \times [0,+\infty) / \frac{1}{4}\gamma(y) \leq t \leq \frac{3}{8}\gamma(y)\} \to C_3^*$ is a homeomorphism, where

$$C_3^* = \{(y,t) \in \partial(X) \times [0,+\infty) / t \leq \frac{3}{8}\gamma(y)\} \subset W_1 \,,$$
$$p_1 g(M_2^*) \subset \ im\ (h), M_2^* \subset U', h^{-1}p_1 g(M_2^*) \subset U' \subset U, (\bar{f})^{-1}h^{-1}p_1 g(M_2^*) \subset C_3^* \,,$$
$$u^{-1}(\bar{f})^{-1}h^{-1}p_1 g(M_2^*) \subset C_3 \subset W_1, \bar{f}(C_3) \subset M_2^*$$

and $\bar{f}u^{-1}(\bar{f})^{-1}h^{-1}p_1 g(M_2^*) \subset M_2^*$. Thus the inverse map of $g_{|M_2^*}$ is the continuous map $\bar{f}u^{-1}\bar{f}^{-1}h^{-1}\ p_{1/g(M_2^*)}$.

d_4) $g(M_1)$, $g(M_3)$, $g(M_2^*)$ are closed sets of $H \times [0,+\infty)$ and therefore $g(X) = g(M_1) \cup g(M_3) \cup g(M_2^*)$ is a closed set and $g : X \to g(X)$ is a homeomorphism.

e) The map $T_x(g) : T_x(X) \to T_{g(x)}(H \times [0,+\infty))$ is an injective map, for every $x \in X$.

e_1) First we note that $M_1 = \{x \in U / \beta(x) \geq \frac{3}{8}\gamma\alpha(x)\}$ and that the map $\lambda : \partial(X) \times [0,+\infty)) \to R$, defined by $\lambda(y,t) = -t + \frac{3}{8}\gamma(y)$, is a map of class $p+1$ such that $\lambda^{-1}(0) \cap (\partial(X) \times \{0\}) = \phi$ and $T_{(x,t)}(\lambda) \neq 0$ for every $(x,t) \in \lambda^{-1}(0)$. Then the set $\lambda^{-1}((-\infty,0])$ is a closed submanifold of class $p+1$ of $\partial(X) \times [0,+\infty)$ such that $\partial(\lambda^{-1}((-\infty,0])) = \lambda^{-1}(0) \cup ((\partial(X) \times \{0\}) \cap \lambda^{-1}((-\infty,0))) = \lambda^{-1}(0)$. But $\lambda^{-1}((-\infty,0]) = \{(y,t) \in \partial(X) \times [0,+\infty) / t \geq \frac{3}{8}\gamma(y)\}$ and $\bar{f}\lambda^{-1}((-\infty,0]) = \{x \in U / \beta(x) \geq \frac{3}{8}\gamma\alpha(x)\} = M_1$. Hence it follows that M_1 is a submanifold of class $p+1$ of X such that $\partial(M_1) = \bar{f}(\partial(\lambda^{-1}((-\infty,0]))) = \bar{f}(\lambda^{-1}(0)) = \{x \in U / \beta(x) = \frac{3}{8}\gamma\alpha(x)\}$, $\mathrm{int}(M_1) = \{x \in U / \beta(x) > \frac{3}{8}\gamma\alpha(x)\}$ is an open set of X and $T_x(j) : T_x(M_1) \to T_x(X)$ is a bijective map for every $x \in M_1$, where $j : M_1 \to X$ is the inclusion map.

Moreover for every $y \in M_1$ it occurs that $g(y) = (h(y), \tau(y))$, $T_y(g_{|M_1})$ is an injective map and $T_y(g)$ is an injective map.

e_2) If $y \in M_2$, then $g(y) = (h\bar{f}u(\bar{f})^{-1}(y), \beta(y))$.

On the other hand $u_{|\overset{\circ}{C}_3} : \overset{\circ}{C}_3 \to \overset{\circ}{C}_3$ is a diffeomorphism of class $p+1$ and $(\bar{f})^{-1}(M_2) = \overset{\circ}{C}_3$. Therefore $T_y(g)$ is an injective map for all $y \in M_2$.

e_3) We have that M_3 is a submanifold of class $p+1$ of X such that $\partial(M_3) = \{x \in U / \beta(x) = \frac{1}{4}\gamma\alpha(x)\}$ and $\mathrm{int}(M_3) = \{x \in U / \beta(x) < \frac{1}{4}\gamma\alpha(x)\}$ is an open set of X. Then for every $x \in M_3$, $T_x(j) : T_x(M_3) \to T_x(X)$ is a bijective map, where $j : M_3 \to X$ is the inclusion map. Moreover $(\bar{f})^{-1}(M_3) = C_1$ and for every $y \in M_3$

it occurs that $g(y) = (h\alpha(y), \beta(y)) = (h_{|\partial(X)\times 1_{[0,+\infty)}})_{|C_1}(\bar{f})^{-1}(y)$, $T_y(g_{|M_3})$ is an injective map and $T_y(g)$ is an injective map.

Then using the formula $T_x(j)T_x(\partial X) = (T_xX)^0 = \partial((T_xX)^i) \subset \ker T_x(\tau)$ for every $x \in \partial(X)$, we have that $\mathrm{ind}(v) = \mathit{ind}\,(T_x(g)(v))$ for every $v \in (T_xX)^i$ and all $x \in X$ and g is an immersion of class p. Hence g is a closed embedding of class p.

Lastly it is straighforward to check that for every $x \in \bar{f}(\mathring{C}_1)$, $T_x^g(X) = H_1 \times R$, where H_1 is closed linear subspace of H. Hence $N_x^g(X) \subset H \times \{0\}$ for every $x \in \bar{f}(\mathring{C}_1)$. □

Proposition 4.7

Let $f : X \to X$ be a differentiable map of class p such that $f.f = f$. Suppose that $f(\partial(X)) \subset \partial(X)$ and $\ker(T_{x_0}(f)) \subset (T_{x_0}X)^i$ for every $x_0 \in f(X) \cap \partial(X)$. Then we have that:

1) f is a subimmersion at every $x_0 \in f(X)$, i.e. f localizes as a linear continuous map whose kernel and image admit topological supplements.

2) $f(X) = \{x \in X/f(x) = x\}$ is a totally neat submanifold of class p of X. Moreover, if X is a Hausdorff manifold, then $f(X)$ is a closed set of X (see [7]).

Proof of Theorem B

a) $\Rightarrow$ b). From 4.7 we have that there are a real Hilbert space $(H, <,>)$, a closed embedding $g : X \to H \times [0, +\infty)$ of class ∞ with $g(\partial(X)) = g(X) \cap (H \times \{0\})$, a collar neighbourhood (f, A) of $\partial(X)$ in X of class ∞ and an open set G in $\partial(X) \times [0, +\infty)$, such that $\partial(X) \times \{0\} \subset G$, $((p_1 g_{|\partial(X)}) \times 1_{[0,+\infty)})_{|G} = g_{|A^0} f/G$, $f(G) = G_1$ is an open set in X with $\partial(X) \subset G_1$ and for every $x \in G_1$, $N_x^g(X) \subset H \times \{0\} = (H \times R)_{p_2}^0$.

Using 3.1 we have that there is an open set Ω of $\tilde{A} \subset N^g(X)$, with $X \times \{0\} \subset \Omega$ and there is an open set W of $H \times [0, +\infty)$, with $g(X) \subset W$ such that $e_{|\Omega} : \Omega \to W$ is a diffeomorphism of class ∞ and $e.\xi = g$, where $\xi : X \to N^g(X)$ is defined by $\xi(y) = (y, 0)$, $\tilde{A}$ is an open set of $N^g(X)$ with $(x, 0) \in \tilde{A}$ for every $x \in X$ and $e : \tilde{A} \to H \times [0, +\infty)$ defined by $e(x, v) = g(x) + v$ is a local diffeomorphism of class ∞ at $(x, 0) \in \tilde{A}$ for every $x \in X$.

Moreover the map $\pi : W \to W$ defined by $\pi = e\xi p_{1|\Omega}(e_{|\Omega})^{-1}$ is a map of class ∞ such that $\pi(W) = g(X)$, $\pi(\partial(W)) \subset \partial(g(X)) = g(\partial(X)) = g(X) \cap (H \times \{0\})$ and $\pi g(x) = g(x)$ for all $x \in X$.

Lastly the map $\pi : W \to g(X)$ is a submersion of class ∞ at every $g(x) \in g(X)$.

Let $(x_0, t_0) \in W$ such that $\pi(x_0, t_0) \in \partial(W)$. Then $\pi(x_0, t_0) = (y_0, 0)$, $(e_{|\Omega})^{-1}(x_0, t_0) = (x_1, u_1, v_1)$ $\pi(x_0, t_0) = g(x_1)$ and $x_1 \in \partial(X)$, $(x_0, t_0) = g(x_1) + (u_1, v_1)$, $(u_1, v_1) \in N_{x_1}^g(X)$, $(y_0, 0) \in g(G_1)$.

Let us consider open neighbourhoods $V^{x_0} \subset G_1, V_R^0 \subset R$, $V^{y_0} \subset H$ and $V^0 \subset [0, +\infty)$ such that $V^{y_0} \times V^0 \subset W$, $V_R^0 \subset V^0$, $N^g(X) \cap (V^{x_1} \times B_\varepsilon^H(0) \times V_R^0) \subset \Omega$, $V^{y_0} - y_0 \subset B_{\varepsilon/2}^H(0)$ and $[V^{y_0} \cap p_1(g(\partial(X)))] \times V^0 \subset g(V^{x_1}) \subset g(G_1) = gf(G)$. Then if $(y, t) \in V^{y_0} \times V^0$ we have that $(y, 0) \in \partial(W)$, $\pi(y, 0) \in g(\partial(X)) =$

$g(X) \cap (H \times \{0\})$ and $\pi(y,0) = g(x_2) = (p_1 g_{|\partial(X)} \times 1_{[0,+\infty)})(x_2, 0) = (p_1 g(x_2), 0)$ with $x_2 \in \partial(X)$. Since $\pi(y_0, 0) = (y_0, 0)$, there are open neighbourhoods $V_1^{y_0} \subset H$, $V_1^0 \subset [0, +\infty)$ such that $\pi(V_1^{y_0} \times V_1^0) \subset V^{y_0} \times V^0$ and $V_1^{y_0} \times V_1^0 \subset V^{y_0} \times V^0$.

Now, if $(y,t) \in V_1^{y_0} \times V_1^0$, then $p_1 g(x_2) \in V^{y_0} \cap p_1 g(\partial(X))$, $(p_1 g(x_2), t) \in g(V^{x_1})$, $(p_1 g(x_2), t) = g(z)$ with $z \in V_1^{x_1}$, $g(x_2) + (0,t) = g(z)$ and $(y, 0) = g(x_2) + (u, 0)$, where $(e_{|\Omega})^{-1}(y,0) = (x_2, (u,0))$, $\pi(y,0) = g(x_2)$ and $(u,0) \in N^g_{x_2}(X)$.

Hence $(y,t) = (y,0) + (0,t) = g(x_2) + (u,0) + (0,t) = g(z) + (u,0)$. On the other hand, from the formula $g.f_{|G} = (p_1 g_{|\partial(X)} \times 1_{[0,+\infty)})_{|G}$, it is straightforward to check that $T^g_{x_2}(X) = T^g_z(X)$ and therefore $N^g_{x_2}(X) = N^g_z(X)$, $(u,0) \in N^g_z(X)$, $(z,(u,0)) \in N^g(X)$, $p_1 g(x_2) + u = y$, $u = (y - y_0) - (p_1 g(x_2) - y_0)$, $\| y - y_0 \| < \frac{\varepsilon}{2}$, $\| p_1 g(x_2) - y_0 \| < \varepsilon/2$ and $\| u \| < \varepsilon$. Finally $u \in B^H_\varepsilon(0)$, $(z, u, 0) \in \Omega$, $\pi(y,t) = g(z) = (p_1 g(x_2), t)$ and $\ker(D\pi(y_0, 0)) \subset H \times \{0\}$.

b) $\Rightarrow$ a) By 4.8, $r(U)$ is a totally neat submanifold of class ∞ of U, which is homeomorphic to X. Thus X admits a Hilbert differentiable structure of class ∞ with $\partial(X) \neq \phi$ and $\partial^2(X) = \phi$.

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