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LANDESMAN – LAZER TYPE PROBLEMS AT AN EIGENVALUE OF ODD MULTIPLICITY

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Abstract. The aim of this paper is to establish some a priori bounds for solutions of Landesman-Lazer problem. We show the application for the solution structure of the nonlinear differential equation of the fourth order

1. The general theory

Let X be a real Banach space with the norm $\|\cdot\|$ and let $D(L) \subset X$ be the domain of the closed Fredholm operator

$$L : D(L) \to X$$

with index zero. We shall suppose that 0 is an isolated eigenvalue of odd multiplicity of L, hence there exists such a $\delta_0 > 0$ that for $\lambda \in (-\delta_0, \delta_0)$, $\lambda \neq 0$, $(L - \lambda \cdot I)^{-1}$ exists on X and that mapping is continuous.

Let there exists a continuous positive definite bilinear form

$$\langle \cdot\, , \cdot \rangle : X \times X \to \mathbb{R}$$

such that $z \in R(L)$ (range of L) iff $\langle z\, , u \rangle = 0$ for all $u \in NS(L)$ (nullspace of the operator L).

Let $P_0 : X \to X$ be a continuous linear projection onto $NS(L)$. We denote the operator

$$L_{P_0} : D(L) \cap NS(P_0) \to R(L)$$

defined by

$$L_{P_0} = L|_{D(L) \cap NS(P_0)}.$$

The operator L_{P_0} is one-to-one on $D(L) \cap NS(P_0)$ and therefore there exists an inverse operator which we denote by $L_{P_0}^{-1} = K_{P_0}$. We suppose that K_{P_0} be a completely continuous. Let

$$F : X \to X$$

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be a L-completely continuous mapping i.e. $P_1 \circ F$ and $K_{P_0} \circ (I - P_1) \circ F$ are completely continuous, where $P_1 : X \to X$ is a continuous linear projection with

$$NS(P_1) = R(L)$$

and $I : X \to X$ is the identity mapping. Let F satisfy

$$\lim_{||u|| \to \infty} \frac{||F(u)||}{||u||} = 0 \tag{1}$$

Let

$$R(L) = NS(P_0) \tag{2}$$

be true. Then we can take $P_0 = P_1$.

Let $h \in X$. We assume that there exists a number $d > 0$ sufficiently small which has following property :

For each $y \in NS(L)$, $||y|| = 1$ each sequence $\{y_n\} \subset NS(L)$, $||y_n|| = 1$, $y_n \to y$ as $n \to \infty$, each sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ and for each sequence $\{z_n\} \subset NS(P_0)$, $||z_n|| < d$

$$\langle h\,,\,y\rangle > \liminf_{n\to\infty} \langle F(t_n y_n + t_n z_n)\,,\,y\rangle \tag{3}$$

or

$$\langle h\,,\,y\rangle < \limsup_{n\to\infty} \langle F(t_n y_n + t_n z_n)\,,\,y\rangle \tag{4}$$

is valid.

In [3] instead of (1) the assumption is considered

$$||F(u)|| \le c_1 ||u||^{\alpha} + d_1$$

for constants $c_1 > 0$, $d_1 \le 0$, $\alpha \in < 0, 1)$ and all $u \in X$ and instead of (3), (4) the hypotheses are considered

$$\langle h\,,\,y\rangle > \liminf_{n\to\infty} \langle F(t_n y_n + t_n^{\alpha} z_n)\,,\,y\rangle$$

or

$$\langle h\,,\,y\rangle < \limsup_{n\to\infty} \langle F(t_n y_n + t_n^{\alpha} z_n)\,,\,y\rangle$$

where the sequences $\{t_n\}$, $\{y_n\}$ have the same meaning as above and $\{z_n\}$ is any bounded sequence in $NS(P_0)$.

We now considere the equation

$$L(u) - \lambda\, u + F(u) = h \tag{5}$$

where λ is a real parameter. First we shall introduce the modification of Theorem 3.6.2 [2] p.99 which we use in the proof of next theorem.

Denote $L_0(x) = L(x) - \lambda\, x$ the operator which maps D onto B. Then for $|\lambda| < \delta_0$, $L_0^{-1} \in \mathcal{L}(B\,,\,D)$.

Lemma 1. *Let B be a Banach space and $D \subset B$ be the subspace (it need not be closed). Let L, $L_0 : D \to B$ be such operators that the inverse operators L^{-1}, $L_0^{-1} \in \mathcal{L}(B, D)$. If $\Delta = |\lambda| \, ||L^{-1}|| < 1$, then $||L^{-1} - L_0^{-1}|| \leq (1 - \Delta)^{-1} \Delta \cdot ||L^{-1}||$.*

Let $0 \leq \Delta \leq \frac{1}{2}$. Then we calculate the norm of the operator L_0^{-1}.

$$||L_0^{-1}|| \leq ||L^{-1}|| + ||L_0^{-1} - L^{-1}|| \leq ||L^{-1}|| + ||L^{-1}|| = 2 \cdot ||K_{P_0}|| \,.$$

Theorem 1. *Let all assumptions given above be satisfied. Let condition (3) and*

$$0 < \delta = \min\left(\delta_0 \, , \, \frac{1}{2 \cdot ||K_{P_0}||}\right) . \tag{7}$$

be satisfied. Then for all λ such that $0 \leq \lambda \leq \delta$ there exists an $R_0 > 0$ for which any solution u of (5) satisfies $||u|| \leq R_0$.

Proof. Let u be a solution of (5) and write $u = u_1 + u_2$, $u_1 \in NS(L)$, $u_2 \in NS(P_0)$. We can write the equation in the following form

$$L(u_2) - \lambda(u_1 + u_2) + F(u_1 + u_2) - h = 0 \tag{8}$$

Then

$$\langle -\lambda \, u_1 + F(u_1 + u_2) - h \, , \, v\rangle = 0 \, , \quad v \in NS(L) \, . \tag{9}$$

Applying $I - P_1$ to the equation (8) we have

$$L(u_2) - \lambda \, u_2 + (I - P_1) \circ F(u_1 + u_2) - (I - P_1)h = 0 \, . \tag{10}$$

Since $NS(P_0) = NS(P_1) = R(L)$ it follows that

$$L - \lambda \cdot I : D(L) \cap NS(P_0) \to NS(P_1)$$

is invertible for $|\lambda| \leq \delta$. By Lemma 1 and (7) we get that

$$||(L - \lambda \cdot I)^{-1}|| \leq 2 \cdot ||K_{P_0}|| \, , \quad \text{if} \quad |\lambda| \cdot ||I|| \leq \frac{1}{2 \cdot ||K_{P_0}||}$$

By (10) we have

$$\begin{aligned} ||u_2|| &\leq ||(L - \lambda \cdot I)^{-1}|| \cdot ||I - P_1|| \cdot ||h - F(u_1 + u_2)|| \\ &\leq 2 \cdot ||K_{P_0}|| \cdot ||I - P_1||(||h|| + ||F(u_1 + u_2)||) \end{aligned} \tag{11}$$

Take ε such that $0 < \varepsilon < d$, where d is given in the assumption (3). There exists such an $\varepsilon_1 > 0$ that

$$\varepsilon_1 < \frac{\varepsilon}{8 \, ||K_{P_0}|| \cdot ||I - P_1||} \tag{12}$$

and moreover

$$2\,||K_{P_0}||\cdot||I-P_1||\cdot\varepsilon_1 = c_1 < \frac{1}{2} \tag{13}$$

Denote by

$$2\,||K_{P_0}||\cdot||I-P_1||\cdot||h|| = c_2 \tag{14}$$

From the property (1) it follows that for $\varepsilon_1 > 0$ there exists such an $R(\varepsilon_1) > 0$ that for each u with the norm $||u|| > R$

$$||F(u)|| \leq \varepsilon_1\,||u||$$

is valid. By (11) and (14) we have

$$||u_2|| \leq c_2 + c_1\,||u_1+u_2||\,, \quad \text{for } u \text{ with the norm } ||u|| > R \tag{15}$$

In the case that the solution u of (5) fulfils the astimate $||u|| < R$ is nothing to prove. We suppose that this astimate is not fulfilled.
1. Let $R < ||u||$ and $0 \leq ||u_1|| \leq R$. By (15) it follows that

$$\begin{aligned}||u|| &\leq ||u_1|| + ||u_2|| \leq ||u_1|| + c_2 + c_1\,||u_1+u_2|| \\ &\leq R + c_2 + c_1\,||u||\end{aligned}$$

Then

$$||u|| \leq \frac{c_2+R}{1-c_1}$$

2. Let $R < ||u||$ and $||u_1|| > R > 0$. By (15) it follows

$$\frac{||u_2||}{||u_1||} \leq \frac{c_2}{1-c_1}\cdot\frac{1}{||u_1||} + \frac{c_1}{1-c_1} \leq \frac{c_2}{1-c_1}\cdot\frac{1}{||u_1||} + 2\,c_1 \tag{16}$$

We suppose that the set of all solution of the equation (5) for $0 \leq \lambda \leq \delta$ is not bounded. Therefore there exists a sequence of $\{u_n\}$ of equation (5) corresponding to values $\lambda = \lambda_n \in [0,\delta]$ such that $||u_n|| \to \infty$. From (16) it follows that necessarily $||u_{1_n}|| \to \infty$. Let $u_{1_n} = t_n\,y_n$, $t_n = ||u_{1_n}||$, $y_n = \dfrac{u_{1_n}}{||u_{1_n}||}$, $y_n \in NS(L)$, $||y_n|| = 1$. Then there exists a subsequence $\{y_{n_k}\}$ in the finite-dimensional space $NS(L)$ which converges to $y \in NS(L)$, $||y|| = 1$. By rewriting y_{n_k} to y_n we have $u_n = t_n\cdot y_n + t_n\cdot z_n$ where $z_n = \dfrac{u_{2_n}}{||u_{1_n}||}$ is the sequence from $NS(P_0)$. By divergence $||u_{1_n}|| \to \infty$ it follows that for chosen $\varepsilon > 0$ there exists such an $n_0 \in N$ that for each $n \geq n_0$

$$\frac{c_2}{1-c_1}\cdot\frac{1}{||u_{1_n}||} < \frac{\varepsilon}{2}$$

is valid. By (12) and (13) we have

$$2\,c_1 < \frac{\varepsilon}{2}$$

Therefore

$$||z_n|| = \frac{||u_{2_n}||}{||u_{1_n}||} < \varepsilon < d$$

(9) implies that

$$\langle -\lambda_n\, t_n\, y_n\,,\, y\rangle + \langle F(t_n\, y_n + t_n\, z_n)\,,\, y\rangle = \langle h\,,\, y\rangle$$

Because $y_n \longrightarrow y$ as $n \longrightarrow \infty$

$$\langle y_n\,,\, y\rangle = \langle y\,,\, y\rangle + \langle y_n - y\,,\, y\rangle > 0$$

for n large. For such n

$$\langle F(t_n\, y_n + t_n\, z_n)\,,\, y\rangle \geq \langle h\,,\, y\rangle$$

and

$$\liminf_{n\to\infty}\langle F(t_n\, y_n + t_n\, z_n)\,,\, y\rangle \geq \langle h\,,\, y\rangle$$

which contradicts with (3). This completes the proof of theorem. □

Using a similar argument we can prove the next theorem.

Theorem 2. *Let all conditions of Theorem 1 be satisfied and let instead of the condition (3) the assumption (4) be satisfied. Then for all λ such $-\delta \leq \lambda \leq 0$ there exists an $R_0 > 0$ such that any solution u of (5) satisfies $||u|| \leq R_0$.*

2. Application to the fourth order differential equation

Let the space $X = L^2([0, 2\pi])$ be provided with the norm $||x||_2$ and let the scalar product

$$\langle x, y\rangle = \int_0^{2\pi} x(t)\cdot y(t)\, dy$$

Let the linear differential operator L be defined by

$$L(x) = x^{(4)} + (m^2 + n^2)\, x''$$

where $0 \leq m \leq n$, $m, n \in N$ and the domain of the operator L is

$$D(L) = \{x(t) \in C^3([0, 2\pi])\,,\ x^{(4)} \in L^2([0, 2\pi])\ :\ x^{(i)}(0) = x^{(i)}(2\pi)\,, i = 0, 1, 2, 3\}$$

Therefore the operator L maps $D(L) \subset X$ into X. A fundamental system of solutions of the equation $L(x) = 0$ is $y_1(t) = 1$, $y_2(t) = t$, $y_3(t) = \cos\sqrt{m^2 + n^2}\, t$, $y_4(t) = \sin\sqrt{m^2 + n^2}\, t$ and so $\lambda = 0$ is the eigenvalue of the operator L. If is $m^2 + n^2 = k^2$, $k \in Z$, $k \neq 0$ holds then

$$NS(L) = \{y \in D(L)\ :\ y(t) = c_1 + c_2\, \cos kt + c_3 \sin kt\,,\quad c_i \in \mathbb{R}\,,\quad i = 0, 1, 2, 3\}.$$

Now we consider this case.

Lemma 2. *Let the operator L be defined on $D(L)$. Then*

$$NS(L) \cap R(L) = \{0\}.$$

Proof. The problem $L(x) = 0$, $x^{(i)}(0) = x^{(i)}(2\pi)$, $i = 0, 1, 2, 3$ is self-adjoint and therefore the assertion of the lemma is true. □

For $\lambda < -\frac{1}{4}(n^2 - m^2)^2 - m^2 n^2$ there exists the inverse operator $(L - \lambda \cdot I)^{-1}$, (Lemma 1 [4] p.) and this operator is completely continuous, (Lemma 4.4, [1] p. 145). The conditions of Theorem 1 [5] p. 555 hold and we have that L is a Fredholm operator of index zero and it is a closed operator.

Now we take continuous projectors

$$P_0 : X \to X, \qquad P_1 : X \to X$$

such that $R(P_0) = NS(L)$, $NS(P_1) = R(L)$. Let $NS(P_0) = R(L)$. Then we can take $P_0 = P_1$. The operator $L|_{D(L) \cap NS(P_1)}$ is one-to-one and therefore there exists the inverse operator $K_{P_0} : R(L) \to D(L) \cap NS(P_0)$. Now we construct the operator K_{P_0}. The Cauchy function for the equation $L(x) = 0$ is

$$K_1(t,s) = k^{-3}[k(t-s) + \sin k(s-t)], \quad \text{for} \quad 0 \le s < t \le 2\pi.$$

Let $x \in D(L) \cap NS(P_0)$ be the solution of the equation $L(x) = y$, $y \in R(L_2)$. Then it has the form

$$x(t) = c_1 + c_2 \cos kt + c_3 \sin kt + \int_0^t K_1(t,s)\, y(s)\, ds. \tag{17}$$

The function $x(t) \in D(L) \cap NS(P_0)$ and it follows that x is orthogonal to all functions belonging to $NS(L)$ and therefore we have

$$\begin{aligned} 0 =& \langle x(t), 1 \rangle = 2\pi c_1 + \int_0^{2\pi} \int_0^t K_1(t,s)\, y(s)\, ds\, dt, \\ 0 =& \langle x(t), \cos kt \rangle = \pi c_2 + \int_0^{2\pi} \int_0^t K_1(t,s)\, y(s)\, ds\, dt, \\ 0 =& \langle x(t), \sin kt \rangle = \pi c_3 + \int_0^{2\pi} \int_0^t K_1(t,s)\, y(s)\, ds\, dt. \end{aligned} \tag{18}$$

From periodic conditions it follows that $y \in R(L)$ if and only if

$$\int_0^{2\pi} \frac{\partial^i K_1(2\pi, s)}{\partial t^i} \cdot y(s)\, ds = 0, \quad \text{for} \quad i = 0, 1, 2, 3.$$

is true. By Fubini's theorem in (18) as well as by putting the constants c_i, $i = 1, 2, 3$ in (17) we get that

$$x(t) = -\frac{1}{\pi}\int_0^{2\pi}\left[\frac{4\pi(\pi+s)-3s^2}{4\,k^2} + \frac{2\pi-s}{2\,k^3}(\sin ks - \cos ks - 2) - \frac{3}{2\,k^4}\sin ks - \pi\cdot K(t,s)\right] y(s)\,ds\,,$$

where

$$K(t,s) = \begin{cases} K_1(t,s)\,, & 0 \le s \le t \le 2\pi\,, \\ 0\,, & 0 \le t < s \le 2\pi \end{cases}$$

Then the operator K_{P_0} is

$$K_{P_0}(y)(t) = -\frac{1}{\pi}\int_0^{2\pi}\left[\frac{4\pi(\pi+s)-3s^2}{4\,k^2} + \frac{2\pi-s}{2\,k^3}(\sin ks - \cos ks - 2) - \frac{3}{2\,k^4}\sin ks - \pi\cdot K(t,s)\right] y(s)\,ds \tag{19}$$

We have the following estimate for the norm of the operator K_{P_0}

$$||K_{P_0}|| \le 4\cdot\left[\frac{2\pi^2}{k^2} + \frac{5\pi + 2k\pi^2}{k^3} + \frac{3}{2\,k^4}\right] < +\infty$$

and the operator K_{P_0} is Hilbert-Schmidt operator. By (19) it follows that K_{P_0} has the continuous kernel on $[0, 2\pi]\times[0, 2\pi]$ and by Lemma 4.4, [1], p.145 we have that the operator K_{P_0} is completely continuous operator on $R(L_2)$.

Now we calculate the algebraic multiplicity of the eigenvalue $\lambda = 0$. To the first corresponding eigenfunction $u_0^1(t) = 1$ we look for such a function $u_1^1(t)$ that the equality

$$L(u_1^1) = u_0^1$$

is valid. From the assertion of Lemma 1 it follows that such a function from $D(L)$ does not exists. Therefore the length of the chain determined by the eigenfunction u_0^1 is equal to one. A similar results holds for the eigenfunctions $u_0^2(t) = \cos kt$, $u_0^3(t) = \sin kt$. And hence the algebraic multiplicity of the eigenvalue $\lambda = 0$ is equal to three.

We shall assume that the function $F : \mathbb{R} \to \mathbb{R}$ is continuous and is such that

$$\lim_{||u||\to\infty}\frac{|F(u)|}{||u||} = 0$$

and $h \in X$ is 2π-periodic function.

We now consider the equation

$$L(x) - \lambda\, m^2 n^2\, x(t) + F(x)(t) = h(t) \tag{20}$$

on $D(L_2)$. The verification of the conditions (3) a (4) may, in general be very difficult. In what follows two theorems we shall show that these conditions can be replased by other two conditions.

Theorem 3. *Let the function F be bounded in $\mathbb{R}$ and let*

$$\limsup_{s\to\infty} F(s) < h(t) < \liminf_{s\to\ -\infty} F(s) \tag{21}$$

be valid. Then the condition (3) is fulfield.

Proof. Let $y \in NS(L)$, $||y|| = 1$. We take a sequence postupnos"t $\{y_n\} \subset NS(L)$, $||y_n|| = 1$, with $y_n \to y$ as $n \to \infty$ and the real sequence $\{t_n\}_{n=1}^{\infty}$, $t_n \to \infty$ as $n \to \infty$. Let $\{z_n\}_{n=1}^{\infty} \subset NS(P_0)$ be such a sequence that $||z_n|| < d$, where d is a number sufficiently small. Choose $\varepsilon > 0$. There exists an $\varepsilon' > 0$, such that

$$1 - \left(\varepsilon' + \frac{d}{d+\varepsilon}\right) > 0 \tag{22}$$

is true. By (21) it follows that there exists an $a > 0$ such that

$$\limsup_{s\to\infty} F(s) + 2\,a < h(t) < \liminf_{s\to-\infty} F(s) - 2\,a$$

is valid. Denote by

$$\begin{aligned} M_1 =& \{t \in [0, 2\pi] : y(t) \geq d + \varepsilon\} \\ M_2 =& \{t \in [0, 2\pi] : y(t) \leq -(d + \varepsilon)\} \\ M_3 =& \{t \in [0, 2\pi] : |y(t)| < d + \varepsilon\} \end{aligned}$$

We shall show the validity of the condition (3).

1. Consider the set M_1 and a sequence

$$\left\{ \frac{t_n\,(y_n(t) + z_n(t))}{t_n\, y(t)} \right\}_{n=1}^{\infty} \tag{23}$$

on it. Because $y_n \to y$ as $n \to \infty$, $||y_n|| = ||y|| = 1$ and $||z_n|| < d$ for $\varepsilon' > 0$, by (22) there exists $n_0' \in N$ such that for all $n \geq n_0'$ the inequality

$$1 - \left(\varepsilon' + \frac{d}{d+\varepsilon}\right) \leq \frac{t_n\,(y_n(t) + z_n(t))}{t_n\, y(t)} \leq 1 + \left(\varepsilon' + \frac{d}{d+\varepsilon}\right)$$

is valid for each $t \in M_1$. Therefore

$$\begin{aligned} t_n(d+\varepsilon)\left[1 - \left(\varepsilon' + \frac{d}{d+\varepsilon}\right)\right] \leq t_n\, y(t) \left[1 - \left(\varepsilon' + \frac{d}{d+\varepsilon}\right)\right] \\ \leq t_n\,(y_n(t) + z_n(t)) \end{aligned}$$

As $n \to \infty$ $t_n(d+\varepsilon)\left[1 - \left(\varepsilon' + \frac{d}{d+\varepsilon}\right)\right] \to \infty$ then $t_n\, y_n(t) + +t_n\, z_n(t) \to \infty$ uniformly on M_1, too. Under the assmption (21) it follows the existence of such constants k_1, h_1 that

$$\limsup_{s\to\infty} F(s) = k_1 < h_1 \leq h(t)$$

It is true that for the above determined $a > 0$ there exists such an s_0' that for each $s \geq s_0'$ and each $t \in [0, 2\pi]$

$$F(s) \leq k_1 + 2\,a < h_1 \leq h(t)$$

We have that for each n sufficiently great

$$F(t_n y_n(t) + t_n z_n(t)) \leq k_1 + 2\,a < h(t)$$

for all $t \in M_1$. Multiplying the last inequality by the function $y(t)$ on M_1 it follows that

$$F(t_n y_n(t) + t_n z_n(t)) \cdot y(t) \leq (k_1 + 2\,a) \cdot y(t) < h(t) \cdot y(t)$$

Integrating these inequality on the set M_1 we get that

$$\int\limits_{M_1} h(t)\, y(t)\, dt - \int\limits_{M_1} F(t_n y_n + t_n z_n(t))\, y(t)\, dt$$
$$\geq \int\limits_{M_1} a\, y(t)\, dt \geq a(d + \varepsilon)\, \mu(M_1) \geq 0$$

Therefore

$$\int\limits_{M_1} h(t)\, y(t)\, dt - \liminf_{n\to\infty} \int\limits_{M_1} F(t_n y_n(t) + z_n(t))\, y(t)\, dt \geq a\,(d + \varepsilon)\, \mu(M_1) \geq 0 \tag{24}$$

2. Consider the set M_2. The function $y(t)$ is negative on M_2. By (23) we obtain that for $\varepsilon' > 0$ choosen at the beginning of the proof, there exists an $n_0'' \in N$ such that for all $n \geq n_0''$ the inequality

$$t_n y(t) \left[1 - \left(\varepsilon' + \frac{d}{d+\varepsilon}\right)\right] \geq t_n(y_n(t) + z_n(t)) \geq$$
$$\geq t_n\, y(t) \left[1 + \left(\varepsilon' + \frac{d}{d+\varepsilon}\right)\right]$$

holds. Therefore

$$-t_n(d+\varepsilon) \left[1 - \left(\varepsilon' + \frac{d}{d+\varepsilon}\right)\right] \geq t_n\, y(t) \left[1 - \left(\varepsilon' + \frac{d}{d+\varepsilon}\right)\right]$$
$$\geq t_n(y_n(t) + z_n(t))$$

As $n \to \infty$ $-t_n(d+\varepsilon) \left[1 - \left(\varepsilon' + \frac{d}{d+\varepsilon}\right)\right] \to -\infty$ so $t_n y_n(t) + +t_n z_n(t) \to -\infty$ uniformly on M_2. By the condition (21) the existence of constants k_2, h_2 follows for which

$$\liminf_{s\to-\infty} F(s) = k_2 > h_2 \geq h(t)$$

It is true that for the above determined $a > 0$ there exists such s_0'' that for each $s \geq s_0''$ and each $t \in [0, 2\pi]$

$$F(s) \geq k_2 - 2\,a > h_2 \geq h(t)$$

We have that for each n sufficiently great

$$F(t_n y_n(t) + t_n z_n(t)) \geq k_2 - 2\,a > h(t)$$

for all $t \in M_2$. Multiplying this inequality by the function $y(t)$ on M_2 it follows that

$$F(t_n y_n(t) + t_n z_n(t)) \cdot y(t) \leq (k_2 - 2\,a) \cdot y(t) < h(t)\, y(t)$$

By the integration on the set M_2 we get that

$$\begin{aligned} &\int\limits_{M_2} h(t)\, y(t)\, dt - \int\limits_{M_2} F(t_n y_n + t_n z_n(t))\, y(t)\, dt \\ &\qquad \geq - \int\limits_{M_2} a\, y(t)\, dt \geq a(d+\varepsilon)\, \mu(M_2) \geq 0 \end{aligned}$$

Therefore

$$\begin{aligned} &\int\limits_{M_2} h(t)\, y(t)\, dt - \liminf_{n \to \infty} \int\limits_{M_2} F(t_n y_n(t) + z_n(t))\, y(t)\, dt \\ &\qquad \geq a\,(d+\varepsilon)\, \mu(M_2) \geq 0 \end{aligned} \tag{25}$$

Adding inequalities (24) a (25) we get

$$\begin{aligned} &\int\limits_{M_1 \cup M_2} h(t)\, y(t)\, dt - \liminf_{n \to \infty} \int\limits_{M_1 \cup M_2} F(t_n y_n(t) + t_n z_n(t)) \cdot y(t)\, dt \\ &\qquad \geq a(d+\varepsilon) \cdot \mu(M_1 \cup M_2) > 0 \end{aligned} \tag{26}$$

3. Consider the set M_3. Now we make following estimations. The function F is bounded and therefore there exists such a $K > 0$ that

$$|F(r)| \leq K\,, \quad \text{for all } r \in \mathbb{R} \tag{27}$$

the function $h \in X$ and therefore there exists such a constant $H > 0$ that

$$|h(t)| \leq H\,, \quad \text{for all } t \in M_3 \subset [0, 2\pi] \tag{28}$$

We denote $d_1 = d+\varepsilon$. The function $y \in NS(L)$ and it is true that if $\lim\limits_{d \to 0^+} \mu(M_3) = 0$ then $\mu(M_1 \cup M_2) \to 2\pi$. So far the proof of the theorem has not depended on the choice of numbers ε, d. We choose ε, d such that

$$(H+K)\, \mu(M_3) < a\, \mu(M_1 \cup M_2)$$

Then by (27), (28) it follows that the estimations

$$\left| \int_{M_3} F(t_n y_n(t) + t_n z_n(t)) \cdot y(t)\, dt \right| < K\,(d+\varepsilon)\mu(M_3)$$

$$\left| \int_{M_3} h(t)\, y(t)\, dt \right| < H \cdot (d+\varepsilon)\mu(M_3)$$

are valid. By the introduced estimates it is true that

$$\left| \int_{M_3} h(t)\, y(t)\, dt - \liminf_{n\to\infty} \int_{M_3} F(t_n y_n(t) + t_n z_n(t)) \cdot y(t)\, dt \right|$$

$$\leq \left| \int_{M_3} h(t)\, y(t)\, dt \right| + \limsup_{n\to\infty} \left| \int_{M_3} F(t_n y_n(t) + t_n z_n(t)) \cdot y(t)\, dt \right|$$

$$\leq (H+K)(d+\varepsilon)\mu(M_3) < a(d+\varepsilon)\mu(M_1 \cup M_2)\,.$$

Adding to the inequality (26) the inequality

$$\int_{M_3} h(t)\, y(t)\, dt - \liminf_{n\to\infty} \int_{M_3} F(t_n y_n(t) + t_n z_n(t))\, y(t)\, dt <$$

$$< a(d+\varepsilon)\mu(M_1 \cup M_2)$$

we have that

$$\int_0^{2\pi} h(t)\, y(t)\, dt - \lim_{n\to\infty} \int_0^{2\pi} F(t_n y_n(t) + t_n z_n(t))\, y(t)\, dt > 0$$

Thus the theorem is completely proved. □

Similary the next theorem can be proved.

Theorem 4. *Let the function F be bounded in $\mathbb{R}$ and let*

$$\limsup_{s\to-\infty} F(s) < h(t) < \liminf_{s\to\infty} F(s) \tag{29}$$

be valid. Then the condition (4) is fulfielled.

We showed that the eigenvalue $\lambda = 0$ has an odd algebraic multiplicity and it is an isolated eigenvalue of the operator L, i. e. that exists such $\delta_0 > 0$ that for $\lambda \in (-\delta_0\,,\, \delta_0)$, $\lambda \neq 0$ there exists $(L - \lambda \cdot I)^{-1}$. From the form of equation (20) we have that the operator L_0 and $\varDelta$ in Lemma 1 are

$$L_0(x) = L(x) - \lambda\, m^2 n^2\, x\,,$$

$$\varDelta = |\lambda\, m^2 n^2| \cdot ||K_{P_0}||\,.$$

If $0 \leq \varDelta \leq \dfrac{1}{2}$ then $0 \leq |\lambda| \leq \dfrac{1}{2m^2n^2||K_{P_0}||}$. Denote by $\delta = \min\Big(\delta_0\,,\, \dfrac{1}{2m^2n^2||K_{P_0}||}\Big) > 0$. By Theorem 1 and Theorem 3 the next theorem follows.

Theorem 5. *Let the condition of Theorem 3 hold and let $0 \leq \lambda \leq \delta$. Then there exists such an $R_0 > 0$ that any solution u of the equation (20) satisfies $||u|| \leq R_0$.*

Similarly by Theorem 2 and Theorem 4 we get Theorem 6.

Theorem 6. *Let the conditions of Theorem 4 hold and let $-\delta \leq \lambda \leq 0$. Then there exists such an $R_0 > 0$ that any solution u of the equation (20) satisfies $||u|| \leq R_0$.*

If we use Theorem 9 [3] p. 144 we obtain a result about a number of solutions of the equation (20) in a neighbourhood of 0.

Corrollary 1. *Let the function F be bounded in $\mathbb{R}$. If (21) holds then there exists such an $\eta_1 > 0$ that*

(1) *for $0 \leq \lambda \leq \delta$ exists at least one 2π-periodic solution of (20)*
(2) *for $-\eta_1 \leq \lambda < 0$ exists at least two 2π-periodic solutions of (20).*
If (27) holds then there exists an $\eta_2 > 0$ such that
(3) *for $-\delta \leq \lambda \leq 0$ exists at least one 2π-periodic solution of (20)*
(4) *for $0 < \lambda \leq \eta_2$ exists at least two 2π- periodic solutions of (20).*

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