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SYMMETRIES OF CONNECTIONS ON FIBERED MANIFOLDS

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Abstract. The (infinitesimal) symmetries of first and second-order partial differential equations represented by connections on fibered manifolds are studied within the framework of certain "strong horizontal" structures closely related to the equations in question. The classification and global description of the symmetries is presented by means of some natural compatible structures, e.g. by vertical prolongations of connections.

1. Introduction

In our recent paper [15], we have introduced a formalism which can be applied for a geometrical description of some indirrect integration methods concerning differential equations for integral sections of connections, i.e. Pfaffian systems for integral mappings of corresponding horizontal distributions. The results, motivated by the variational analysis over one-dimensional base [13], [14], suggest a generalization either of the "method of characterististics" from the theory of exterior differential systems or the "method of fields of extremals" (Hamilton-Jacobi method) from mechanics. This is done in terms of *characterizable connections* on $\pi_{1,0}$, their *characteristic* 2-*connections* on π and *fields of paths*, which are local connections on π. The crucial idea of reasoning on equations through the corresponding connections and their interrelations rests upon certain results on natural transformations [5] and it takes full advantage of the relationships between the associated horizontal distributions.

In this paper, we study in some sense complementary "strong horizontal" distributions which appear within the framework of considerations on the symmetries of such equations. It should be mentioned here that analogous structures appear in a particular case of $J^1\pi = \mathbb{R} \times TM$ e.g. in [3], [27], or in a slightly more general situation on $J^1\pi$ over an arbitrary one-dimensional base [26]. The importance of these structures either for equations themselves or for the calculus of forms along $\pi_{1,0}\colon \mathbb{R} \times TM \to \mathbb{R} \times M$ is discussed e.g. in [23].

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The notions and results are twofold, considered either "locally" for a single (mostly integrable) connection Γ on π or "globally" for a characterizable connection Ξ on $\pi_{1,0}$. It turns out that the complement to the Cartan distribution $C^{\Gamma}_{\pi_{1,0}}$ of a connection Γ on π in the decomposition of the tangent bundle $T\Gamma(Y)$, expressed in terms of the *vertical prolongation* $\mathcal{V}\Gamma$ of Γ, has a global counterpart in the so-called *reduced connection* associated to any characterizable connection on $\pi_{1,0}$. It should be mentioned that some of these ideas are partially motivated by [2], [28], [29]. The importance of the decompositions will become apparent in the last section, where the infinitesimal symmetries of connections are studied. We hope that the presented classification creates a contribution to the transparent description of the situation. In the last part of this section, we reap the benefit of the relations between 2-connections and their fields of paths to the description of the relations between their symmetries by means of the corresponding vertical prolongations.

Our formalism and notation of the fibered manifolds theory, jet prolongations and connections is mostly in accordance with [16], [17], [24]. For jet prolongations of vector fields and groups of transformations see e.g. [9], [10], [11], [19], [24]. The motivations and results from the theory of prolongations of connections and related topics are due to [7], [8], [18]. For the theory of natural operations and invariants we refer to [9], [12]. Further theories we are dealing with are the geometry of vector distributions and differential equations and their symmetries [1], [25] and mainly [6], [19], [24]. An interesting approach to these problems can be found in [2], [28], [29].

2. Connections and equations

Let $\pi\colon Y \to X$ be a fibered manifold with local fibered coordinates (x^i, y^σ), $i = 1, \ldots, n = \dim X$, $\sigma = 1, \ldots, m = \dim Y - \dim X$. The first jet prolongation of π as the set of all 1–jets $j^1_x\gamma$ $(x \in X)$ of local sections $\gamma\colon X \supset U \to Y$ of π (the set of such sections we denote by $S_U(\pi)$ or generally $S_{\mathrm{loc}}(\pi)$) is denoted by $J^1\pi$. The induced coordinates on $J^1\pi$ are denoted by $(x^i, y^\sigma, y^\sigma_i)$. The manifold $J^1\pi$ will be viewed as the total space both of the fibered manifold $\pi_1\colon J^1\pi \to X$ and of the affine bundle $\pi_{1,0}\colon J^1\pi \to Y$.

A *connection* on π is a section (generally defined on some open subset of Y) $\Gamma\colon Y \to J^1\pi$ of $\pi_{1,0}$. In local fibered coordinates, the equations of Γ are $y^\sigma_i \circ \Gamma = \Gamma^\sigma_i(x^i, y^\lambda)$, where Γ^σ_i are the *components* of Γ. The *horizontal form* h_Γ of Γ is $h_\Gamma = D_{\Gamma i} \otimes dx^i$, where $D_{\Gamma i} = \partial/\partial x^i + \Gamma^\sigma_i \partial/\partial y^\sigma$ is the i-th (absolute) derivative with respect to Γ. The complementary projection to h_Γ is the *vertical form* $v_\Gamma = I - h_\Gamma$ and the connection Γ is then identified with a canonical decomposition $TY = V_\pi Y \oplus H_\Gamma$, where the n-dimensional π–horizontal distribution $H_\Gamma = \mathrm{Im}\ h_\Gamma$ is locally generated by the vector fields $D_{\Gamma i}$ for $i = 1, \ldots, n$ or equivalently by the forms $dy^\sigma - \Gamma^\sigma_j\, dx^j$ for $\sigma = 1, \ldots, m$.

By a *first-order differential equation* on π (more precisely, a (nonlinear) system of partial differential equations (PDE)) we mean any (embedded) submanifold $\mathcal{E}$ of $J^1\pi$. A *solution* of $\mathcal{E}$ is a section $\gamma \in S_{\mathrm{loc}}(\pi)$ such that $j^1\gamma \subset \mathcal{E}$. An intrinsic object related to $\mathcal{E}$ is the *Cartan distribution* $C^{\mathcal{E}}_{\pi_{1,0}}$ *of* $\mathcal{E}$ defined by $C^{\mathcal{E}}_{\pi_{1,0}} = C_{\pi_{1,0}} \cap T\mathcal{E}$, where $C_{\pi_{1,0}}$ is the canonical *Cartan distribution* on $J^1\pi$.

Evidently, the $(n+m)$-dimensional submanifold $\Gamma(Y) \subset J^1\pi$ can be viewed as a global counterpart of a first-order differential equation "explicitly solved with respect to derivatives". Thus a section $\gamma \in S_{\mathrm{loc}}(\pi)$ is called a *path* or an *integral section* of the connection Γ if $j^1\gamma = \Gamma \circ \gamma$, which locally means the equations for γ on its domain

$$\frac{\partial \gamma^\sigma}{\partial x^i} = \Gamma^\sigma_i(x^j, \gamma^\lambda) \ .$$

Recall that γ is an integral section of Γ if and only if the image of γ is an integral manifold of the horizontal distribution H_Γ. Accordingly, the Frobenius integrability conditions for Γ, locally expressed by

$$\frac{\partial \Gamma^\sigma_i}{\partial x^j} + \frac{\partial \Gamma^\sigma_i}{\partial y^\lambda}\,\Gamma^\lambda_j = \frac{\partial \Gamma^\sigma_j}{\partial x^i} + \frac{\partial \Gamma^\sigma_j}{\partial y^\lambda}\,\Gamma^\lambda_i \ ,$$

means equivalently that there exists a unique maximal integral section of Γ passing through each point $y \in Y$ or that the distribution H_Γ is involutive or finally that for the curvature R_Γ of the connection Γ it holds $R_\Gamma = 0$.

The second (holonomic) prolongation of π is the set $J^2\pi$ of 2–jets of local sections of π with local coordinates $(x^i, y^\sigma, y^\sigma_i, y^\sigma_{ij})$.

The first jet prolongation $J^1\pi_{1,0}$ of the bundle $\pi_{1,0}\colon J^1\pi \to Y$ is the set of 1-jets of local connections on π with the induced coordinates $(x^i, y^\sigma, y^\sigma_i, z^\sigma_{ij}, z^\sigma_{i\lambda})$.

A *2-connection* on π is a section $\Gamma^{(2)}\colon J^1\pi \to J^2\pi$ of $\pi_{2,1}$. In coordinates, $y^\sigma_{ij} \circ \Gamma^{(2)} = \Gamma^\sigma_{ij}$, where $\Gamma^\sigma_{ij} = \Gamma^\sigma_{ji}$ are the components of $\Gamma^{(2)}$. The horizontal form of $\Gamma^{(2)}$ is locally expressed by $h_{\Gamma^{(2)}} = D_{\Gamma^{(2)}i} \otimes dx^i$, where $D_{\Gamma^{(2)}i} = \partial/\partial x^i + y^\sigma_i \partial/\partial y^\sigma + \Gamma^\sigma_{ij}\partial/\partial y^\sigma_j$ is the i-th absolute derivative with respect to $\Gamma^{(2)}$. The canonical decomposition generated by $h_{\Gamma^{(2)}}$ is $TJ^1\pi = V_{\pi_1}J^1\pi \oplus H_{\Gamma^{(2)}}$, where the n-dimensional π_1–horizontal distribution $H_{\Gamma^{(2)}} = \mathrm{Im}\ h_{\Gamma^{(2)}}$ is locally generated by the vector fields $D_{\Gamma^{(2)}i}$ or by the forms $dy^\sigma = y^\sigma_j\, dx^j$, $dy^\sigma_i = \Gamma^\sigma_{ij} dx^j$.

Since $H_{\Gamma^{(2)}} \subset C_{\pi_{1,0}}$, a *path* or an *integral section* of the connection $\Gamma^{(2)}$ is defined to be a section $\gamma \in S_{\mathrm{loc}}(\pi)$ such that $j^2\gamma = \Gamma^{(2)} \circ j^1\gamma$. Consequently, 2-connections on π are geometric counterparts of systems of second-order differential equations solved with respect to highest derivatives of the form

$$\frac{\partial^2 \gamma^\sigma}{\partial x^i \partial x^j} = \Gamma^\sigma_{ij}\left(x^k, \gamma^\lambda, \frac{\partial \gamma^\lambda}{\partial x^k}\right) \ .$$

Recall that γ is an integral section of $\Gamma^{(2)}$ if and only if the image of $j^1\gamma$ is an integral manifold of the horizontal distribution $H_{\Gamma^{(2)}}$.

Finally, a connection on $\pi_{1,0}$ is a section $\Xi\colon J^1\pi \to J^1\pi_{1,0}$ of $(\pi_{1,0})_{1,0}$. The horizontal form of Ξ is locally expressed by $h_\Xi = D_{\Xi j} \otimes dx^j + D_{\Xi\lambda} \otimes dy^\lambda$, where $D_{\Xi j} = \partial/\partial x^j + \Xi^\sigma_{ij}\partial/\partial y^\sigma_i$ and $D_{\Xi\lambda} = \partial/\partial y^\lambda + \Xi^\sigma_{i\lambda}\partial/\partial y^\sigma_i$ for $j = 1, \ldots, n$ and $\lambda = 1, \ldots, m$, are the j-th or the λ-th absolute derivative with respect to Ξ, respectively. The canonical decomposition generated by h_Ξ is $TJ^1\pi = V_{\pi_{1,0}}J^1\pi \oplus H_\Xi$, where the $(n+m)$-dimensional $\pi_{1,0}$–horizontal distribution $H_\Xi = \mathrm{Im}\, h_\Xi$ is locally generated by the vector fields $D_{\Xi j}$ and $D_{\Xi\lambda}$ or equivalently by the forms $dy_i^\sigma = \Xi_{ij}^\sigma\, dx^j + \Xi_{i\lambda}^\sigma\, dy^\lambda$.

Evidently, the integral sections of connections on $\pi_{1,0}$ are just local connections on π. Such a connection $\Gamma \in S_{\mathrm{loc}}(\pi_{1,0})$ is an integral section of Ξ if and only if $j^1\Gamma = \Xi \circ \Gamma$, which in fibered coordinates means a system of $(n+m)$ first order PDE of the form

$$\frac{\partial \Gamma_i^\sigma}{\partial x^j} = \Xi_{ij}^\sigma(x^k, y^\nu, \Gamma_k^\nu), \qquad \frac{\partial \Gamma_i^\sigma}{\partial y^\lambda} = \Xi_{i\lambda}^\sigma(x^k, y^\nu, \Gamma_k^\nu)$$

for the components of Γ, or equivalently if and only if the image of Γ is an integral manifold of the horizontal distribution H_Ξ.

We refer to [15] for the more detailed discussion of the following notions and results.

It was shown in [5] that there is a unique 2-connection $\Gamma^{(2)}$ on π naturally assigned to any connection Ξ on $\pi_{1,0}$. A connection Ξ on $\pi_{1,0}$ is then called *characterizable* if $H_{\Gamma^{(2)}} \subset H_\Xi$ or equivalently $H_{\Gamma^{(2)}} = H_\Xi \cap C_{\pi_{1,0}}$ for the above 2-connection $\Gamma^{(2)}$, called now the *characteristic connection* of Ξ. Since the local conditions for Ξ to be characterizable are $\Xi_{ij}^\sigma - \Xi_{ji}^\sigma + \Xi_{i\lambda}^\sigma y_j^\lambda - \Xi_{j\lambda}^\sigma y_i^\lambda = 0$, the components of its characteristic connection are $\Gamma_{ij}^\sigma = \Xi_{ij}^\sigma + \Xi_{i\lambda}^\sigma y_j^\lambda$. The integral manifolds of $H_{\Gamma^{(2)}}$ of maximal dimension, which are just first jet prolongations of integral sections of the characteristic connection, are called *characteristics* of the connection Ξ and the most important fact for the theory of equations under consideration is that the integral sections of Ξ are foliated by the characteristics.

A connection $\Gamma\colon Y \supset V \to J^1\pi$ on π is called a *field of paths* of a 2-connection $\Gamma^{(2)}$ on π if on V

$$J^1(\Gamma, \mathrm{id}_X) \circ \Gamma = \Gamma^{(2)} \circ \Gamma\ , \tag{2.1}$$

where $J^1(\Gamma, \mathrm{id}_X)$ is the first jet prolongation of the fibered morphism Γ over X. An arbitrary field of paths of $\Gamma^{(2)}$ is integrable and a connection $\Gamma\colon Y \supset V \to J^1\pi$ is a field of paths of $\Gamma^{(2)}$ if and only if the submanifold $\Gamma(V) \subset J^1\pi$ is foliated by first jet prolongations of integral sections of $\Gamma^{(2)}$, i.e. if $H_{\Gamma^{(2)}}|_{\Gamma(V)} \equiv C_{\pi_{1,0}}^\Gamma$. In fiber coordinates, (2.1) reads $\Gamma_{ij}^\sigma \circ \Gamma = D_{\Gamma j}(\Gamma_i^\sigma)$. Since $\Gamma^{(2)} \circ j^1\gamma = \Gamma^{(2)} \circ \Gamma \circ \gamma = J^1(\Gamma, \mathrm{id}_X) \circ \Gamma \circ \gamma = j^1(\Gamma \circ \gamma) = j^1(j^1\gamma) = j^2\gamma$, if Γ is a field of paths of $\Gamma^{(2)}$ and γ is an integral section of Γ then γ is an integral section (a path) of $\Gamma^{(2)}$. Thus if $\Gamma\colon Y \supset V \to J^1\pi$ is a field of paths of $\Gamma^{(2)}$ then H_Γ defines a foliation of V such that each leaf of this foliation is the image of an integral section of $\Gamma^{(2)}$.

By the above results, the problem of finding all the integral sections of a given integrable 2-connection is equivalent to the problem of solving all fields of paths of $\Gamma^{(2)}$. This is not much surprising within the context of the theory of *prolongations of differential equations*. Clearly, the submanifold $J^1(\Gamma, \mathrm{id}_X) \circ \Gamma(Y) \subset J^2\pi$ is just the first prolongation of the equation $\Gamma(Y) \subset J^1\pi$. Conversely, each field of paths Γ of $\Gamma^{(2)}$ represents a (local) reduction of the order of the equation $\Gamma^{(2)}(J^1\pi) \subset J^2\pi$.

It is easy to prove that if Ξ is a characterizable connection on $\pi_{1,0}$ and $\Gamma^{(2)}$ its characteristic 2-connection on π then each integral section Γ of Ξ is a field of paths of $\Gamma^{(2)}$. Consequently, if Ξ is integrable then $\Gamma^{(2)}$ is integrable and each integral section of $\Gamma^{(2)}$ is locally imbedded in a field of paths Γ, which is an integral section of Ξ.

3. Strong horizontal distributions

First recall the structure of vertical bundles on Y and $J^1\pi$. The vertical subbundle $V_\pi Y$ to π of TY, representing the family of vectors tangent to the fibers of π, can be viewed as the total space of the fibered manifold $\rho\colon V_\pi Y \to X$, $\rho = \pi \circ \tau_Y$. Consequently, there is an identification of the tangent space TY_x of Y_x with the fiber $\rho^{-1}(x)$ of ρ. There are two canonical vertical subbundles of $TJ^1\pi$, namely $V_{\pi_1}J^1\pi$ or $V_{\pi_{1,0}}J^1\pi$ of π_1 or $\pi_{1,0}$–vertical vectors on $J^1\pi$, respectively. To relate them with $V_\pi Y$, we have to recall that there is a canonical isomorphism between $J^1\rho$ and $V_{\pi_1}J^1\pi$ over $\pi^*_{1,0}(V_\pi Y)$. The isomorphism is represented locally by a rearrangement; if the induced coordinates on $J^1\rho$ are $(x^i, y^\sigma, \dot{y}^\sigma, y^\sigma_i, \dot{y}^\sigma_i)$ then those on $V_{\pi_1}J^1\pi$ are $(x^i, y^\sigma, y^\sigma_i, \dot{y}^\sigma, \dot{y}^\sigma_i)$. Thus there is a natural identification of the tangent space $T(J^1\pi)_x$ of a fiber $(J^1\pi)_x = \pi_1^{-1}(x)$ with $\rho_1^{-1}(x)$ and $T(J^1\pi)_y \cong \rho^{-1}_{1,0}(y)$.

Using a canonical vertical functor V for a connection $\Gamma\colon Y \to J^1\pi$ on π one obtains a mapping $V\Gamma\colon V_\pi Y \to V_{\pi_1}J^1\pi$, representing a π_1-vertical lift of π-vertical vectors by

$$\zeta^\sigma \frac{\partial}{\partial y^\sigma}\Big|_y \xmapsto{V\Gamma} \zeta^\sigma \frac{\partial}{\partial y^\sigma}\Big|_{\Gamma(y)} + \frac{\partial \Gamma^\sigma_i}{\partial y^\lambda}\zeta^\lambda \frac{\partial}{\partial y^\sigma_i}\Big|_{\Gamma(y)} . \tag{3.1}$$

Using the above identification $V_{\pi_1}J^1\pi \cong J^1\rho$ we get the so-called *vertical prolongation* $\mathcal{V}\Gamma$ of Γ, realizing the only connection on ρ naturally determined by the given connection Γ on π [7]. This vertical prolongation finds wide application for example in the theory of prolongations of connections on π to connections on π_1 (see [7], [8], [9], [18] and references therein). In what follows we would like to show that it also provides a natural description of the geometry related to the equation $\Gamma(Y) \subset J^1\pi$ in terms of its symmetries. It is to be remarked that this approach is not quite so original (cf. [28]).

First we will be interested in a decomposition of the tangent bundle $T\Gamma(Y)$, expressing the *internal* geometry of the corresponding equation $\Gamma(Y) \subset J^1\pi$. Since we will have frequent occasion for expressing the results with reference to jet prolongations of vector fields, we recall that ζ being a vector field on Y, locally expressed by $\zeta = \zeta^i\partial/\partial x^i + \zeta^\sigma\partial/\partial y^\sigma$, its first jet prolongation $\mathcal{J}^1\zeta$ is

$$\mathcal{J}^1\zeta = \zeta^i\frac{\partial}{\partial x^i} + \zeta^\sigma\frac{\partial}{\partial y^\sigma} + \left(D_i(\zeta^\sigma) - D_i(\zeta^j)y^\sigma_j\right)\frac{\partial}{\partial y^\sigma_i} . \tag{3.2}$$

For a π-projectable vector field ζ on Y, the flow of $\mathcal{J}^1\zeta$ is the jet prolongation of that of ζ, which consists of fibered isomorphisms of π.

Since $\mathcal{J}^1 D_{\Gamma i} \circ \Gamma = T\Gamma \circ D_{\Gamma i}$ if and only if Γ is integrable, the following assertion can be easily proved.

Lemma 3.1. *Let Γ be an integrable connection on π. Then for any Γ-horizontal vector field ζ on Y holds*

$$\mathcal{J}^1\zeta \circ \Gamma = T\Gamma \circ \zeta \in C^{\Gamma}_{\pi_{1,0}} . \tag{3.3}$$

It is trivial to verify that the tangent bundle $T\Gamma(Y)$ splits into the direct sum $T\Gamma(Y) = C^{\Gamma}_{\pi_{1,0}} \oplus \mathcal{V}\Gamma(V_\pi Y)$ accordingly to the decomposition $TY = H_\Gamma \oplus V_\pi Y$ corresponding to Γ.

Considering $T\Gamma(Y)$ in obvious sense as an subbundle of $TJ^1\pi$ we find the assertion concerning a natural decomposition associated to any integrable connection on π. This proposition is to be considered as an inner version of Proposition 3.3.

Proposition 3.1. *Let Γ be an integrable connection on π and ζ a vector field on Y. Then*

$$\mathcal{J}^1\zeta \circ \Gamma = \mathcal{J}^1(h_\Gamma \circ \zeta) \circ \Gamma + \mathcal{V}\Gamma \circ v_\Gamma \circ \zeta + (\mathcal{J}^1\zeta \circ \Gamma)^{\pi_{1,0}} ,$$

where

$$\begin{aligned} \mathcal{J}^1(h_\Gamma \circ \zeta) \circ \Gamma &\in C^{\Gamma}_{\pi_{1,0}} \\ \mathcal{V}\Gamma \circ v_\Gamma \circ \zeta &\in \mathcal{V}\Gamma(V_\pi Y) \\ (\mathcal{J}^1\zeta \circ \Gamma)^{\pi_{1,0}} &\in V_{\pi_{1,0}} J^1\pi . \end{aligned}$$

Moreover,

$$\begin{aligned} \mathcal{V}\Gamma \circ v_\Gamma \circ \zeta + (\mathcal{J}^1\zeta \circ \Gamma)^{\pi_{1,0}} &= \mathcal{J}^1(v_\Gamma \circ \zeta) \circ \Gamma \\ \mathcal{J}^1 D_{\Gamma i} \circ \Gamma &= D_i \circ J^1(\Gamma, \mathrm{id}_X) \circ \Gamma . \end{aligned}$$

The whole situation will become apparent in terms of decompositions of characterizable connections on $\pi_{1,0}$. Recall that a non-vanishing (1,1) tensor field F of constant rank on a manifold M is called an *f(3,-1) structure* on M if $F^3 - F = 0$, which yields a canonical direct sum decomposition of TM induced by any such f(3,-1) structure. The eigenspaces corresponding to the eigenvalues 0, -1, $+1$ are $\mathrm{Im}(F^2 - I)$, $\mathrm{Im}(F^2 - F)$, $\mathrm{Im}(F^2 + F)$, respectively.

Proposition 3.2. *Let Ξ be a characterizable connection on $\pi_{1,0}$ and $\Gamma^{(2)}$ its characteristic 2-connection on π. Then $F_\Xi = 2h_\Xi - h_{\Gamma^{(2)}} - I$ is an f(3,-1) structure on $J^1\pi$ of rank $m(n+1)$.*

Proof. Since $h_\Xi \circ h_{\Gamma^{(2)}} = h_{\Gamma^{(2)}} \circ h_\Xi = h_{\Gamma^{(2)}}$, it is easy to see that $F_\Xi^2 = v_{\Gamma^{(2)}}$, which immediately gives $F_\Xi^3 - F_\Xi = 0$. The rank of F_Ξ is evident from the local expression

$$F_\Xi = \left(F^\sigma_{ij} \frac{\partial}{\partial y^\sigma_i} - y^\sigma_j \frac{\partial}{\partial y^\sigma} \right) \otimes dx^j + \\ + \frac{\partial}{\partial y^\sigma} \otimes dy^\sigma - \frac{\partial}{\partial y^\sigma_i} \otimes dy^\sigma_i + F^\sigma_{i\lambda} \frac{\partial}{\partial y^\sigma_i} \otimes dy^\lambda ,$$

where the functions $F^\sigma_{ij} = \Xi^\sigma_{ij} - \Xi^\sigma_{i\lambda} y^\lambda_j$ and $F^\sigma_{i\lambda} = 2\,\Xi^\sigma_{i\lambda}$ are the components of F_Ξ. □

Since $F_\Xi^2 - I = -h_{\Gamma^{(2)}}$, $F_\Xi^2 + F_\Xi = 2(h_\Xi - h_{\Gamma^{(2)}})$ and $F_\Xi^2 - F_\Xi = 2\,v_\Xi$, the assertion follows.

Proposition 3.3. *Let Ξ be a characterizable connection on $\pi_{1,0}$ and $\Gamma^{(2)}$ its characteristic 2-connection on π. Then there is a canonically determined direct sum decomposition*

$$TJ^1\pi = V_{\pi_{1,0}}J^1\pi \oplus H_{\Gamma^{(2)}} \oplus H_{F_\Xi} \ , \tag{3.4}$$

given for any $\zeta^{(1)} \in TJ^1\pi$ by

$$\zeta^{(1)} = v_\Xi(\zeta^{(1)}) + h_{\Gamma^{(2)}}(\zeta^{(1)}) + (h_\Xi - h_{\Gamma^{(2)}})(\zeta^{(1)}) \ ,$$

where $H_{\Gamma^{(2)}} \oplus H_{F_\Xi} = H_\Xi$.

The m-dimensional $H_{F_\Xi} = \mathrm{Im}(h_\Xi - h_{\Gamma^{(2)}})$ will be called a *strong horizontal distribution.* By the strong horizontality of H_Ξ we mean the decomposition

$$V_{\pi_1}J^1\pi = V_{\pi_{1,0}}J^1\pi \oplus H_{F_\Xi} \ .$$

Using coordinates, for $\zeta^{(1)} = \zeta^i\partial/\partial x^i + \zeta^\sigma\partial/\partial y^\sigma + \zeta^\sigma_j\partial/\partial y^\sigma_j$ the decomposition (3.4) means

$$\begin{aligned}
v_\Xi(\zeta^{(1)}) &= \left(\zeta^\sigma_j - \Xi^\sigma_{ji}\zeta^i - \Xi^\sigma_{j\lambda}\zeta^\lambda\right)\frac{\partial}{\partial y^\sigma_j} \ ,\\
h_{\Gamma^{(2)}}(\zeta^{(1)}) &= \zeta^i\left(\frac{\partial}{\partial x^i} + y^\sigma_i\frac{\partial}{\partial y^\sigma} + (\Xi^\sigma_{ij} + \Xi^\sigma_{i\lambda}y^\lambda_j)\frac{\partial}{\partial y^\sigma_j}\right) \ ,\\
(h_\Xi - h_{\Gamma^{(2)}})(\zeta^{(1)}) &= (\zeta^\lambda - \zeta^i y^\lambda_i)\left(\frac{\partial}{\partial y^\lambda} + \Xi^\sigma_{j\lambda}\frac{\partial}{\partial y^\sigma_j}\right) \ .
\end{aligned}$$

It appears that there is further interesting object closely related with the results just referred to (cf. [2], [28], [29]).

Definition 3.1. A *reduced connection of type (1,0)* on π is a section

$$\Gamma_{(1,0)} \colon \pi^*_{1,0}(V_\pi Y) \to V_{\pi_1}J^1\pi$$

which is linear in $\dot y^\sigma$, i.e. whose local expression is $\dot y^\sigma_i \circ \Gamma_{(1,0)} = \Gamma^\sigma_{i\lambda}(x^j, y^\sigma, y^\sigma_i)\,\dot y^\lambda$.

In other words, $\Gamma_{(1,0)}$ represents a lift of vector fields expressed by

$$\left(j^1_x\gamma, \zeta^\sigma\frac{\partial}{\partial y^\sigma}\,|_{\gamma(x)}\right) \xrightarrow{\Gamma_{(1,0)}} \zeta^\sigma\frac{\partial}{\partial y^\sigma}\,|_{j^1_x\gamma} + \Gamma^\sigma_{i\lambda}\,\zeta^\lambda\frac{\partial}{\partial y^\sigma_i}\,|_{j^1_x\gamma} \ ,$$

and thus it generates a decomposition

$$V_{\pi_1}J^1\pi = V_{\pi_{1,0}}J^1\pi \oplus H_{\Gamma_{(1,0)}} \ ,$$

with $H_{\Gamma_{(1,0)}} = \mathrm{Im}\,\Gamma_{(1,0)}$ generated by the vector fields $\partial/\partial y^\lambda + \Gamma^\sigma_{i\lambda}\partial/\partial y^\sigma_i$ for $\lambda = 1, \ldots, m$.

Definition 3.2. A π-vertical vector field ζ on Y will be called an *integral section* of a reduced connection $\Gamma_{(1,0)}$ on π if for any $j^1_x\gamma$ holds

$$\mathcal{J}^1\zeta(j^1_x\gamma) = \Gamma_{(1,0)}\left(j^1_x\gamma, \zeta(\gamma(x))\right) \ . \tag{3.5}$$

Using coordinates, (3.5) means

$$\frac{\partial\zeta^\sigma}{\partial x^i} + \frac{\partial\zeta^\sigma}{\partial y^\lambda}\, y^\lambda_i = \Gamma^\sigma_{i\lambda}\,\zeta^\lambda \ . \tag{3.6}$$

Remark that if we are given $\Gamma_{(1,0)}$ then obviously each connection Γ on π generates a connection $\Gamma_{(1,0)}\circ\Gamma$ on $\rho\colon V_\pi Y \to X$ by the composition

$$V_\pi Y \xrightarrow{\tau_Y\times\mathrm{id}} Y\times_Y V_\pi Y \xrightarrow{\Gamma\times\mathrm{id}} J^1\pi\times_Y V_\pi Y \xrightarrow{\Gamma_{(1,0)}} V_{\pi_1}J^1\pi \ .$$

Locally

$$\zeta^\sigma\frac{\partial}{\partial y^\sigma}\,|_y \xrightarrow{\Gamma_{(1,0)}\circ\Gamma} \zeta^\sigma\frac{\partial}{\partial y^\sigma}\,|_{\Gamma(y)} + \Gamma^\sigma_{i\lambda}\,\zeta^\lambda\frac{\partial}{\partial y^\sigma_i}\,|_{\Gamma(y)} \ . \tag{3.7}$$

Considering $h_\Xi\colon \pi^*_{1,0}(TY)\to TJ^1\pi$ as the horizontal lift with respect to a given characterizable connection Ξ on $\pi_{1,0}$, Prop. 3.3 can be reformulated.

Theorem 3.1. *Each characterizable connection on $\pi_{1,0}$ splits into the direct sum of a 2-connection on π and a reduced connection of type $(1,0)$ on π. The decomposition is given by*

$$H_\Xi = H_{\Gamma^{(2)}} \oplus H_{\Gamma_{(1,0)}} \ ,$$

where $\Gamma^{(2)}$ is the characteristic connection of Ξ and $\Gamma_{(1,0)} = h_\Xi|_{\pi^*_{1,0}(V_\pi Y)}$, i.e. locally

$$\Gamma^\sigma_{i\lambda} = \Xi^\sigma_{i\lambda} \ . \tag{3.8}$$

The last assertion of this section imply the meaning of reduced connections (or in other words of the corresponding strong horizontal subbundles) in the theory of symmetries. By (3.1), (3.7) and (3.8):

Proposition 3.4. *Let Ξ be a characterizable connection on $\pi_{1,0}$, $\Gamma_{(1,0)}$ the associated reduced connection of type $(1,0)$ on π. Let $\Gamma\in S_V(\pi_{1,0})$ be an integral section of Ξ. Then $\Gamma_{(1,0)}\circ\Gamma \equiv \mathcal{V}\Gamma$ on $\tau_Y^{-1}(V)\subset V_\pi Y$.*

4. Symmetries

In this section we will discuss the symmetries of connections in keeping with previous considerations. It should be noticed that we follow the *infinitesimal* approach, so that symmetries will be vector fields as the generators of the groups of invariant transformations.

First recall the symmetries of distributions in accordance with [6].

Let P be a regular distribution on a smooth manifold M. A vector field ζ on M is called a *symmetry of the distribution* P if its flow consists of diffeomorphisms which preserve P, i.e. if $\{\alpha_t\}$ is the corresponding one-parametric group of diffeomorphisms then $T\alpha_t(P_x) = P_{\alpha_t(x)}$ for $x \in M$. The set $\mathrm{Sym}(P)$ of all symmetries of P is a Lie algebra with respect to Lie bracket of vector fields and

$$\zeta \in \mathrm{Sym}(P) \Longleftrightarrow [\zeta, \xi] \in P \qquad \text{for each } \xi \in P \tag{4.1}$$

or equivalently

$$\zeta \in \mathrm{Sym}(P) \Longleftrightarrow \mathcal{L}_\zeta \omega \in P^* \qquad \text{for each } \omega \in P^*, \tag{4.2}$$

where by P^* we denote the ideal of differentiable forms on M, annihilating P. Denoting by v_i the local generators of P, $i = 1, \dots, \dim P$, then (4.1) means the existence of functions $k_{ij} \in C^\infty(M)$ such that $[\zeta, v_i] = \sum_j k_{ij} v_j$ for each i. Using the definition of the Lie derivative we can see that it is equivalent to the condition $T\alpha_t(v_i) = \sum_j \lambda_{ij}(t) v_j$ for the flow $\{\alpha_t\}$ of ζ where $\lambda_{ij} \in C^\infty(M)$ are smoothly parameterized by t and

$$k_{ij} = -\frac{d}{dt}\bigg|_{t=0} \lambda_{ij}(t) \ .$$

The importance of symmetries of a completely integrable distribution rests upon the fact that the corresponding flows preserve the set of maximal integral manifolds of P.

Example 4.1. Let P be a one-dimensional distribution locally generated by a nowhere vanishing vector field ξ. Then (4.1) reads $\zeta \in \mathrm{Sym}(P) \Longleftrightarrow [\zeta, \xi] = k\xi$ for $k \in C^\infty(M)$. The notion of the maximal integral manifold of P coincides locally with that of the maximal integral curves of ξ.

Example 4.2. Considering the Cartan distribution $C_{\pi_{1,0}}$ on $J^1\pi$, the symmetries of $C_{\pi_{1,0}}$ are called *contact vector fields*. In this case (4.2) means that the Lie derivative of any contact form with respect to a contact vector field is again a contact form. It can be shown (e.g. [24]) that if a vector field $\zeta^{(1)}$ on $J^1\pi$ is projectable onto Y, then $\zeta^{(1)}$ is a contact vector field if and only if it is a prolongation of a vector field on Y. Moreover, if $m > 1$ then every contact vector field is projectable and thus it is a prolongation. Notice that the singularity of the case $m = 1$ is induced by the same dimension n of the fibers of $\pi_{1,0}$ and of the integral submanifolds $j^1\gamma(U)$ of $C_{\pi_{1,0}}$.

The set of symmetries of P lying within P is denoted $\mathrm{Char}(P)$. The flow of any such *characteristic symmetry* of P moves integral manifolds along itself. Since $\mathrm{Char}(P)$ is an ideal in $\mathrm{Sym}(P)$, the quotient algebra $\mathrm{Shuf}(P) = \mathrm{Sym}(P)/\,\mathrm{Char}(P)$ of the so-called *shuffling symmetries* of P can be constructed. Thus any $\zeta \in \mathrm{Shuf}(P)$ represents the whole class of symmetries whose flow rearrange the integral manifolds of P in the same way.

Secondly, the symmetries of equations can be studied. Let $\mathcal{E} \subset J^1\pi$ be a first order differential equation on π. They are two different ways for understanding the symmetries of $\mathcal{E}$, closely related to their internal and external geometry. The *external symmetry* of $\mathcal{E}$ is any contact vector field on $J^1\pi$ tangent to $\mathcal{E}$. The flow of such a symmetry preserve $\mathcal{E}$ and obviously the restriction of ζ to $\mathcal{E}$ defines a symmetry of $C^{\mathcal{E}}_{\pi_{1,0}}$. Just the symmetries of the distribution $C^{\mathcal{E}}_{\pi_{1,0}}$ (vector fields on $\mathcal{E}$) are called *internal symmetries* of $\mathcal{E}$.

Definition 4.1. Let Γ be a connection on π. A vector field ζ on Y is called a *symmetry* of Γ if and only if ζ and $\mathcal{J}^1\zeta$ are Γ-related, i.e.

$$\mathcal{J}^1\zeta \circ \Gamma = T\Gamma \circ \zeta \ . \tag{4.3}$$

Corollary 4.1. *If Γ is integrable, then each Γ-horizontal vector field ζ on Y is a symmetry of Γ.*

Proof. See Lemma 3.1. □

Proposition 4.1. *Let Γ be an integrable connection on π. Then a vector field ζ on Y is a symmetry of Γ if and only if*

$$\mathcal{J}^1(v_\Gamma \circ \zeta) \circ \Gamma = \mathcal{V}\Gamma \circ v_\Gamma \circ \zeta \ . \tag{4.4}$$

Proof. Since $\zeta = h_\Gamma(\zeta) + v_\Gamma(\zeta)$ and due to Prop. 3.1 and Corollary 4.1, ζ is a symmetry of Γ if and only if $v_\Gamma(\zeta)$ is a symmetry. Since evidently $\mathcal{V}\Gamma \circ v_\Gamma \circ \zeta = T\Gamma \circ v_\Gamma \circ \zeta$, the assertion is then an immediate consequence of Prop. 3.1. □

The condition (4.3) can be represented in terms of (generalized) Lie derivative [9]. Actually, a vector field ζ on Y is a symmetry of Γ if and only if a *generalized Lie derivative*

$$\widetilde{\mathcal{L}}_{(\zeta,\mathcal{J}^1\zeta)}\Gamma = T\Gamma \circ \zeta - \mathcal{J}^1\zeta \circ \Gamma$$

of Γ with respect to ζ and $\mathcal{J}^1\zeta$ vanishes. In this context, previous proposition can be obtained (again using Lemma 3.1) by

$$\begin{aligned} \widetilde{\mathcal{L}}_{(\zeta,\mathcal{J}^1\zeta)}\Gamma &= \widetilde{\mathcal{L}}_{(h_\Gamma\circ\zeta+v_\Gamma\circ\zeta,\mathcal{J}^1(h_\Gamma\circ\zeta)+\mathcal{J}^1(v_\Gamma\circ\zeta))}\Gamma = \\ &= \widetilde{\mathcal{L}}_{(h_\Gamma\circ\zeta,\mathcal{J}^1(h_\Gamma\circ\zeta))}\Gamma + \widetilde{\mathcal{L}}_{(v_\Gamma\circ\zeta,\mathcal{J}^1(v_\Gamma\circ\zeta))}\Gamma = \widetilde{\mathcal{L}}_{(v_\Gamma\circ\zeta,\mathcal{J}^1(v_\Gamma\circ\zeta))}\Gamma \ . \end{aligned} \tag{4.5}$$

The relation (4.4) locally means a system of PDE for the *generating functions* $\varphi^\sigma = \zeta^\sigma - \Gamma^\sigma_j\zeta^j$ of $v_\Gamma(\zeta) = \varphi^\sigma\partial/\partial y^\sigma$ expressed by $D_{\Gamma i}(\varphi^\sigma) = \mathcal{L}_{v_\Gamma(\zeta)}\Gamma^\sigma_i$ or equivalently

$$\frac{\partial\varphi^\sigma}{\partial x^i} + \frac{\partial\varphi^\sigma}{\partial y^\lambda}\,\Gamma^\lambda_i = \frac{\partial\Gamma^\sigma_i}{\partial y^\lambda}\,\varphi^\lambda \tag{4.6}$$

for $i = 1,\dots,n$, $\sigma = 1,\dots,m$.

Directly by calculations in coordinates we see that :

Corollary 4.2. *A vector field ζ on Y is a symmetry of an integrable connection Γ if and only if*

$$\mathcal{L}_{v_\Gamma(\zeta)} h_\Gamma = 0 \ . \tag{4.7}$$

In other words, due to the canonical vertical splitting

$$V_{\pi_{1,0}} J^1\pi \cong J^1\pi \times_Y (V_\pi Y \otimes \pi^*(T^*X))$$

and accordingly to (4.5), the generalized Lie derivative $\widetilde{\mathcal{L}}_{(\zeta, \mathcal{J}^1\zeta)}\Gamma$ of an integrable connection Γ reads

$$\widetilde{\mathcal{L}}_{(\zeta, \mathcal{J}^1\zeta)}\Gamma = \left(\Gamma, \mathcal{L}_{v_\Gamma(\zeta)} h_\Gamma\right) \ . \tag{4.8}$$

If we are asked to give an explicit formula for the symmetry ζ of an integrable Γ then if $\zeta = \zeta^i \partial/\partial x^i + \zeta^\sigma \partial/\partial y^\sigma$, we can see that $[v_\Gamma(\zeta), D_{\Gamma i}] = [\zeta - h_\Gamma(\zeta), D_{\Gamma i}] = [\zeta, D_{\Gamma i}] - [\zeta^j D_{\Gamma j}, D_{\Gamma i}]$ and since $\mathcal{L}_{v_\Gamma(\zeta)} h_\Gamma = [v_\Gamma(\zeta), D_{\Gamma i}] \otimes dx^i$, ζ is a symmetry of Γ if and only if

$$[\zeta, D_{\Gamma i}] = -D_{\Gamma i}(\zeta^j) D_{\Gamma j} \tag{4.9}$$

for any $i = 1, \ldots, n$. In this arrangement, symmetries of Γ are vector fields ζ such that $\mathcal{J}^1\zeta|_{\Gamma(Y)}$ are the symmetries of $C^\Gamma_{\pi_{1,0}}$. Accordingly, the symmetries of an integrable connection Γ are just the symmetries of the horizontal distribution H_Γ.

An additional characterization can be given in the case of *projectable* symmetries, the flow of which consists of π-morphisms. By means of direct calculations we obtain :

Lemma 4.1. *Let ζ be a vector field on Y. Then $\mathcal{L}_{h_\Gamma(\zeta)} h_\Gamma = 0$ if and only if ζ is projectable.*

Previous assertion means that $\mathcal{L}_\zeta h_\Gamma = \mathcal{L}_{v_\Gamma(\zeta)} h_\Gamma$ and equivalently

$$\widetilde{\mathcal{L}}_{(\zeta, \mathcal{J}^1\zeta)}\Gamma = (\Gamma, \mathcal{L}_\zeta h_\Gamma) \tag{4.10}$$

if and only if ζ is projectable.

Corollary 4.3. *A projectable vector field ζ on Y is a symmetry of Γ if and only if*

$$\mathcal{L}_\zeta h_\Gamma = 0 \ . \tag{4.11}$$

Notice that in the case of projectable symmetry ζ of Γ, (4.9) reads $[\zeta, D_{\Gamma i}] = -D_i(\zeta^j) D_{\Gamma j}$.

Remark 4.1. To prove the importance of the projectability, we recall a result of [24]. Namely; if ζ is a vector field on Y with the flow $\{\alpha_t\}$ and R is a (1,1) tensor field on Y then $\mathcal{L}_\zeta R = 0$ if and only if $T\alpha_t \circ R = R \circ T\alpha_t$ for each t.

Corollary 4.4. *A projectable vector field ζ on Y is a symmetry of an integrable connection Γ on π if and only if its flow permutes the integral sections of Γ.*

Proof. Let $\gamma \in S_U(\pi)$ and denote by $h_\Gamma|_\gamma$ the restriction of h_Γ to $T_{\gamma(x)}Y$ for any $x \in U$. Evidently γ is an integral section of Γ if and only if $h_\Gamma|_\gamma = T\gamma \circ T\pi$. Let ζ be a projectable vector field on Y with the flow $\{\alpha_t\}$ and suppose $\alpha_t \circ \gamma$ to be defined. Clearly $\widetilde{\gamma} = \alpha_t \circ \gamma \in S_{\widetilde{U}}(\pi)$ is an integral section of Γ if and only if $h_\Gamma|_{\widetilde{\gamma}} = T\alpha_t \circ T\gamma \circ T\pi \circ T\alpha_t^{-1}$. The assertion is due to Corollary 4.3 and Remark 4.1, since ζ is a symmetry of Γ if and only if $h_\Gamma|_{\widetilde{\gamma}} = T\alpha_t \circ h_\Gamma|_\gamma \circ T\alpha_t^{-1}$. □

Previous results shows that in further considerations we can assume symmetries of Γ to be π-vertical. The flow of such a symmetry permutes the integral sections of Γ without changing their parameterization since the flow consists of isomorphisms of π over X; and (4.9) reads $[\zeta, D_{\Gamma i}] = 0$. The set of all symmetries of Γ will be denoted by $\mathrm{Sym}(\Gamma)$ and the set of π-vertical symmetries by $\mathrm{Sym}_v(\Gamma)$.

Proposition 4.2. *Let Γ be an integrable connection on π. Then it holds*

$$\mathrm{Sym}_v(\Gamma) \cong \mathrm{Shuf}(C^\Gamma_{\pi_{1,0}}) \ .$$

Proof. The assertion accords with the evident fact that

$$H_\Gamma \cong \mathrm{Char}(C^\Gamma_{\pi_{1,0}}) \ .$$

Analogously as in [6] it can be shown that $\mathrm{Shuf}(C^\Gamma_{\pi_{1,0}}) \cong \ker \Delta_\Gamma$, where

$$\Delta_\Gamma \colon (C^\infty(Y))^m \to (C^\infty(Y))^{nm}$$

is an operator defined for each $\varphi = (\varphi^\sigma)$ by $\Delta_\Gamma(\varphi) = (\Delta^\sigma_{\Gamma i}(\varphi))$, where

$$\Delta^\sigma_{\Gamma i}(\varphi) = D_{\Gamma i}(\varphi^\sigma) - \frac{\partial \Gamma^\sigma_i}{\partial y^\lambda}\, \varphi^\lambda \ .$$

On the other hand, also $\mathrm{Sym}_v(\Gamma) \cong \ker \Delta_\Gamma$. This isomorphism is given by the identification of π-vertical vector field with its generating functions family $\varphi = (\zeta^\sigma)$ by (4.4). □

Next assertions can be viewed as a confirmation of a deep relation between Γ and $\mathcal{V}\Gamma$ in terms of vertical symmetries, sections and fields along sections. A connection Γ is again supposed to be integrable.

Proposition 4.3. *Let $\zeta \in \mathrm{Sym}_v(\Gamma)$. Then a section γ of π is an integral section of Γ if and only if the section $\xi = \zeta \circ \gamma$ of ρ is an integral section of $\mathcal{V}\Gamma$.*

Proof. Let γ be an integral section of Γ. Since ζ is vertical, (4.4) reads $\mathcal{J}^1\zeta \circ \Gamma = \mathcal{V}\Gamma \circ \zeta$. Then $\mathcal{V}\Gamma \circ \xi = \mathcal{V}\Gamma \circ \zeta \circ \gamma = \mathcal{J}^1\zeta \circ \Gamma \circ \gamma = \mathcal{J}^1\zeta \circ j^1\gamma = j^1\xi$. Conversely, if $\mathcal{V}\Gamma \circ \xi = j^1\xi$, then $\mathcal{V}\Gamma \circ \zeta \circ \gamma = \mathcal{J}^1\zeta \circ j^1\gamma$, which implies $\mathcal{J}^1\zeta \circ \Gamma \circ \gamma = \mathcal{J}^1\zeta \circ j^1\gamma$. Since $\mathcal{J}^1\zeta$ is a section, we have $\Gamma \circ \gamma = j^1\gamma$. □

Corollary 4.5. *For any vertical ζ it holds $\zeta \in \mathrm{Sym}_v(\Gamma)$ if and only if $\xi = \zeta \circ \gamma$ is an integral section of $\mathcal{V}\Gamma$ for each integral section γ of Γ.*

Proof. Let $y \in Y$ be an arbitrary point, let γ be a maximal integral section of Γ such that $\gamma(x) = y$ for $x = \pi(y)$. Then $\mathcal{J}^1\zeta \circ \Gamma(y) = \mathcal{J}^1\zeta \circ j^1\gamma(x) = j^1\xi(x) = \mathcal{V}\Gamma \circ \xi(x) = \mathcal{V}\Gamma \circ \zeta \circ \gamma(x) = \mathcal{V}\Gamma \circ \zeta(y)$. The converse is evident from Prop. 4.3. □

Remark 4.2. Since $\zeta \colon Y \to V_\pi Y$ is a fibered morphism between π and ρ over X and $\mathcal{J}^1\zeta = J^1(\zeta, \mathrm{id}_X)$, the so-called *covariant derivative* $\nabla_{(\Gamma,\mathcal{V}\Gamma)}\zeta$ *of* ζ with respect to the pair of connections Γ and $\mathcal{V}\Gamma$ can be considered, see [8], [17]. This is defined as $\nabla_{(\Gamma,\mathcal{V}\Gamma)}\zeta = \nabla_{\mathcal{V}\Gamma} \circ \mathcal{J}^1\zeta \circ \Gamma$. Using a canonical fibered morphism

$$V_{\pi_1} J^1\pi \times_{\pi^*_{1,0}(V_\pi Y)} V_{\pi_1} J^1\pi \to V_{\pi_{1,0}} J^1\pi$$

(realizing an ordinary difference of vectors in the same fiber), we can easily see that $\nabla_{(\Gamma,\mathcal{V}\Gamma)}\zeta \colon Y \to V_{\pi_{1,0}} J^1\pi$ and moreover $\nabla_{(\Gamma,\mathcal{V}\Gamma)}\zeta = -\widetilde{\mathcal{L}}_{(\zeta,\mathcal{J}^1\zeta)}\Gamma = (\Gamma, -\mathcal{L}_\zeta h_\Gamma)$. Consequently $\zeta \in \mathrm{Sym}_v(\Gamma)$ if and only if $\nabla_{(\Gamma,\mathcal{V}\Gamma)}\zeta = 0$. Recall [8], where the last condition was proved to be equivalent with ζ-relativity of the vertical lifts $h_\Gamma(\xi)$ and $h_{\mathcal{V}\Gamma}(\xi)$ for any vector field ξ on X.

This characterization can be compared to that of [28] expressing the situation by means of the so-called *covariant derivative* with respect to the *canonical Bliznikas reduced connection.*

Remark 4.3. At the very end of the previous section, we have suggested the importance of the reduced connection $\Gamma_{(1,0)}$ associated to a given characterizable connection Ξ on $\pi_{1,0}$. From Def. 3.2 and Prop. 3.4 we deduce that if ζ is an integral section of $\Gamma_{(1,0)}$ then $\zeta|_V$ is a symmetry of an arbitrary integral section $\Gamma \in S_V(\pi_{1,0})$ of Ξ.

Definition 4.2. Let $\Gamma^{(2)}$ be an integrable 2-connection on π. A vector field $\zeta^{(1)}$ on $J^1\pi$ will be called a *a first-order symmetry* of $\Gamma^{(2)}$ if $\mathcal{L}_{v_{\Gamma^{(2)}}(\zeta^{(1)})} h_{\Gamma^{(2)}} = 0$ or equivalently

$$[\zeta^{(1)}, D_{\Gamma^{(2)}i}] = -D_{\Gamma^{(2)}i}(\zeta^j) D_{\Gamma^{(2)}j} \; . \tag{4.12}$$

Locally, $\zeta^{(1)}$ must be such a vector field on $J^1\pi$ that

$$v_{\Gamma^{(2)}}(\zeta^{(1)}) = \varphi^\sigma \frac{\partial}{\partial y^\sigma} + D_{\Gamma^{(2)}i}(\varphi^\sigma) \frac{\partial}{\partial y^\sigma_i} \; ,$$

where the equations for the generating functions $\varphi^\sigma = \zeta^\sigma - y^\sigma_i \zeta^i$ on $J^1\pi$ are

$$D^2_{\Gamma^{(2)}ij}(\varphi^\sigma) = \frac{\partial \Gamma^\sigma_{ij}}{\partial y^\lambda} \varphi^\lambda + \frac{\partial \Gamma^\sigma_{ij}}{\partial y^\lambda_k} D_{\Gamma^{(2)}k}(\varphi^\lambda) \; , \tag{4.13}$$

where $D^2_{\Gamma^{(2)}ij}(f) = D_{\Gamma^{(2)}j}(D_{\Gamma^{(2)}i}(f))$. The right side of (4.13) may be written as $v_{\Gamma^{(2)}}(\zeta^{(1)})(\Gamma^\sigma_{ij})$ and the set of these symmetries we denote by $\mathrm{Sym}^{(1)}(\Gamma^{(2)})$.

Remark 4.4. As before, the first-order symmetries of an integrable connection $\Gamma^{(2)}$ are just the symmetries of $H_{\Gamma^{(2)}}$; the symbol of $v_{\Gamma^{(2)}}$ can be in omitted for π_1-projectable symmetries and this is equivalent with the permutation of the first jet prolongations of integral sections of $\Gamma^{(2)}$. The set of π_1-vertical symmetries of the first order of $\Gamma^{(2)}$ will be denoted by $\mathrm{Sym}_v^{(1)}(\Gamma^{(2)})$ and according to Prop. 4.2 it can be shown that in view of $H_{\Gamma^{(2)}} \cong \mathrm{Char}(C_{\pi_{2,1}}^{\Gamma^{(2)}})$ it holds $\mathrm{Sym}_v^{(1)}(\Gamma^{(2)}) \cong \mathrm{Shuf}(C_{\pi_{2,1}}^{\Gamma^{(2)}})$, where $C_{\pi_{2,1}}^{\Gamma^{(2)}} = C_{\pi_{2,1}} \cap T\Gamma^{(2)}(J^1\pi)$ is the Cartan distribution of the second-order equation $\Gamma^{(2)}(J^1\pi) \subset J^2\pi$.

Let us return to vector fields on Y.

Definition 4.3. Let $\Gamma^{(2)}$ be a 2-connection on π. A vector field ζ on Y will be called a *zeroth-order symmetry* or briefly a *symmetry* of $\Gamma^{(2)}$ if

$$\mathcal{L}_{v_{\Gamma^{(2)}}(\mathcal{J}^1\zeta)} h_{\Gamma^{(2)}} = 0 \ . \tag{4.14}$$

Denote the set of these symmetries by $\mathrm{Sym}(\Gamma^{(2)})$. If $\Gamma^{(2)}$ is integrable then (4.14) means

$$[\mathcal{J}^1\zeta, D_{\Gamma^{(2)}i}] = -D_i(\zeta^j) D_{\Gamma^{(2)}j} \ . \tag{4.15}$$

Such a symmetry is determined by the functions $\varphi^\sigma = \zeta^\sigma - y_i^\sigma \zeta^i$ (again on $J^1\pi$) satisfying

$$D_{\Gamma^{(2)}j}\left(D_i(\varphi^\sigma)\right) = \frac{\partial \Gamma_{ij}^\sigma}{\partial y^\lambda}\varphi^\lambda + \frac{\partial \Gamma_{ij}^\sigma}{\partial y_k^\lambda} D_{\Gamma^{(2)}k}(\varphi^\lambda) \ , \tag{4.16}$$

where on the right side is just $v_{\Gamma^{(2)}}(\mathcal{J}^1\zeta)(\Gamma_{ij}^\sigma)$.

Remark 4.5. For $\zeta^{(1)} = \zeta^i \partial/\partial x^i + \zeta^\sigma \partial/\partial y^\sigma + \zeta_j^\sigma \partial/\partial y_j^\sigma$, (4.12) means that

$$\zeta_j^\sigma = D_{\Gamma^{(2)}j}(\zeta^\sigma) - y_i^\sigma D_{\Gamma^{(2)}j}(\zeta^i) \ .$$

Consequently, if $\zeta^{(1)}$ is $\pi_{1,0}$-projectable then $\zeta_j^\sigma = D_j(\zeta^\sigma) - y_i^\sigma D_j(\zeta^i)$ and (according to Example 4.2) a $\pi_{1,0}$- projectable vector field $\zeta^{(1)}$ on $J^1\pi$ is a first-order symmetry of $\Gamma^{(2)}$ if and only if it is a prolongation of the symmetry $\zeta = T\pi_{1,0}(\zeta^{(1)})$.

Proposition 4.4. *A projectable vector field ζ on Y is a symmetry of an integrable 2-connection $\Gamma^{(2)}$ on π if and only if its flow permutes the integral sections of $\Gamma^{(2)}$.*

Proof. Definition 4.3 means that ζ on Y is a symmetry of $\Gamma^{(2)}$ if and only if $\mathcal{J}^1\zeta$ is a symmetry of the first order. By previous considerations it is for an projectable ζ equivalent to the fact that the flow of $\mathcal{J}^1\zeta$ permutes the first jet prolongations $j^1\gamma$ of integral sections γ of $\Gamma^{(2)}$. But this flow is the prolongation of the flow of ζ. Hence if α_t permutes the integral sections of $\Gamma^{(2)}$ and if $j^1\gamma$ is the prolongation of an integral section then ($J^1(\alpha_t)$ denotes $J^1(\alpha_t, \mathrm{id}_X)$) $J^1(\alpha_t)\circ j^1\gamma$ is by definition just $j^1(\alpha_t \circ \gamma)$ and thus again a jet prolongation of an integral section.

Conversely, if $J^1(\alpha_t)$ permutes the prolongations $j^1\gamma$ of integral sections and if we are given such a section γ, then $j^2(\alpha_t\circ\gamma) = j^1(j^1(\alpha_t\circ\gamma)) = j^1(J^1(\alpha_t)\circ j^1\gamma) = \Gamma^{(2)}\circ(J^1(\alpha_t)\circ j^1\gamma) = \Gamma^{(2)}\circ j^1(\alpha_t\circ\gamma)$ and consequently $\alpha_t\circ\gamma$ is an integral section. □

The remainder of the section is devoted to the study of the algebras $\mathrm{Sym}_v(\Gamma^{(2)})$ or $\mathrm{Sym}_v(\Gamma)$ of the vertical symmetries of $\Gamma^{(2)}$ or Γ, respectively. Accordingly, the word "vertical" will be omitted if there is no danger of confusion.

In (4.4), the vertical prolongation $\mathcal{V}\Gamma$ of Γ on π allowed us to characterize the symmetries of Γ. In the case of 2-connections an analogous concept may be introduced. In order to relate it with the symmetries of $\Gamma^{(2)}$, the second jet prolongation of a π-vertical vector field on Y must be recalled.

For any ζ being a vertical vector field on Y, the second jet prolongation $\mathcal{J}^2\zeta = J^2(\zeta,\mathrm{id}_X)$ is a vector field on $J^2\pi$ with the flow which is the second jet prolongation of the flow of ζ. In coordinates

$$\mathcal{J}^2\zeta = \zeta^\sigma\frac{\partial}{\partial y^\sigma} + D_i\zeta^\sigma\frac{\partial}{\partial y^\sigma_i} + D^2_{ij}\zeta^\sigma\frac{\partial}{\partial y^\sigma_{ij}}\ , \tag{4.17}$$

where $D^2_{ij}f$ denotes a composition $D_j(D_if)$ of total derivatives with respect to base coordinates, where the inner component is a vector field along $\pi_{1,0}$ while the outer one is a vector field along $\pi_{2,1}$.

The *vertical prolongation* $\mathcal{V}\Gamma^{(2)}$ of a 2-connection $\Gamma^{(2)}$ on π can be defined analogously to $\mathcal{V}\Gamma$ of Γ by means of the vertical functor V, which gives a mapping $V\Gamma^{(2)}\colon V_{\pi_1}J^1\pi \to V_{\pi_2}J^2\pi$, and thus defines a lift

$$\begin{aligned}\zeta^\sigma\frac{\partial}{\partial y^\sigma}\Big|_z + \zeta^\sigma_i\frac{\partial}{\partial y^\sigma_i}\Big|_z \xrightarrow{V\Gamma^{(2)}} \zeta^\sigma\frac{\partial}{\partial y^\sigma}\Big|_{\Gamma^{(2)}(z)} + \zeta^\sigma_i\frac{\partial}{\partial y^\sigma_i}\Big|_{\Gamma^{(2)}(z)} +\\ + \left(\frac{\partial\Gamma^\sigma_{ij}}{\partial y^\lambda}\zeta^\lambda + \frac{\partial\Gamma^\sigma_{ij}}{\partial y^\lambda_k}\zeta^\lambda_k\right)\frac{\partial}{\partial y^\sigma_{ij}}\Big|_{\Gamma^{(2)}(z)}\ .\end{aligned} \tag{4.18}$$

Composing $V\Gamma^{(2)}$ with a canonical diffeomorphism between $V_{\pi_2}J^2\pi$ and $J^2\rho$ for $\rho\colon V_\pi Y \to X$, one obtains $\mathcal{V}\Gamma^{(2)}$ as a 2-connection on ρ.

From (4.16), (4.17) and (4.18), where $\varphi^\sigma = \zeta^\sigma$ are now functions on Y, we have

Proposition 4.5. *A vertical vector field ζ on Y is a symmetry of a 2-connection $\Gamma^{(2)}$ on π if and only if*

$$\mathcal{J}^2\zeta\circ\Gamma^{(2)} = \mathcal{V}\Gamma^{(2)}\circ\mathcal{J}^1\zeta\ . \tag{4.19}$$

Remark 4.6. Following Remark 4.2, it is possible to present that $\zeta\in\mathrm{Sym}_v(\Gamma^{(2)})$ if and only if

$$0 = \nabla_{(\Gamma^{(2)},\mathcal{V}\Gamma^{(2)})}\mathcal{J}^1\zeta = -\widetilde{\mathcal{L}}_{(\mathcal{J}^1\zeta,\mathcal{J}^2\zeta)}\Gamma^{(2)} = (\Gamma^{(2)}, -\mathcal{L}_{\mathcal{J}^1\zeta}h_{\Gamma^{(2)}})\colon J^1\pi \to V_{\pi_{2,1}}J^2\pi\ .$$

Analogously to Corollary 4.5 we obtain

Corollary 4.6. *Let $\Gamma^{(2)}$ be an integrable 2-connection on π. Then $\zeta \in \mathrm{Sym}_v(\Gamma^{(2)})$ if and only if $\xi = \zeta \circ \gamma$ is an integral section of $\mathcal{V}\Gamma^{(2)}$ for each integral section γ of $\Gamma^{(2)}$.*

In what follows, $\Gamma \in S_U(\pi_{1,0})$ is supposed to be a field of paths of a 2-connection $\Gamma^{(2)}$. Since any integral section of Γ is an integral section of $\Gamma^{(2)}$, due to Prop. 4.4 we get an expected result.

Corollary 4.7. *If ζ is a symmetry of $\Gamma^{(2)}$ then $\zeta|_U$ is a symmetry of Γ.*

One might ask on the relationships between the vertical prolongations $\mathcal{V}\Gamma^{(2)}$ and $\mathcal{V}\Gamma$.

Proposition 4.6. *A connection Γ on π is a field of paths of a 2-connection $\Gamma^{(2)}$ on π if and only if $\mathcal{V}\Gamma$ is a field of paths of $\mathcal{V}\Gamma^{(2)}$.*

Proof. Let Γ be a field of paths of $\Gamma^{(2)}$, i.e. $\Gamma^{(2)} \circ \Gamma = J^1(\Gamma, \mathrm{id}_X) \circ \Gamma$. Relative to the naturality of all morphisms, functors and structures, it is easy to see that $\mathcal{V}\Gamma^{(2)} \circ \mathcal{V}\Gamma = V(\Gamma^{(2)} \circ \Gamma) = V(J^1(\Gamma, \mathrm{id}_X) \circ \Gamma) = VJ^1(\Gamma, \mathrm{id}_X) \circ V\Gamma = J^1(\mathcal{V}\Gamma, \mathrm{id}_X) \circ \mathcal{V}\Gamma$ which means that $\mathcal{V}\Gamma$ is a field of paths of $\mathcal{V}\Gamma^{(2)}$. Note that we use the above mentioned identifications of various prolongations of $\rho\colon V_\pi Y \to X$ with the corresponding vertical bundles, if necessary. The converse can be obtained analogously. □

As an immediate consequence we obtain the well-known result affirming that an arbitrary vertical symmetry of an equation is a symmetry of its prolongation.

Corollary 4.8. *If ζ is a symmetry of Γ, then ζ is a symmetry of $\Gamma^{(2)} \circ \Gamma$.*

Proof. What we want to prove is (following (4.19))

$$\mathcal{J}^2\zeta \circ \Gamma^{(2)} \circ \Gamma = \mathcal{V}\Gamma^{(2)} \circ \mathcal{J}^1\zeta \circ \Gamma \qquad \text{on } U \,. \tag{4.20}$$

Let $y \in U$ be an arbitrary point, γ the maximal integral section of Γ passing through y, $\gamma(x) = y$. Then $\mathcal{V}\Gamma^{(2)} \circ \mathcal{J}^1\zeta \circ \Gamma(y) = \mathcal{V}\Gamma^{(2)} \circ \mathcal{J}^1\zeta \circ \Gamma \circ \gamma(x) = \mathcal{V}\Gamma^{(2)} \circ \mathcal{J}^1\zeta \circ j^1\gamma(x) = \mathcal{V}\Gamma^{(2)} \circ j^1(\zeta \circ \gamma)(x) = \mathcal{V}\Gamma^{(2)} \circ j^1\xi(x) = j^2\xi(x) = \mathcal{J}^2\zeta \circ j^2\gamma(x) = \mathcal{J}^2\zeta \circ \Gamma^{(2)} \circ j^1\gamma(x) = \mathcal{J}^2\zeta \circ \Gamma^{(2)} \circ \Gamma(y)$. □

Clearly, the whole situation may be described diagrammatically:

$$\begin{array}{ccccc}
V_\pi Y & \xrightarrow{\mathcal{V}\Gamma} & V_{\pi_1} J^1\pi & \xrightarrow{\mathcal{V}\Gamma^{(2)}} & V_{\pi_2} J^2\pi \\
{\scriptstyle \zeta}\uparrow & & \uparrow{\scriptstyle \mathcal{J}^1\zeta} & & \uparrow{\scriptstyle \mathcal{J}^2\zeta} \\
Y & \xrightarrow{\Gamma} & J^1\pi & \xrightarrow{\Gamma^{(2)}} & J^2\pi \;.
\end{array}$$

Example 4.3. First consider a time-dependent vector field $v = \Gamma^\sigma(t, q^\lambda)\partial/\partial q^\sigma$ on M, identified with the connection Γ on $\pi : \mathbb{R} \times M \to \mathbb{R}$ by $D_\Gamma = \partial/\partial t + v$. A *symmetry* of v is then a vector field $\zeta = \zeta^0 \partial/\partial t + \zeta^\sigma \partial/\partial q^\sigma$ on $\mathbb{R} \times M$, such that $[\zeta, D_\Gamma] = -D_\Gamma(\zeta^0)D_\Gamma$. Considering π-vertical symmetries we obtain again (generally time-dependent) vector fields ζ on M satisfying $[\zeta, \partial/\partial t + v] = 0$.

Let now

$$D_{\Gamma^{(2)}} = \frac{\partial}{\partial t} + q^\sigma_{(1)} \frac{\partial}{\partial q^\sigma} + \Gamma^\sigma_{(2)} \frac{\partial}{\partial q^\sigma_{(1)}}$$

be a (global) semispray on $\mathbb{R} \times TM$, identified with the 2-connection $\Gamma^{(2)}\colon \mathbb{R} \times TM \to \mathbb{R} \times T^2M$. A vector field $\zeta^{(1)} = \zeta^0 \partial/\partial t + \zeta^\sigma \partial/\partial q^\sigma + \zeta^\sigma_{(1)} \partial/\partial q^\sigma_{(1)}$ is then a *first-order symmetry* of the semispray $D_{\Gamma^{(2)}}$ if

$$[\zeta^{(1)}, D_{\Gamma^{(2)}}] = -D_{\Gamma^{(2)}}(\zeta^0) D_{\Gamma^{(2)}} \tag{4.21}$$

which locally means

$$\zeta^\sigma_{(1)} = D_{\Gamma^{(2)}}(\zeta^\sigma) - q^\sigma_{(1)} D_{\Gamma^{(2)}}(\zeta^0)$$

and

$$D^2_{\Gamma^{(2)}}(\varphi^\sigma) = \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda} \varphi^\lambda + \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}} D_{\Gamma^{(2)}}(\varphi^\lambda)$$

for the generating functions $\varphi^\sigma = \zeta^\sigma - q^\sigma_{(1)}\zeta^0$. In particular, for $\zeta^{(1)}$ π_1-projectable or π_1-vertical (4.21) reads $[\zeta^{(1)}, D_{\Gamma^{(2)}}] = -D(\zeta^0) D_{\Gamma^{(2)}}$ or $[\zeta^{(1)}, D_{\Gamma^{(2)}}] = 0$, respectively.

If $\zeta^{(1)}$ is projectable onto $\mathbb{R} \times M$ then it is a prolongation of the (zeroth-order) *symmetry* of $D_{\Gamma^{(2)}}$, which is a vector field ζ satisfying $[\mathcal{J}^1\zeta, D_{\Gamma^{(2)}}] = -D(\zeta^0) D_{\Gamma^{(2)}}$. Provided $D_{\Gamma^{(2)}} = \partial/\partial t + w$, where $w(q^\lambda, q^\lambda_{(1)})$ is a semispray on TM, then $\zeta^{(1)}$ on TM is a *first order symmetry* of w if and only if $[\zeta^{(1)}, w] = 0$ and ζ on M is a *symmetry* of w if and only if $[\zeta^c, w] = 0$, where ζ^c is a complete lift of ζ.

It is well-known that the study of symmetries play a intrinsic role in the geometry both of autonomous and time-dependent dynamics; we refer e.g. to [3], [4], [19], [20], [21], [22] (and references therein). From this point of view, the above classification of symmetries for connections might be useful e.g. in the geometry of dynamics on an arbitrary fibered manifold over one-dimensional base (e.g. [14] and references therein). In fact, first order symmetries of the Hamilton vector field associated to a first-order regular lagrangian $\lambda = L\,dt$ on $J^1\pi$ are nothing else that the so-called *dynamical symmetries* of λ and accordingly the (zeroth-order) symmetries are just the *Lie symmetries* of λ (cf. [20], [21]).

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