ARCHIVUM MATHEMATICUM (BRNO)
Tomus 30 (1994), 207 – 213

PROPERTY (B) OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

Jozef Džurina

Abstract. The aim of this paper is to deduce oscillatory and asymptotic behavior of delay differential equation

$$L_n u(t) - p(t)u(\tau(t)) = 0,$$

from the oscillation of a set of the first order delay equations.

In this paper we are concerned with the oscillatory and asymptotic behavior of solutions of the differential equations of the form

$$L_n u(t) - p(t)u(\tau(t)) = 0, \tag{1}$$

where $n \geq 3$ and L_n denotes the general disconjugate differential operator of the form

$$L_n = \frac{1}{r_n(t)}\frac{d}{dt}\frac{1}{r_{n-1}(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{r_1(t)}\frac{d}{dt}\frac{\cdot}{r_0(t)}. \tag{2}$$

It is always assumed that $r_i(t)$, $0 \leq i \leq n$, $p(t)$ and $\tau(t)$ are continuous on $[t_0, \infty)$, $p(t) > 0$, $\tau(t) < t$, $\tau(t) \to \infty$ as $t \to \infty$ and

$$\int^{\infty} r_i(s)\, ds = \infty \quad \text{for} \quad 1 \leq i \leq n-1. \tag{3}$$

The operator L_n satisfying (3) is said to be in canonical form. It is well-known that any differential operator of the form (2) can always be represented in a canonical form in an essentially unique way (see Trench [7]). In the sequel we will assume that the operator L_n is in canonical form.

1991 *Mathematics Subject Classification*: Primary 34C10.
Key words and phrases: property (B), comparison theorem, deviating argument...
Received June 2, 1993.

We introduce the following notation:

$$L_0u(t) = \frac{u(t)}{r_0(t)},$$
$$L_iu(t) = \frac{1}{r_i(t)}\frac{d}{dt}L_{i-1}u(t), \quad 1 \le i \le n.$$

The domain $D(L_n)$ of L_n is defined to be the set of all functions $u : [T_u, \infty) \to \mathbb{R}$ such that $L_iu(t)$, $0 \le i \le n$ exist and are continuous on $[T_u, \infty)$. A nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

If $u(t)$ is a nonoscillatory solution of (1) then according to a generalization of a lemma of Kiguradze [4, Lemma 3] there is an integer ℓ, $0 \le \ell \le n$ such that $\ell \equiv n \pmod 2$ and

$$\begin{aligned} u(t)L_iu(t) &> 0, \quad 0 \le i \le \ell, \\ (-1)^{i-\ell}u(t)L_iu(t) &> 0, \quad \ell \le i \le n \end{aligned} \tag{4}$$

for all sufficiently large t. A function $u(t)$ satisfying (4) is said to be a function of degree ℓ. The set of all nonoscillatory solutions of degree ℓ of (1) is denoted by N_ℓ. If we denote by N the set of all nonoscillatory solution of (1), then

$$N = N_1 \cup N_3 \cup \cdots \cup N_n \qquad \text{if } n \text{ is odd,}$$

and

$$N = N_0 \cup N_2 \cup \cdots \cup N_n \qquad \text{if } n \text{ is even.}$$

Following Kiguradze, we say that equation (1) has property (B) if for n even $N = N_0 \cup N_n$ and for n odd $N = N_n$.

In a recent paper [5] Kusano and Naito has presented a useful comparison principle which enables us to deduce property (B) of a delay equation of the form (1) from that of ordinary differential equation

$$L_nu(t) - \frac{p(\tau^{-1}(t))r_n(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))r_n(t)}u(t) = 0,$$

where $\tau^{-1}(t)$ is the inverse function to $\tau(t)$. The objectives of this paper is to show that this comparison theorem may fail and then it is a good idea to compare equation (1) with a set of the first order delay equations

$$y'(t) + q_i(t)y(\tau(t)) = 0, \tag{E_i}$$

We present the relationship between property (B) of equation (1) and oscillation of equations (E_i).

We begin by formulating a preparatory result (Lemma 1) which can be found in [5] and is needed in the sequel.

Let $i_k \in \{1, 2, \cdots, n-1\}$, $1 \leqslant k \leqslant n-1$ and $t, s \in [t_0, \infty)$, we define

$$I_0 = 1,$$
$$I_k(t, s; r_{i_k}, \cdots, r_{i_1}) = \int_s^t r_{i_k}(x) I_{k-1}(x, s; r_{i_{k-1}}, \cdots, r_{i_1})\, dx.$$

It is easy to verify that for $1 \leqslant k \leqslant n-1$

$$\begin{aligned} I_k(t, s; r_{i_k}, \cdots, r_{i_1}) &= (-1)^k I_k(s, t; r_{i_1}, \cdots, r_{i_k}), \\ I_k(t, s; r_{i_k}, \cdots, r_{i_1}) &= \int_s^t r_{i_1}(x) I_{k-1}(t, x; r_{i_k}, \cdots, r_{i_2})\, dx. \end{aligned} \tag{5}$$

Lemma 1. *If $u \in \mathcal{D}(L_n)$ then the following formula holds for $0 \leqslant i \leqslant k \leqslant n-1$ and $t, s \in [T_u, \infty)$:*

$$\begin{aligned} L_i u(t) &= \sum_{j=i}^{k} (-1)^{j-i} L_j u(s) I_{j-i}(s, t; r_j, \cdots, r_{i+1}) \\ &\quad + (-1)^{k-i+1} \int_t^s I_{k-i}(x, t; r_k, \cdots, r_{i+1})\, r_{k+1}(x) L_{k+1} u(x)\, dx. \end{aligned} \tag{6}$$

This lemma is a generalization of Taylor's formula. The proof is immediate.

For convenience and further references we make use the following notations:

$$\begin{aligned} q_i(t) = r_{i+1}(t) &\int_t^\infty r_n(s) r_0(\tau(s)) p(s) I_{n-i-2}(s, t; r_{n-1}, \cdots, r_{i+2}) \\ &\times \int_{t_1}^{\tau(t)} I_{i-1}(\tau(s), x; r_1, \cdots, r_{i-1})\, r_i(x)\, dx ds, \end{aligned}$$

for sufficiently large t_1 with $\tau(t) > t_1$ and $i = 1, 2, \cdots, n-2$.

The following theorem presents an important relationship between equation (1) and corresponding first order differential inequalities.

Lemma 2. *Let*

$$\tau(t) \quad \textit{be nondecreasing.} \tag{7}$$

Assume that for $\ell \in \{1, 2, \cdots, n-2\}$, such that $n+\ell$ is even the linear differential inequalities

$$y'(t) + q_\ell(t) y(\tau(t)) \leqslant 0 \tag{$\widetilde{E}_\ell$}$$

have no eventually positive solutions. Then equation (1) has property (B).

Proof. For the sake of contradiction we assume that (1) has a nonoscillatory solution, which belongs to the class N_ℓ,where $\ell \in \{1, 2, \cdots, n-2,\}$, such that $n+\ell$

is even. We may assume that $u(t)$ is positive. Then there exists a t_1 such that (4) holds for all $t \geqslant t_1$.

Putting $i = \ell + 1$, $k = n - 1$ and $s \geqslant t \geqslant t_1$ in (6), we have in view of (4) and (5)

$$-L_{\ell+1}u(t) \geqslant \int_t^s r_n(x) I_{n-\ell-2}(x, t; r_{n-1}, \cdots, r_{\ell+2})\, L_n u(x)\, dx,$$

letting $s \to \infty$ and using (1), we obtain

$$-L_{\ell+1}u(t) \geqslant \int_t^\infty r_n(x) I_{n-\ell-2}(x, t; r_{n-1}, \cdots, r_{\ell+2})\, p(x) u(\tau(x))\, dx, \tag{8}$$

for $t \geqslant t_1$. Now by Lemma 1, with $i = 0$, $k = \ell - 1$ and $t \geqslant s = t_1$ taking (4) and (5) into account, one gets

$$L_0 u(t) \geqslant \int_{t_1}^t I_{\ell-1}(t, x; r_1, \cdots, r_{\ell-1})\, r_\ell(x) L_\ell u(x)\, dx \tag{9}$$

for $t \geqslant t_1$. Combining (8) with (9) we have with respect to (7)

$$\begin{aligned} -L_{\ell+1}u(t) &\geqslant \int_t^\infty r_n(s) p(s) r_0(\tau(s)) I_{n-\ell-2}(s, t; r_{n-1}, \cdots, r_{\ell+2}) \\ &\quad \times \int_{t_1}^{\tau(s)} I_{\ell-1}(\tau(s), x; r_1, \cdots, r_{\ell-1})\, r_\ell(x) L_\ell u(x)\, dx ds \\ &\geqslant \int_t^\infty r_n(s) p(s) r_0(\tau(s)) I_{n-\ell-2}(s, t; r_{n-1}, \cdots, r_{\ell+2}) \\ &\quad \times \int_{t_1}^{\tau(t)} I_{\ell-1}(\tau(s), x; r_1, \cdots, r_{\ell-1})\, r_\ell(x) L_\ell u(x)\, dx ds. \end{aligned}$$

Since $L_\ell u(t)$ is decreasing ($\ell \geqslant 1$) we obtain from the last inequality

$$-L_{\ell+1}u(t) \geqslant L_\ell(\tau(t)) \frac{q_\ell(t)}{r_{\ell+1}(t)}, \tag{10}$$

for $t \geqslant t_1$. Let $y(t)$ be given by

$$y(t) \equiv L_\ell u(t),$$

then $y(t) > 0$ on $[t_1, \infty)$ and $y'(t) = r_{\ell+1}(t) L_{\ell+1} u(t)$ and by (10) we have

$$y'(t) + q_\ell(t) y(\tau(t)) \leqslant 0, \quad t \geqslant t_1.$$

which contradicts with the fact that differential inequality $(\widetilde{E}_\ell)$ has no positive solutions.

Theorem 1. *Let (7) hold. Define a function $f = f(\lambda)$ for $0 \leqslant \lambda \leqslant 1/\mathrm{e}$ by*

$$f\,\mathrm{e}^{-\lambda f} = 1, \quad 1 \leqslant f \leqslant \mathrm{e}\,.$$

Assume that for $\ell \in \{1, 2, \cdots, n-2\}$ such that $n + \ell$ is even either

$$d = \liminf_{t\to\infty} \int_{\tau(t)}^{t} q_\ell(s)\,ds > \frac{1}{\mathrm{e}}, \tag{11}$$

or

$$c = \limsup_{t\to\infty} \int_{\tau(t)}^{t} q_\ell(s)\,ds > 1, \tag{12}$$

or when $0 < d \leqslant 1/\mathrm{e}$ and $c \leqslant 1$ the folowing condition hold

$$f(d)\left(1 - \sqrt{1-c}\right)^2 > 1. \tag{13}$$

Then equation (1) has property (B).

Proof. It is known (see e.g. [2, Corollary 4] or [1, Theorem 1] or [6, Theorem 2.1.1]) that conditions (11) or (12) or (13) are sufficient for differential inequalities $(\widetilde{E}_i)$ to have no positive solutions. Our assertion follows by Lemma 2.

Lemma 3. *Suppose that $q(t) \in C([t_0, \infty))$ is positive. Equation*

$$y'(t) + q(t)y(\tau(t)) = 0$$

has a positive solution if and only if the differential inequality

$$y'(t) + q(t)y(\tau(t)) \leqslant 0$$

has a positive solution.

This lemma can be found in [3, Corollary 3.2.2].

Theorem 2. *Let (7) hold. Assume that for $\ell \in \{1, 2, \cdots, n-2\}$ such that $n + \ell$ is even the delay differential equations*

$$y'(t) + q_\ell(t)y(\tau(t)) = 0 \tag{E_ℓ}$$

are oscillatory. Then equation (1) has property (B).

Proof. This theorem immediately follows from Lemma 2 and Lemma 3.

Example 1. Let us consider the third order equation

$$y'''(t) - \frac{a}{t^2\sqrt[3]{t}}\,y(\sqrt[3]{t}) = 0, \quad a > 0, \quad \text{and} \quad t \geqslant 1, \tag{14}$$

We have $\tau(t) = \sqrt[3]{t}$, $p(t) = \dfrac{a}{t^2\sqrt[3]{t}}$ and $q_1(t) = (\tau(t) - t_1)\displaystyle\int_t^\infty p(s)\,ds$. By Theorem 1 equation (14) has property (B) for all $a > 0$. On the other hand, by the abovementioned result of Kusano and Naito one gets that (14) has property (B) if the ordinary equation without delay

$$y'''(t) - \frac{3a}{t^5}y(t) = 0 \tag{15}$$

has property (B). But as (15) has not property (B) criterion of Kusano and Naito fails for (14).

We can extend our previous results to more general nonlinear differential equation of the form

$$L_n u(t) - f(t, u(\tau(t))) = 0, \tag{16}$$

where L_n and τ are the same as in (1) and f is a continuos function such that $f : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ and $xf(t, x) > 0$ for $x \neq 0$.

Lemma 4. *Let all conditions of Lemma 2 are satisfied. Moreover we assume that*

$$|f(t, x)| \geqslant p(t)|x| \quad \text{for } t \in [t_0, \infty) \quad \text{and } x \neq 0. \tag{17}$$

Then equation (16) has property (B).

Proof. By Lemma 2 equation (1) has property (B). Applying the well known comparison theorem of Kusano and Naito [5, Theorem 1] one gets that condition (17) guarantees property (B) of (16).

Theorem 3. *Let all conditions of Theorem 2 hold and (17) is satisfied. Then equation (16) has property (B).*

Proof. The assertion of Theorem 3 follows by Theorem 2 and Theorem 1 in [5].

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Jozef Džurina
Department of Mathematical Analysis
Šafarik University
Jesenná 5
041 54 Košice, SLOVAKIA

E-mail: Dzurina@turing.upjs.sk