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CHARACTERIZING TOLERANCE TRIVIAL FINITE ALGEBRAS

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Abstract. An algebra A is tolerance trivial if $\operatorname{Tol} A = \operatorname{Con} A$ where $\operatorname{Tol} A$ is the lattice of all tolerances on A. If A contains a Mal'cev function compatible with each T $\operatorname{Tol} A$, then A is tolerance trivial. We investigate finite algebras satisfying also the converse statement.

Let R be a binary relation on a set A and f be an n-ary function on A. We say that f is *compatible with* R or that R is *compatible with* f if

$$\langle a_i, b_i \rangle \in R \quad \text{for} \quad i = 1, \dots, n \quad \text{imply } \langle f(a_1, \dots, a_n), f(b_1, \dots, b_n) \rangle \in R .$$

Let $\mathcal{A} = (A, F)$ be an algebra. A reflexive and symmetric binary relation T on A compatible with each $f \in F$ is called a *tolerance* on A. The set of all tolerances on A is denoted by $\operatorname{Tol} A$. It is well-known that $\operatorname{Tol} A$ is an algebraic lattice with respect to set inclusion. In general, the congruence lattice $\operatorname{Con} A$ is not a sublattice of $\operatorname{Tol} A$. If $\operatorname{Tol} \mathcal{A} = \operatorname{Con} A$, we say that an algebra A is *tolerance trivial.* A variety $\mathcal{V}$ is tolerance trivial if each $A \in \mathcal{V}$ has this property.

Tolerance trivial algebras were studied by numerous authors, see [7] for the first essential results and [2] for the almost complete survey. Recent results on tolerance trivial BCK-algebras were published by J. G. Raftery, I. G. Rosenberg and T. Sturm in [14].

If $a, b \in A$, denote by $T(a, b)$ the least tolerance on the algebra A containing the pair $\langle a, b \rangle$. Denote by $\vee$ the operation join in $\operatorname{Tol} A$; meet coincides with the set intersection. A function g, $g : A^n \to A$, is called an *n-ary algebraic function over* A if there exists an $(n + m)$-ary term t over A $(m \geq 0)$ and elements $a_1, \dots, a_m$ of A such that

$$g(x_1, \dots, x_n) = t(x_1, \dots, x_n, a_1, \dots, a_m) .$$

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Lemma 1 (see [2]). *Let c, d, $a_i, b_i \in A$ for $i = 1, \dots, n$. Then $\langle c, d\rangle \in \vee\{T(a_i, b_i) : i = 1, \dots, n\}$ if and only if there exists a $2n$-ary algebraic function g over A such that*

$$c = g(a_1, \dots a_n, b_1, \dots, b_n)\,, \quad d = g(b_1, \dots, b_n, a_1, \dots, a_n)\,.$$

Corollary (see [14]). *An algebra A is tolerance trivial if and only if for each $a, b, c \in A$ there exists a 4-ary algebraic function g over A such that*

$$a = g(c, c, a, b)\,, \quad b = g(a, b, c, c)\,.$$

Remark. This g is obtained from that G of Corollary 1.3 in [14] by the permutation of variables:

$$g(x_1, x_2, x_3, x_4) = G(x_3, x_2, x_1, x_4)\,,$$

in order to obtain a form similar to that in the following:

Definition. A ternary function f on the set A is called a *Mal'cev function* if it satisfies

$$f(a, b, b) = a\,, \quad f(a, a, c) = c$$

for each a, b, c of A: f is called a *Pixley function* if it is a Mal'cev function and, moreover,

$$f(a, b, a) = a\,.$$

Definition. An algebra A is *congruence-permutable* if $\Theta \circ \Phi = \Phi \circ \Theta$ for each two Θ, $\Phi \in \operatorname{Con} A$; A is *arithmetical* if t is congruence-permutable and $\operatorname{Con} A$ is distributive. A variety $\mathcal{V}$ is *congruence-permutable* (or *arithmetical*) if each $A \in \mathcal{V}$ has this property.

Recall three well known results:

Lemma 2 (Mal'cev [11]). *A variety $\mathcal{V}$ is congruence-permutable if and only if there exists a ternary term t in $\mathcal{V}$ which is a Mal'cev function.*

Lemma 3 (Pixley [13]). *A variety $\mathcal{V}$ is arithmetical if and only if there exists a ternary term t in $\mathcal{V}$ which is a Pixley function.*

Lemma 4 ([2] or [17]). *A variety $\mathcal{V}$ is tolerance trivial if and only if $\mathcal{V}$ is congruence-permutable.*

A. F. Pixley in [12] gave the answer to the question whether the statement of Lemma 3 remains true if a single algebra instead of a variety is considered:

Proposition. *A finite algebra A is arithmetical if and only if there exists a ternary Pixley function compatible with every $\Theta \in \operatorname{Con} A$.*

With respect to Lemma 2 and Lemma 4, it rises a question if also finite tolerance trivial algebras can be characterized by Mal'cev functions compatible with tolerances. The aim of this paper is to give a partial answer to this question.

At first we note that H. -P. Gumm [10] proved that for congruence-permutable algebras, the assertion similar to that of the Proposition for arithmetical algebras does not hold. He gave an example of finite (25 element) congruence-permutable algebra which has not a Mal'cev function compatible with every its congruence. On the contrary, G. Czédli with the author proved that for algebras of cardinality less or equal to 8, congruence permutability can be characterized by the existence of a Mal'cev function compatible with all congruences (for A with card$A \leq 4$, see also [5]).

Lemma 5. *Let A be an algebra. If there exists a Mal'cev function an A compatible with all tolerances of A, then A is tolerance trivial.*

Proof. Let p be a Mal'cev function on A compatible with each $T \in \mathrm{Tol}\, A$. Let $T \in \mathrm{Tol}\, A$ and $a, b, c \in A$. If $\langle a, b\rangle \in T$ and $\langle b, c\rangle \in T$ then also

$$\langle a, c\rangle = \langle p(a, b, b), p(b, b, c)\rangle \in T$$

proving $T \in \mathrm{Con}\, A$. □

Theorem 1. *Let A be a finite algebra with card $A \leq 8$. The following conditions are equivalent:*

(1) A is tolerance trivial;

(2) there exists a ternary Mal'cev function compatible with each $T \in \mathrm{Tol}\, A$.

Proof. (2) ⇒ (1) by Lemma 5. Prove (1) ⇒ (2). By Theorem 1.4 in [14], every tolerance trivial algebra is congruence-permutable. By the Theorem of [6], for A with card $A \leq 8$ there exists a Mal'cev function p compatible with each $\Theta \in \mathrm{Con}\, A$. Since A is tolerance trivial, p is compatible with every $T \in \mathrm{Tol}\, A$ proving (2). □

Remark. As it was mentioned above, every tolerance trivial algebra is congruence permutable. However, the converse statement does not hold in a general case, see e.g. [3] or [4] for some examples.

Remark. It is an open question if the assertion of Theorem 1 is true also for finite algebras with card $A > 8$. However, for a special type of finite algebras, we are able to give the complete answer:

Theorem 2. *Let A be a finite algebra such that* $\mathrm{Con}\, A$ *is distributive. Then A is tolerance trivial if and only if there exists Mal'cev function (properly Pixley function) compatible with each $T \in \mathrm{Tol}\, A$.*

Proof. Let A be a finite algebra with distributive $\mathrm{Con}\, A$. If A is tolerance trivial, then A is congruence permutable and hence arithmetical. By the Proposition, there exists a Pixley function (and hence Mal'cev function) compatible with every $\Theta \in \mathrm{Con}\, A = \mathrm{Tol}\, A$. The converse implication is done by Lemma 5. □

Corollary. *A finite lattice L is tolerance trivial if and only if there exists a Mal'cev function (Pixley function) compatible with each $T \in \mathrm{Tol}\, L$.*

Now, we can apply this approach to the implication algebras (see [1] and [8]). An *implication algebra* is a groupoid $(A; \circ)$ satisfying the axioms

$$(x \circ y) \circ x = x, \quad (x \circ y) \circ y = (y \circ x) \circ x, \quad x \circ (y \circ z) = y \circ (x \circ z).$$

Every implication algebra has nullary operation 1 defined by

$$x \circ x = 1$$

(for each $x \in A$). If $(A; \circ)$ is an implication algebra, then $(A, \leq)$ is a poset with the greatest element 1, where $\leq$ is defined as follows

$$a \leq b \quad \text{if and only if} \quad a \circ b = 1\,.$$

It was proved in [1] that $(A, \leq)$ is a $\vee$-semilattice which is union of Boolean algebras. Moreover,

$$a \vee b = (a \circ b) \circ b\,.$$

If there exists some lower bound p of $a, b \in A$, then there exists also the infimum

$$\inf(a, b) = a \wedge b$$

and

$$a \wedge b = [a \circ (b \circ p)] \circ p\,;$$

for some detailes see [1]. Henceforth, if A is an implication algebra which is downward directed, then A is a distributive lattice. If, moreover, A is finite, then A has the least element 0 and A is a Boolean algebra ($a \circ x$ is a complement of an element a in the interval $[x, 1]$, see [1]). Thus, we are ready to state

Theorem 3. *Let A be a finite implication algebra with at least two elements. The following conditions are equivalent:*

(i) *A is tolerance trivial;*

(ii) *there exists a Mal'cev function (Pixley function) which is a term over A (and hence compatible with each $T \in \operatorname{Tol} A$);*

(iii) *A is a Boolean algebra (with respect to the above derived operations).*

Proof. (iii) $\Rightarrow$ (ii): If A is a Boolean algebra (with derived operations), we can put

$$p(x, y, z) = (x \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y' \wedge z)\,.$$

Then p is a term over the implication algebra $(A; \circ)$ and it is a Pixley function (thus also a Mal'cev function).

(ii) $\Rightarrow$ (i) by Lemma 5.

It remains to prove (i) $\Rightarrow$ (iii). Suppose A is tolerance trivial and let A not be a Boolean algebra with respect to the derived operations. Since A is finite, there exist different minimal elements $x_1, \dots, x_n$ (with respect to the induced order $\leq$) such that every interval $[x_i, 1]$ is a Boolean algebra and

$$A = [x_1, 1] \cup \dots \cup [x_n, 1]\,;$$

for some details, see [1]. Introduce a binary relation T on A as follows:

$$T = [x_1, 1]^2 \cup \dots \cup [x_n, 1]^2\,.$$

It is trivial that T is reflective and symmetric. Suppose $\langle a,b\rangle \in T$ and $\langle c,d\rangle \in T$. Then $c,d \in [x_j,1]$ for some $j \in \{1,\dots,n\}$. By the axioms of implications algebra, we have

$$x\circ(y\circ x) = y\circ(x\circ x) = y\circ 1 = 1\,,$$

i.e. $x \le y\circ x$ in the induced order for any x,y of A. Thus

$$x_j \le c \le a\circ c \le 1 \qquad \text{and} \qquad x_j \le d \le b\circ d \le 1\,,$$

which give $a\circ c,\, b\circ d \in [x_j,1]$, i.e. $\langle a\circ c, b\circ d\rangle \in T$ proving $T \in \operatorname{Tol} A$. It is evident from definition of T that $\langle x_i,1\rangle \in T$, $\langle x_j,1\rangle \in T$ but $\langle x_i,x_j\rangle \notin T$ for $i \ne j$. By (i) we infer $n=1$, i.e. $A=[x_1,1]$ thus A is a Boolean algebra. $\square$

References

[1] Abbot, J. C., *Semi-boolean algebra*, Matem. Vestnik **4** (1967), 177–198.
[2] Chajda, I., *Algebraic Theory of Tolerance Relations*, Publ. Palacký University Olomouc 1991 (Czech Republic).
[3] Chajda, I., *Tolerances in permutable algebras*, Czech. Math. J. **38** (1988), 218–225.
[4] Chajda, I., *On the existence of non-trivial tolerances in permutable algebras*, Czech. Math. J. **40** (1990), 598–600.
[5] Chajda, I., *Every at most four element algebra has a Mal'cev theory for permutability*, Math. Slovaca **41** (1991), 35–39.
[6] Chajda, I., Czédli, G., *Maltsev functions on small algebras*, Studia Sci. Math. Hungarica (Budapest) **28** (1993), 339–348.
[7] Chajda, I., Niederle, J., Zelinka, B., *On the existence conditions for compatible tolerances*, Czech. Math. J. **26** (1976), 304–311.
[8] Chajda, I., Zelinka, B., *Tolerances and congruences in implication algebras*, Czech. Math. J. **38** (1988), 207–217.
[9] Cornish, W. H., *On Iséki's BCK-algebras*, Lectures Notes in Pure and Appl. Math. **74** (1982), 101–122, New York.
[10] Gumm, H.-P., *Is there a Mal'cev theory for single algebras?*, Algebra Univ. **8** (1978), 320–329.
[11] Mal'cev, A. I, *On the general theory of algebraic systems (Russian)*, Matem. Sbornik **35** (1954), 3–20.
[12] Pixley, A. F., *Completeness in arithmetical algebras*, Algebra Univ. **2** (1972), 177–192.
[13] Pixley, A. F., *Distributivity and permutability of congruence relations in equational classes of algebras*, Proc. Amer. Math. Soc. **14** (1963).
[14] Raftery, J. G., Rosenberg, I. G., Sturm, T., *Tolerance relations and BCK-algebras*, Math. Japon. **36** (1991).
[15] Sturm, T., *On commutative BCK-algebras*, Math. Japon. **27** (1982), 197–212.
[16] Tanaka, S., *On ∧-commutative algebras*, Math. Semin. Notes Kobe Univ. **3** (1975), 59–64.
[17] Werner, H., *A Mal'cev condition for admissible relations*, Algebra Univ. **3** (1973), 263.

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