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NATURAL LIFTINGS OF (0,2)–TENSOR FIELDS TO THE TANGENT BUNDLE

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Abstract. We determine all first order natural operators transforming $(0,2)$–tensor fields on a manifold M into $(0,2)$–tensor fields on TM.

1. Introduction

There are classical constructions of tensor fields on the tangent bundle TM from a tensor field on the base manifold M, namely the vertical and the complete lifts, cf. [9] and [12]. Moreover, if M is endowed with a linear connection, then one can also define the horizontal lift of a tensor field to TM. From a general point of view, geometrical constructions are natural differential operators. Then the full list of such operators gives the complete list of all possible geometric constructions.

The aim of this paper is to determine all first order natural operators $T^*\otimes T^* \rightsquigarrow (T^*\otimes T^*)T$ transforming $(0,2)$–tensor fields on M into $(0,2)$–tensor fields on TM. For comparison's sake, we point out that Kowalski and Sekizawa [7] determined all natural operators transforming Riemannian metrics on M into metrics on TM. Recently Janyška [3] has classified all first order natural operators from Riemannian metrics into 2–forms on the tangent bundle. In both of these examples the regularity of the original $(0,2)$–tensor field on the base manifold M is essential, while we shall consider arbitrary $(0,2)$–tensor fields without any additional requirement. In what follows we shall use the concept of a natural operator from [6].

We remark that liftings of tensor fields to the tangent bundle play an important role in the analytical mechanics, see e.g. [2]. All manifolds and maps are assumed to be infinitely differentiable.

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2. The canonical liftings

Let M be a manifold of dimension m. We denote by $p_M : TM \to M$ the tangent bundle and by $q_M : T^*M \to M$ the cotangent bundle of M. The canonical coordinates (x^i) on M induce the additional coordinates $y^i = dx^i$ on TM and p_i on T^*M. The coordinates on TTM will be denoted by $(x^i, y^i, X^i = dx^i, Y^i = dy^i)$, on TT^*M by $(x^i, p_i, \xi^i = dx^i, P_i = dp_i)$ and on T^*TM by $(x^i, w^i = dx^i, r_i dx^i + s_i dw^i)$.

If $f : M \to \mathbb{R}$ is a function, then the *vertical lift* of f to TM is a function $f^V : TM \to \mathbb{R}$ defined by $f^V = f \circ p_M$. The *complete lift* f^C of f is defined by $f^C(y) = df(x)(y)$, $x = p_M(y)$.

Let $X = \xi^i(x)\frac{\partial}{\partial x^i}$ be a vector field on M. The *vertical lift* of X to TM is a vertical vector field X^V on TM determined by the translations in the individual fibres of TM. The *complete lift* of X to TM is the flow prolongation X^C of X, $X^C = \frac{\partial}{\partial t}\big|_0 T(\exp tX)$, where $\exp tX$ means the flow of X. In coordinates, $X^V = \xi^i(x)\frac{\partial}{\partial y^i}$, $X^C = \xi^i(x)\frac{\partial}{\partial x^i} + \frac{\partial \xi^i(x)}{\partial x^j} y^j \frac{\partial}{\partial y^i}$. Let us remark that X^V and X^C can also be defined by means of their actions on functions: $X^V(f^C) = (Xf)^V$, $X^C(f^C) = (Xf)^C$ for every function $f : M \to \mathbb{R}$.

To define the vertical and the complete lift of a tensor field we shall use the following assertion (see e.g. [2] and [9]).

Lemma 1. *If G and G' are $(0,r)$–tensor fields on TM such that for all vector fields $X_1, \dots, X_r$ on M we have*

$$G(X_1^C, \dots, X_r^C) = G'(X_1^C, \dots, X_r^C),$$

then $G = G'$.

Definition 1. Let G be a tensor field of type $(0,2)$ on M. The *vertical lift* of G to TM is a tensor field G^V of type $(0,2)$ on TM defined by $G^V(X_1^C, X_2^C) = (G(X_1, X_2))^V$ for all vector fields X_1, X_2 on M. The *complete lift* of G to TM is a tensor field G^C of type $(0,2)$ on TM given by $G^C(X_1^C, X_2^C) = (G(X_1, X_2))^C$ for all vector fields X_1, X_2 on M.

If $G = g_{ij} dx^i \otimes dx^j$ is the coordinate expression of G, then

$$G^V = g_{ij} dx^i \otimes dx^j,$$

$$G^C = \left(\frac{\partial g_{ij}}{\partial x^k} y^k\right) dx^i \otimes dx^j + g_{ij} dx^i \otimes dy^j + g_{ij} dy^i \otimes dx^j.$$

Remark 1. Our concept of a complete lift to the tangent bundle coincides with the definition due to Yano and Ishihara [12] and Morimoto [9]. Morimoto even introduced liftings of tensor fields of type (p,q) to the bundle $T_1^r M$ of 1–dimensional velocities. Moreover, liftings of tensor fields to the bundle $T_k^r M = J_0^r(\mathbb{R}^k, M)$ are studied in [10].

If $G = a_{ij} dx^i \wedge dx^j$ is a 2–form on M, then $G^V = p_M^* G$, i.e. the vertical lift is exactly the pull-back of G to TM. Further, the vertical lift of a Riemannian metric is a degenerated metric of rank m on TM. One can easily prove

Lemma 2. *Let G be a $(0,2)$–tensor field on M. We have*

1. *If G has rank r, then G^C has rank $2r$.*
2. *If G is symmetric (or skew-symmetric), then G^C is symmetric (or skew-symmetric) as well.*
3. *If G is a Riemannian metric on M, then G^C is a pseudoriemannian metric on TM of signature (m,m).*
4. *If G is a 2–form on M, then G^C is a 2–form on TM and we have $dG^C = (dG)^C$.*
5. *$G^C(X^V,Y^C) = G^C(X^C,Y^V) = G(X,Y)^V$, $G^C(X^V,Y^V) = 0$ for all vector fields X, Y on M.*
6. *If G is a symplectic form on M, then G^C is a symplectic form on TM.*
7. *$G^V(X^V,Y^V) = 0$ for all vector fields X, Y on M.*

Denote by $\kappa_M : TTM \to TTM$ the canonical involution and by $s_M : TT^*M \to T^*TM$ the canonical isomorphism [8], [11]. The coordinate expression of s_M is $w^i = \xi^i$, $r_i = P_i$, $s_i = p_i$. It is well-known that the complete lift X^C of a vector field X can be described by $X^C = \kappa_M \circ TX$. We show that a similar characterization holds also for the complete lift of $(0,2)$–tensor fields to TM. There is a canonical isomorphism $\psi\colon A^*\otimes B^* \to \mathrm{Lin}(A,B^*)$, $\psi(a^*\otimes b^*)(c) = \langle b^*,c\rangle a^*$. Hence we can identify every $(0,2)$–tensor field G on M with a linear map $G_L : TM \to T^*M$ over the identity of M, which is defined by $\langle G_L(y),z\rangle_x = G_x(z,y)$, $y,z\in T_xM$. The coordinate expression of G_L is

$$p_i = g_{ij}y^j.$$

Analogously, denote by $\overline{G}_L\colon TTM \to T^*TM$ the linear map over id_{TM} corresponding to a $(0,2)$–tensor field $\overline{G}$ on TM.

Proposition 1. *Let G be an arbitrary $(0,2)$–tensor field on M. Then the complete lift G^C is the only tensor field $\overline{G}$ on TM satisfying*

$$\overline{G}_L = s_M \circ TG_L \circ \kappa_M. \tag{1}$$

Proof. The natural equivalence s can be distinguished among all natural transformations $TT^* \to T^*T$ by the following geometric construction, [5]. If X is a vector field and $\omega\colon M\to T^*M$ is a 1-form on M, then $\langle\omega,X\rangle\colon M\to\mathbb{R}$. By [5], s is the only natural transformation $TT^*\to T^*T$ over the identity of T satisfying $\langle sT\omega, X^C\rangle = \langle\omega,X\rangle^C$. Then the assertion follows from the definitions of G_L and G_L^C and from Lemma 1. □

Let $\alpha = p_i dx^i$ be the Liouville 1–form on T^*M. Then the pull-back $\beta := (G_L)^*\alpha$ is a 1–form on TM, $\beta = g_{ij}y^j\,dx^i$. (We can also define β by $\beta(y) = (G(-,y))^V$.)

Definition 2. Let G be a $(0,2)$–tensor field on M. The *antisymmetric lift* of G to TM is a 2–form G^A on TM defined by $G^A = d\beta$.

Obviously, G^A is the pull-back $G_L^*\Omega$ of the canonical symplectic form $\Omega = d\alpha$ on T^*M. In coordinates,

$$G^A = \left(\frac{\partial g_{jm}}{\partial x^i}y^m\right)dx^i\wedge dx^j - g_{ij}\,dx^i\wedge dy^j.$$

The corresponding matrix expression of G^A is

$$\begin{pmatrix} \left(\frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{im}}{\partial x^j}\right) y^m & -g_{ij} \\ g_{ji} & 0 \end{pmatrix}$$

If G is a Riemannian metric on M, then the antisymmetric lift G^A is exactly the canonical symplectic 2–form on TM defined by Janyška in [3]. In general, we have

Proposition 2. *Let G be a regular $(0,2)$–tensor field on M. Then TM, T_1^rTM, TT_1^rM are symplectic manifolds.*

Proof. Clearly, if G is regular, then $G_L\colon TM \to T^*M$ is an isomorphism and G^A is a symplectic form on TM. By [1], $T_1^rT^*M$ is a symplectic manifold. If ω is the corresponding symplectic form on $T_1^rT^*M$, then the pull-back $(T_1^rG_L)^*\omega$ is a symplectic form on T_1^rTM. Finally, the well-known identification $T_1^rTM \approx TT_1^rM$ determined by the exchange homomorphism of Weil algebras, [6], defines a symplectic structure on TT_1^rM. □

Let Γ be a linear connection on M with the local Christoffel symbols Γ^i_{jk}. Then the tangent space of TM at any point $y \in TM$ splits into the horizontal and vertical subspace with respect to Γ, $T_yTM = H_y \oplus V_y$, and we have a linear isomorphism $T_xM \to H_y$, $x = p_M(y)$. This isomorphism defines the *horizontal lift* of a vector field X on M into a vector field X^H on TM.

Definition 3. Let G be a $(0,2)$–tensor field on M. The *horizontal lift* of G to TM is a tensor field G^H of the same type on TM given by $G^H(X^V,Y^V) = G^H(X^H,Y^H) = 0$, $G^H(X^H,Y^V) = G^H(X^V,Y^H) = (G(X,Y))^V$ for all vector fields X, Y on M.

We have

$$G^H = (g_{is}\Gamma^s_{kj}y^k + g_{sj}\Gamma^s_{ki}y^k)dx^i \otimes dx^j + g_{ij}dx^i \otimes dy^j + g_{ij}dy^i \otimes dx^j.$$

Proposition 3. *Let G be a $(0,2)$–tensor field on M. Then $G^H = G^C$ if and only if $\nabla G = 0$.*

Proof. A direct calculation gives $\nabla G = 0$ iff $\frac{\partial g_{ij}}{\partial x^k} = g_{is}\Gamma^s_{kj} + g_{sj}\Gamma^s_{ki}$. □

It is interesting to point out that the same assertion holds also for the horizontal and complete lift of $(0,1)$–tensor fields (i.e. 1– forms) provided we define the horizontal lift by $\alpha^H(X^H) = 0$, $\alpha^H(X^V) = (\alpha(X))^V$ for every vector field X on M.

3. Invariant functions on $J^1(T^* \otimes T^*) \oplus TT$

The aim of this section is to determine all first order natural operators transforming $(0,2)$–tensor fields on M into functions on TTM. Such functions will then play the role of coefficients of natural transformations $TTM \to T^*TM$ (see Proposition 6 in the next section).

Denote by $Q = \otimes^2\mathbb{R}^{m*} \times \otimes^3\mathbb{R}^{m*} \times \times^3\mathbb{R}^m$ the standard fibre of the bundle functor $J^1(T^*\otimes T^*)\oplus TT$ and by G^r_m the group of all invertible r–jets of $\mathbb{R}^m$ into $\mathbb{R}^m$ with source and target zero. We shall denote by (a^i_j, a^i_{jk}) the canonical coordinates in G^2_m and by tilde the coordinates of the inverse element. If $(g_{ij}, g_{ij,k} := \frac{\partial g_{ij}(x)}{\partial x^k}, y^i, X^i, Y^i)$ are the canonical coordinates on Q, then the action of G^2_m on Q is given by

$$\begin{aligned}\bar g_{ij} &= \tilde a^k_i \tilde a^\ell_j g_{k\ell}, \quad \bar g_{ij,k} = \tilde a^m_i \tilde a^n_j \tilde a^p_k g_{mn,p} + (\tilde a^m_{ik}\tilde a^n_j + \tilde a^m_i \tilde a^n_{jk})g_{mn},\\ \bar y^i &= a^i_j y^j, \quad \bar X^i = a^i_j X^j, \quad \bar Y^i = a^i_j Y^j - a^i_\ell \tilde a^\ell_{mn} a^m_j a^n_k y^j X^k.\end{aligned} \tag{2}$$

Denote further

$$\begin{aligned} &I_1 = g_{ij}y^iy^j, \quad I_2 = g_{ij}X^iX^j, \quad I_3 = g_{ij}X^iy^j, \quad I_4 = g_{ij}y^iX^j,\\ &I_5 = g_{ij,k}y^iy^jX^k + g_{ij}y^iY^j + g_{ij}Y^iy^j,\\ &I_6 = g_{ij,k}X^iX^jy^k + g_{ij}X^iY^j + g_{ij}Y^iX^j.\end{aligned} \tag{3}$$

The geometrical construction of $I_1,\dots,I_6$ is straightforward. Denote by $G(u,v)$ the full contraction of G with $u,v\in TM$. On the iterated tangent bundle we have two canonical projections $p_{TM}, Tp_M : TTM \to TM$. Then $I_1 = G(p_{TM}(A), p_{TM}(A))$, $I_2 = G(Tp_M(A), Tp_M(A))$, $I_3 = G(Tp_M(A), p_{TM}(A))$, $I_4 = G(p_{TM}(A), Tp_M(A))$, $A\in TTM$. Further, differentiating I_1 we get I_5, and $I_6 = I_5\circ\kappa_M$, where $\kappa_M : TTM \to TTM$ is the canonical involution. Obviously, $I_1,\dots,I_6$ are invariants of G^2_m. Now we prove that $I_1,\dots,I_6$ generate all G^2_m–invariants defined on Q.

Proposition 4. *For $m = dim M \geqq 3$, all first order natural operators transforming $(0,2)$–tensor fields on M into functions on TTM are of the form*

$$\varphi(I_1,\dots,I_6)$$

where φ is an arbitrary smooth function of six variables.

Proof. According to the general theory of natural operators, [6], we have to determine all G^2_m–invariant maps $f\colon Q\to\mathbb{R}$, $f = f(y^i, X^i, Y^i, g_{ij}, g_{ij,k})$. Using the tensor evaluation theorem from [6] we get $f = \varphi(P_1,\dots,P_{36})$, where φ is an arbitrary smooth function of 36 variables

$$P_1 = g_{ij}y^iy^j\,, P_2 = g_{ij}X^iX^j\,, P_3 = g_{ij}Y^iY^j\,, P_4 = g_{ij}y^iX^j\,, P_5 = g_{ij}X^iy^j\,,$$
$$P_6 = g_{ij}y^iY^j\,, P_7 = g_{ij}Y^iy^j\,, P_8 = g_{ij}X^iY^j\,, P_9 = g_{ij}Y^iX^j\,, P_{10} = g_{ij,k}y^iy^jy^k\,,$$
$$P_{11} = g_{ij,k}X^iX^jX^k\,, P_{12} = g_{ij,k}Y^iY^jY^k\,, P_{13} = g_{ij,k}y^iX^jX^k\,, P_{14} = g_{ij,k}y^iY^jY^k\,,$$
$$P_{15} = g_{ij,k}y^iy^jX^k\,, P_{16} = g_{ij,k}y^iX^jy^k\,, P_{17} = g_{ij,k}y^iy^jY^k\,, P_{18} = g_{ij,k}y^iY^jy^k\,,$$
$$P_{19} = g_{ij,k}y^iX^jY^k\,, P_{20} = g_{ij,k}y^iY^jX^k\,, P_{21} = g_{ij,k}X^iy^jy^k\,, P_{22} = g_{ij,k}X^iY^jY^k\,,$$
$$P_{23} = g_{ij,k}X^iy^jX^k\,, P_{24} = g_{ij,k}X^iX^jy^k\,, P_{25} = g_{ij,k}X^iy^jY^k\,, P_{26} = g_{ij,k}X^iY^jy^k\,,$$
$$P_{27} = g_{ij,k}X^iX^jY^k\,, P_{28} = g_{ij,k}X^iY^jX^k\,, P_{29} = g_{ij,k}Y^iy^jy^k\,, P_{30} = g_{ij,k}Y^iX^jX^k\,,$$
$$P_{31} = g_{ij,k}Y^iy^jX^k\,, P_{32} = g_{ij,k}Y^iX^jy^k\,, P_{33} = g_{ij,k}Y^iy^jY^k\,, P_{34} = g_{ij,k}Y^iY^jy^k\,,$$
$$P_{35} = g_{ij,k}Y^iX^jY^k\,, P_{36} = g_{ij,k}Y^iY^jX^k\,.$$

Replace (P_{15}, P_6, P_7) by a new triple of independent variables $P'_{15} := P_{15} - P_6 - P_7$, $I_5 = P_{15} + P_6 + P_7$, $P'_7 := P_6 - P_7$. Analogously, we replace (P_{24}, P_8, P_9) by $P'_{24} := P_{24} - P_8 - P_9$, $I_6 = P_{24} + P_8 + P_9$, $P'_9 := P_8 - P_9$. Then φ is of the form

$$\varphi(I_1, \dots, I_6, P_3, P'_7, P'_9, P_{10}, \dots, P_{14}, P'_{15}, P_{16}, \dots, P_{23}, P'_{24}, P_{25}, \dots, P_{36}). \tag{4}$$

It suffices to deduce that φ is independent of all $P's$. Consider the equivariance of (4) on the kernel of the jet projection $G^2_m \to G^1_m$, which is characterized by $a^i_j = \delta^i_j$, and put $y = (1, 0, \dots, 0)$, $X = (0, 1, 0, \dots, 0)$, $Y = (0, 0, 1, 0, \dots, 0)$. We obtain

$$\begin{aligned}&\varphi(I_1, \dots, I_6, P_3, P'_7, P'_9, P_{10}, \dots, P_{14}, P'_{15}, P_{16}, \dots, P_{23}, P'_{24}, P_{25}, \dots, P_{36})\\ &\quad= \varphi(I_1, \dots, I_6, \bar P_3, \bar P'_7, \bar P'_9, \bar P_{10}, \dots, \bar P_{14}, \bar P'_{15}, \bar P_{16}, \dots, \bar P_{23}, \bar P'_{24}, \bar P_{25}, \dots, \bar P_{36})\end{aligned}$$

where $P_3 = g_{33}$, $\bar P_3 = g_{33}(1 + a^3_{12})(1 + a^3_{12}), \dots, P_{36} = g_{33,2}$, $\bar P_{36} = (g_{33,2} + \tilde a^m_{32} g_{m3} + \tilde a^n_{32} g_{3n})(1 + a^3_{12})(1 + a^3_{12})$. Setting $a^3_{12} = -1$ we get that φ does not depend on all $P's$ except P_{10}, P_{11}, P_{13}, P'_{15}, P_{16}, P_{21}, P_{23}, P'_{24}. By the choice of $\tilde a^m_{11}$ we prove that φ is independent of P_{10}, P_{16}, P_{21}. Analogously, by means of $\tilde a^m_{22}$ we get that φ does not depend on P_{11}, P_{13}, P_{23} and the choice of $\tilde a^m_{12}$ yields the independence of φ on P'_{15} and P'_{24}. □

In the case $m = 2$, the same result holds if we restrict ourselves to tensor fields which are either symmetric or antisymmetric.

Proposition 5. *For $m = 2$, all first order natural operators transforming symmetric or antisymmetric $(0, 2)$–tensor fields on M into functions on TTM are of the form*

$$\varphi(I_1, \dots, I_6) \tag{5}$$

where φ is an arbitrary smooth function of six variables.

Proof. Consider the function $f(y^i, X^i, Y^i, g_{ij}, g_{ij,k})$ from the proof of Proposition 4 and define φ by the formula $\varphi(z_1, \dots, z_6) = f(1, 0; 0, 1; 0, 0; g_{11} = z_1, g_{22} =$

$z_2, g_{12} = z_3, g_{21} = z_4; g_{11,2} = z_5, g_{22,1} = z_6, 0,0,0,0,0,0)$. There is a linear transformation transforming independent vectors y and X into $(1,0)$ and $(0,1)$. Next, (2) with $a^i_j = \delta^i_j$ yields $\bar{Y}^i = Y^i - \tilde{a}^i_{12}$, $\bar{g}_{ij,k} = g_{ij,k} + \tilde{a}^m_{ik} g_{mj} + \tilde{a}^n_{jk} g_{in}$. By the choice of $\tilde{a}^1_{12}$ and $\tilde{a}^2_{12}$ we obtain $\bar{Y}^i = 0$. Further, for $g_{ij} \neq 0$ the choice of $\tilde{a}^1_{11}$ and $\tilde{a}^2_{11}$ gives $\bar{g}_{11,1} = 0$, $\bar{g}_{12,1} = 0$. Analogously, using $\tilde{a}^1_{22}$ and $\tilde{a}^2_{22}$ we get $\bar{g}_{22,2} = 0$, $\bar{g}_{12,2} = 0$. By symmetry or antisymmetry we have $\bar{g}_{21,1} = \bar{g}_{12,1} = 0$, $\bar{g}_{21,2} = \bar{g}_{12,2} = 0$. Then $I_1 = g_{11}$, $I_2 = g_{22}$, $I_3 = g_{12}$, $I_4 = g_{21}$, $I_5 = g_{11,2}$, $I_6 = g_{22,1}$. Thus φ is of the form (5) on an open dense subset. □

4. The classification theorem

We first prove the following auxiliary assertion, which has also a number of interesting features in its own right (see Remark 2).

Proposition 6. *For $m \geqq 3$, all first order natural operators $T^* \otimes T^* \rightsquigarrow (TT, T^*T)$ transforming $(0,2)$–tensor fields on M into morphisms $TTM \to T^*TM$ are of the form*

$$
\begin{aligned}
w^i &= A_1 y^i + A_2 X^i, \\
s_i &= A_3 g_{ji} y^j + \overline{A}_3 g_{ij} y^j + A_4 g_{ji} X^j + \overline{A}_4 g_{ij} X^j, \\
r_i &= (A_1 A_4 + A_2 A_3) g_{ji} Y^j + (A_1 \overline{A}_4 + A_2 \overline{A}_3) g_{ij} Y^j + A_2 A_3 g_{ji,k} y^j X^k \\
&\quad + A_2 \overline{A}_3 g_{ij,k} y^j X^k + A_1 A_4 g_{ji,k} X^j y^k + A_1 \overline{A}_4 g_{ij,k} X^j y^k \\
&\quad + A_5 g_{ji} y^j + \overline{A}_5 g_{ij} y^j + A_6 g_{ji} X^j + \overline{A}_6 g_{ij} X^j \qquad (6)\\
&\quad - B_1 g_{ji} Y^j - B_2 g_{ji} Y^j - B_1 g_{ij} Y^j - B_2 g_{ij} Y^j \\
&\quad - B_1 g_{ji,k} y^j X^k - B_2 g_{ij,k} y^j X^k - B_2 g_{ji,k} X^j y^k \\
&\quad - B_1 g_{ij,k} X^j y^k + B_1 g_{jk,i} y^j X^k + B_2 g_{jk,i} X^j y^k \\
&\quad + C_1 g_{jk,i} y^j y^k + C_2 g_{jk,i} X^j X^k
\end{aligned}
$$

where A_i,$\overline{A}_i$ and B_i are arbitrary smooth functions of the invariants $I_1, \dots, I_6$ and

$$C_1 = A_1 A_3 = A_1 \overline{A}_3, \quad C_2 = A_2 A_4 = A_2 \overline{A}_4.$$

Proof. Let $S = \mathbb{R}^m \times \mathbb{R}^{m*} \times \mathbb{R}^{m*}$ be the standard fibre of T^*T with the canonical coordinates (w^i, s_i, r_i). Then we have to determine all G^2_m–equivariant maps $Q \to S$,

$$
\begin{aligned}
w^i &= w^i(y^i, X^i, Y^i, g_{ij}, g_{ij,k}), \\
s_i &= s_i(y^i, X^i, Y^i, g_{ij}, g_{ij,k}), \\
r_i &= r_i(y^i, X^i, Y^i, g_{ij}, g_{ij,k}).
\end{aligned}
$$

Using standard evaluations we find that the action of G^2_m on S is

$$\bar{w}^i = a^i_j w^j, \quad \bar{s}_i = \tilde{a}^j_i s_j, \quad \bar{r}_i = \tilde{a}^j_i r_j - \tilde{a}^j_i a^\ell_{jk} \tilde{a}^m_\ell s_m w^k$$

while the action of G_m^2 on Q is given by (2). First we discuss $w^i(y^i, X^i, Y^i, g_{ij}, g_{ij,k})$. Let us introduce new variables $u_i \in \mathbb{R}^{m*}$, $\bar{u}_i = \tilde{a}_i^j u_j$. Then $\varphi := w^i u_i$ is a G_m^2–invariant function, $\varphi = \varphi(y^i, X^i, Y^i, g_{ij}, g_{ij,k}, u_i)$ and $I_7 = y^i u_i$, $I_8 = X^i u_i$ are further G_m^2–invariants. In the same way as in the proof of Proposition 4 we deduce $\varphi = \varphi(I_1, \dots, I_8)$, so that $w^i u_i = \varphi(I_1, \dots, I_6, y^i u_i, X^i u_i)$. Differentiating with respect to u_i and then setting $u_i = 0$ we obtain $w^i = A_1(I_1, \dots, I_6) y^i + A_2(I_1, \dots, I_6) X^i$. This corresponds to the first equation of (6). Using a similar procedure for $s_i(y^i, X^i, Y^i, g_{ij}, g_{ij,k})$ we deduce the second equation of (6).

Finally, assume $r_i(y^i, X^i, Y^i, g_{ij}, g_{ij,k})$ in the form

$$\begin{aligned} r_i = {} & \alpha_1 g_{ji} Y^j + \overline{\alpha}_1 g_{ij} Y^j + \alpha_2 g_{ji,k} y^j X^k + \overline{\alpha}_2 g_{ij,k} y^j X^k + \alpha_3 g_{ji,k} X^j y^k \\ & + \overline{\alpha}_3 g_{ij,k} X^j y^k + \beta_1 g_{jk,i} y^j X^k + \beta_2 g_{jk,i} X^j y^k + \gamma_1 g_{ij,k} y^j y^k + \gamma_2 g_{ij,k} X^j X^k \\ & + \gamma_3 g_{ji,k} y^j y^k + \gamma_4 g_{ji,k} X^j X^k + \gamma_5 g_{jk,i} y^j y^k + \gamma_6 g_{jk,i} X^j X^k \\ & + \tilde{r}_i(y^i, X^i, Y^i, g_{ij}, g_{ij,k}). \end{aligned}$$

Applying equivariance on the kernel of the jet projection $G_m^2 \to G_m^1$ we get the following relations: $A_2 A_3 = \alpha_2 + \beta_1$, $A_2 \overline{A}_3 = \overline{\alpha}_2 + \beta_2$, $A_1 A_4 = \alpha_3 + \beta_2$, $A_1 \overline{A}_4 = \overline{\alpha}_3 + \beta_1$, $\alpha_1 = \alpha_2 + \alpha_3$, $\overline{\alpha}_1 = \overline{\alpha}_2 + \overline{\alpha}_3$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$, $A_1 A_3 = A_1 \overline{A}_3 = \gamma_5$, $A_2 A_4 = A_2 \overline{A}_4 = \gamma_6$. Then the full equivariance reads $\tilde{a}_i^j \tilde{r}_j(y^i, \dots, g_{ij,k}) = \tilde{r}_i(\bar{y}_i, \dots, \bar{g}_{ij,k})$ so that $\tilde{r}_i$ has the same transformation law as s_i. Thus $\tilde{r}_i = A_5 g_{ji} y^j + \overline{A}_5 g_{ij} y^j + A_6 g_{ji} X^j + \overline{A}_6 g_{ij} X^j$. □

Let $G = g_{ij} dx^i \otimes dx^j$ be a $(0,2)$–tensor field on M. Then G induces a symmetric tensor field $S = S_{ij} dx^i \otimes dx^j$ and an antisymmetric tensor field $R = R_{ij} dx^i \otimes dx^j$ by $S_{ij} = \frac{1}{2}(g_{ij} + g_{ji})$, $R_{ij} = \frac{1}{2}(g_{ij} - g_{ji})$. Denote further

$$G' = g_{ji} dx^i \otimes dx^j .$$

Clearly, $G = S + R$, $G' = S - R$. Now we prove the main result of this paper.

Theorem. *For $m \geqq 3$, all first order natural operators $T^* \otimes T^* \rightsquigarrow (T^* \otimes T^*)T$ transforming $(0,2)$–tensor fields on M into $(0,2)$–tensor fields on TM are of the form*

$$G \mapsto K_1 (G')^C + K_2 G^C + K_3 (G')^V + K_4 G^V + K_5 (G')^A + K_6 G^A \tag{7}$$

where $K_i = K_i(g_{ij} y^i y^j)$ are arbitrary smooth functions of the invariant I_1 and G^C, G^V and G^A denote the canonical liftings.

Proof. Taking into account an identification of every $(0,2)$–tensor field $\overline{G}$ on TM with a linear map $\overline{G}_L \colon TTM \to T^*TM$ over the identity of TM, it suffices to choose suitable morphisms (6) from Proposition (6). Clearly, all such linear maps are of the form

$$
\begin{aligned}
&w^i = y^i\,,\\
&s_i = A_4 g_{ji} X^j + \overline{A}_4 g_{ij} X^j\,,\\
&r_i = A_4(g_{ji}Y^j + g_{ji,k}X^j y^k) + \overline{A}_4(g_{ij}Y^j + g_{ij,k}X^j y^k)\\
&\quad + B_1(g_{jk,i} - g_{ji,k})y^j X^k + B_2(g_{kj,i} - g_{ij,k})y^j X^k\\
&\quad - B_1 g_{ij,k}X^j y^k - B_2 g_{ji,k}X^j y^k\\
&\quad - B_1 g_{ji}Y^j - B_2 g_{ji}Y^j - B_1 g_{ij}Y^j - B_2 g_{ij}Y^j + A_6 g_{ji}X^j + \overline{A}_6 g_{ij}X^j\,.
\end{aligned}
\tag{8}
$$

This can be rewritten as

$$
\begin{aligned}
&w^i = y^i\,,\\
&s_i = (A_4 - B_2)g_{ji}X^j + (\overline{A}_4 - B_1)g_{ij}X^j + B_2 g_{ji}X^j + B_1 g_{ij}X^j\,,\\
&r_i = (A_4 - B_2)(g_{ji}Y^j + g_{ji,k}X^j y^k) + (\overline{A}_4 - B_1)(g_{ij}Y^j + g_{ij,k}X^j y^k)\\
&\quad + B_1(g_{jk,i} - g_{ji,k})y^j X^k + B_2(g_{kj,i} - g_{ij,k})y^j X^k\\
&\quad - B_1 g_{ji}Y^j - B_2 g_{ij}Y^j + A_6 g_{ji}X^j + \overline{A}_6 g_{ij}X^j
\end{aligned}
$$

which is nothing but the coordinate form of (7), where $K_1 = A_4 - B_2$, $K_2 = \overline{A}_4 - B_1$, $K_3 = A_6$, $K_4 = \overline{A}_6$, $K_5 = B_1$, $K_6 = B_2$. Finally, on the standard fibre $V = \otimes^2\mathbb{R}^{m*} \times \otimes^3\mathbb{R}^{m*} \times \mathbb{R}^m$ of $J^1(T^*\otimes T^*) \oplus T$ we have only one invariant $I_1 = g_{ij}y^i y^j$, so that the coefficients K_i are smooth functions of I_1 only (this also follows from the linearity of $\overline{G}_L$). $\square$

Using the symmetric tensor field S and antisymmetric tensor field R one can also express (7) in the form

$$G \mapsto K_1 S^C + K_2 R^C + K_3 S^V + K_4 R^V + K_5 S^A + K_6 R^A\,.$$

Corollary 1. *For $m \geqq 3$, all first order natural operators transforming symmetric or antisymmetric $(0,2)$–tensor fields on M into $(0,2)$–tensor fields on TM are of the form*

$$G \mapsto K_1 G^C + K_2 G^V + K_3 G^A$$

where $K_i = K_i(I_1)$ are arbitrary smooth functions of the invariant I_1.

Corollary 2. *For $m \geqq 3$, all first order natural $\mathbb{R}$–linear operators $T^*\otimes T^* \rightsquigarrow (T^*\otimes T^*)T$ are of the form (7), where K_i are arbitrary real numbers.*

Remark 2. Janyška, [3], has described some natural transformations $TTM \to T^*TM$ on a Riemannian manifold M. He has in fact constructed certain first order natural operators $\mathrm{Reg}S^2T^* \rightsquigarrow (TT, T^*T)$, where $\mathrm{Reg}S^2T^*$ denotes the bundle functor of Riemannian metrics. In Proposition 6 we have determined the analytical form of all first order natural operators $T^* \otimes T^* \rightsquigarrow (TT, T^*T)$, provided $\dim M \geqq 3$. Such operators were then essentially used in the proof of our classification theorem. By Kolář and Radziszewski [5] there is no natural equivalence $TTM \to T^*TM$. This is due to the essentially different character of natural transformations $TTM \to TTM$ and $T^*TM \to T^*TM$. On the other hand, from (6) we can see that a $(0,2)$–tensor field on M induces a 'wide' class of natural transformations $TTM \to T^*TM$. Now we give the geometrical construction of some morphisms $TTM \to T^*TM$ from Proposition 6.

1. The choice $A_1 = 0$, $A_2 = 1$, $A_3 = 0$, $\overline{A}_3 = 0$, $C_1 = 0$, $C_2 = 0$ in (6) gives (8). Then $\overline{A}_4$ or A_4 correspond to the complete lift of $G = g_{ij}dx^i \otimes dx^j$ or $G' = g_{ji}dx^i \otimes dx^j$, respectively. Analogously, A_6 and $\overline{A}_6$ correspond to the vertical lift and B_1 and B_2 correspond to the antisymmetric lift.

2. Each map $f\colon TM \to T^*M$ defines a function $\tilde{f}\colon TM \to \mathbb{R}$ given by $\tilde{f}(y) = \langle f(y), y\rangle$, so that a $(0,2)$– tensor field G on M determines a function $\widetilde{G}_L : TM \to \mathbb{R}$, $\widetilde{G}_L(y) = g_{ij}y^iy^j$. Its exterior differential $d\widetilde{G}_L$ is a 1–form on TM, in coordinates $d\widetilde{G}_L = g_{jk,i}y^jy^kdx^i + (g_{ij}y^j + g_{ji}y^j)dy^i$. Then the morphisms $d\widetilde{G}_L \circ p_{TM}$ and $d\widetilde{G}_L \circ Tp_M : TTM \to T^*TM$ correspond to the terms with C_1 and C_2 in (6).

3. All the morphisms $TTM \to T^*TM$ from (6) with $B_1 = B_2 = C_1 = C_2 = 0$ can be constructed as follows. Denote by $t_M : T^*TM \to T^*TM$ any natural transformation over the identity of TM determined by Kolář and Radziszewski in [5]. Further, let $h_M : TTM \to TTM$ be any natural transformation by Kolář [4]. Moreover, we denote by $s_M : TT^*M \to T^*TM$ the canonical isomorphism [8], [11]. Take the map $G_L\colon TM \to T^*M$ which canonically corresponds to a $(0,2)$–tensor field G on M and evaluate the composition $t_M \circ s_M \circ TG_L : TTM \to T^*TM$. Quite similarly, the tensor field G' induces a map $t_M \circ s_M \circ TG'_L : TTM \to T^*TM$. Next, for $Z \in TTM$ the sum of $(t_M \circ s_M \circ TG_L)(Z)$ and $(t_M \circ s_M \circ TG'_L)(Z)$ with respect to the vector bundle structure $T^*TM \to TM$ determines a map $f\colon TTM \to T^*TM$. Then $f \circ h_M$ is exactly (6) with $B_1 = B_2 = C_1 = C_2 = 0$. If G is symmetric or antisymmetric, then the whole construction is much easier. In fact, in this case it suffices to evaluate $s_M \circ TG_L \circ h_M$ (compare with $s_M \circ TG_L \circ \kappa_M$ in (1)).

Remark 3. The proof of our classification theorem was based on the identification of $(0,2)$–tensor fields with linear maps $TM \to T^*M$. A similar procedure can be used for liftings of $(1,1)$–tensor fields to TM which we identify with linear maps $TM \to TM$.

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