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SPECIAL SOLUTIONS OF LINEAR DIFFERENCE EQUATIONS WITH INFINITE DELAY

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Abstract. For the difference equation (ϵ) $x_{n+1} = Ax_n + \epsilon \sum_{k=-\infty}^{n} R_{n-k}x_k$, where $x_n \in Y$, Y is a Banach space, ϵ is a parameter and A is a linear, bounded operator. A sufficient condition for the existence of a unique special solution $y = \{y_n\}_{n=-\infty}^{\infty}$ passing through the point $x_0 \in Y$ is proved. This special solution converges to the solution of the equation (0) as $\epsilon \to 0$.

The paper [2] contains a result on the existence of so-called two-sided solutions of linear integrodifferential equations of the form

$$\frac{dx(t)}{dt} = Ax(t) + \epsilon \int_{-\infty}^{t} R(t-s)x(s)ds, \tag{1}$$

where $x \in R^n$, $A \in M_n$ – the set of all $n \times n$ matrices, $\epsilon \in R$ is a parameter and $R(t)$ is a continuous matrix function satisfying the inequality

$$||R(t)|| \leqq c\frac{\exp\{-\gamma t\}}{t^{1-\alpha}}, \tag{2}$$

where c, γ, α are positive constants and $0 < \alpha < 1$. It is proven there that if $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of A and $\min\{\operatorname{Re}\lambda_j : 1 \leqq j \leqq n\} > -\gamma$ then there is an $\epsilon_0 > 0$ such that for any $x_0 \in R^n$ there exists a unique solution $x_\epsilon(t)$ (so-called two-sided solution) of (1) defined on the whole interval $(-\infty, \infty)$ satisfying the initial condition $x_\epsilon(0) = x_0$ and $\lim_{\epsilon\to 0} ||x_\epsilon - x||_L = 0$ for any $L > 0$, where $||x_\epsilon - x||_L = \sup\{||x_\epsilon(t) - x(t)|| : -L \leqq t \leqq L\}$, $x(t) = \exp\{At\}x_0$. In the paper [1] a generalization of this result, including the case when $A = A(t)$ is periodic, is proved, where the proof differs from that presented in [2].

If we substitute in (1) the difference $x_{n+1} - x_n$ instead of $\frac{dx(t)}{dt}$ and discretize the integral (more precisely, we put the natural numbers n, i instead of t and s,

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respectively) we obtain a difference equation with infinite delay. Let us consider such a difference equation in a Banach space Y, i. e. the equation

$$x_{n+1} = Ax_n + \epsilon(R_0 x_n + R_1 x_{n-1} + \cdots + R_k x_{n-k} + \dots), \tag{3}$$

where $A, R_i \in L(Y)$ –the space of continuous, linear mappings from Y into Y ($i = 0, 1, \dots$), A is invertible and $A^{-1} \in L(Y)$. We shall prove the following theorem on the existence of special solutions of the equation (3) determined uniquely by the initial value which is a point in Y and defined for all integers.

Theorem. *Let the following conditions be satisfied:*

$$||R_0|| = 1, \quad ||R_n|| \leqq \frac{\gamma^{-n}}{n^{1-\alpha}}, n = 1, \dots, \tag{4}$$

where γ, α are constants, $\epsilon \leqq \gamma$, $0 < \alpha < 1$.

$$||A^{-1}|| < 1, \quad \frac{\gamma^{-1}||A^{-1}||}{1 - ||A^{-1}||} < 1. \tag{5}$$

Then there exists an $\epsilon_o > 0$ such that the following assertions are valid:

(a) *For any $\epsilon \in (0, \epsilon_0]$ there exists an operator solution of the equation (3) of the form*

$$X_n(\epsilon) = D(\epsilon)^n, \tag{6}$$

where $D(\epsilon)$ is independent of n and

$$\lim_{\epsilon \to 0} D(\epsilon) = A, \text{ i. e. } \lim_{\epsilon \to 0} ||D(\epsilon) - A|| = 0.$$

(b) *For any $\epsilon \in (0, \epsilon_0]$ and any $x_0 \in Y$ there exists a unique solution $x = \{x_n(\epsilon)\}_{n=-\infty}^{\infty}$ of the equation (3) satisfying the condition $x_0(\epsilon) = x_0$ such that $x \in B := \{z = \{z_n\}_{n=-\infty}^{\infty} : z_n \in Y, \sup\{|z_n| : -\infty < n \leqq 0\} < \infty\}$, where $|.|$ is the norm on Y. Moreover, $\sup\{|x_n(\epsilon) - A^n x_0| : -L \leqq n \leqq L\} \to 0$ as $\epsilon \to 0$ for any $L > 0$.*

Proof. The operator sequence $\{D^n\}_{n=-\infty}^{\infty}$ is a solution of the equation (3) if and only if

$$D^{n+1} = AD^n + \epsilon(R_o D^n + R_1 D^{n-1} + \cdots + R_k D^{n-k} + \dots). \tag{7}$$

Let us look the matrix D in the form $D = A + Q$, where $Q \in L(Y)$ is an unknown operator such that D is invertible. The equation (7) is obviously equivalent to the equation

$$Q = \mathcal{F}_\epsilon(Q) := \epsilon[R_0 + R_1(A+Q)^{-1} + \cdots + R_k(A+Q)^{-k} + \dots]. \tag{8}$$

Define the mapping $\mathcal{F}_\epsilon : V \to L(Y)$ via the formula (8), where $V = \{Q \in L(Y) : ||Q|| \leqq 1\}$, $||Q|| := \sup\{||Qx|| : ||x|| \leqq 1\}$. If $Q_1,\, Q_2 \in L(Y)$ then

$$||\mathcal{F}_\epsilon(Q_1) - \mathcal{F}_\epsilon(Q_2)|| = \epsilon ||R_1[(A+Q_1)^{-1} - (A+Q_2)^{-1}] +$$

$$+ R_2[(A+Q_1)^{-2} - (A+Q_2)^{-2}] + \dots + R_k[(A+Q_1)^{-k} - (A+Q_2)^{-k}] + \dots ||.$$

Since $(A+Q_j) = A(I + A^{-1}Q_j)$, $j = 1, 2$ where I is the identity operator, we have

$$(9) \quad ||\mathcal{F}_\epsilon(Q_1) - \mathcal{F}_\epsilon(Q_2)|| \leqq \epsilon\{||R_1|| ||A^{-1}|| ||(I + A^{-1}Q_1)^{-1} - (I + A^{-1}Q_2)^{-1}|| +$$

$$+ ||R_2|| ||A^{-1}||^2 ||(I + A^{-1}Q_1)^{-2} - (I + A^{-1}Q_2)^{-2}|| + \dots +$$

$$+ ||R_k|| ||A^{-1}||^k ||(I + A^{-1}Q_1)^{-k} - (I + A^{-1}Q_2)^{-k}|| + \dots \}.$$

We have the following estimation:

$$||(I + A^{-1}Q_1)^{-k} - (I + A^{-1}Q_2)^{-k}|| = ||[(I + A^{-1}Q_1)^{-1}]^k - [(I + A_{-1}Q_2)^{-1}]^k|| \leqq$$

$$\leqq ||(I + A^{-1}Q_1)^{-1} - (I + A^{-1}Q_2)^{-1}|| ||[(I + A^{-1}Q_1)^{-1}]^{k-1} +$$

$$+ [(I + A^{-1}Q_1)^{-1}]^{k-2}(I + A^{-1}Q_2)^{-1} + \dots + [(I + A^{-1}Q_2)^{-1}]^{k-1}||$$

$$\leqq |(I + A^{-1}Q_1)^{-1} - (I + A^{-1}Q_2)^{-1}|| \{||(I + A^{-1}Q_1)^{-1}||^{k-1} +$$

$$||(I + A^{-1}Q_1)^{-1}||^{k-2} ||(I + A^{-1}Q_2)^{-1}|| + \dots + ||(I + A^{-1}Q_2)^{-1}||^{k-1}\}.$$

If $Q_1, Q_2 \in V$ then

$$||(I + A^{-1}Q_i)^{-1}|| = ||I - (A^{-1}Q_i) + (A^{-1}Q_i)^2 - \dots || \leqq 1 + \nu + \nu^2 + \dots = \frac{1}{1-\nu},$$

$i = 1, 2$, where $\nu = ||A^{-1}||$.

Therefore using the above estimation we obtain the inequality:

$$(10) \qquad ||(I + A^{-1}Q_1)^{-k} - (I + A^{-1}Q_2)^{-k}|| \leqq$$

$$\leqq ||(I + A^{-1}Q_1)^{-1} - (I + A^{-1}Q_2)^{-1}|| \frac{k}{(1-\nu)^{k-1}}.$$

Now applying this inequality to the estimation (9) we obtain the estimation:

$$(11) \qquad ||\mathcal{F}_\epsilon(Q_1) - \mathcal{F}_\epsilon(Q_2)|| \leqq$$

$$\leqq \epsilon\{||R_1|| ||A^{-1}|| + ||R_2|| ||A^{-1}||^2 \frac{2}{1-\nu} + \dots +$$

$$+ ||R_n|| ||A^{-1}||^n \frac{n}{(1-\nu)^{n-1}} + \dots \} ||(I + A_{-1}Q_n)^{-1} - (I + A^{-1}Q_2)^{-1}||$$

For all $Q_1, Q_2 \in V$.

We need the following estimation:

$$\|(I+A^{-1}Q_1)^{-1}-(I+A^{-1}Q_2)^{-1}\| = \tag{12}$$

$$= \|(I-A^{-1}Q_1+(A^{-1}Q_1)^2-\dots)-(I-A^{-1}Q_2+(A^{-1}Q_2)^2-\dots)\| \leqq$$

$$\leqq \|A^{-1}Q_1-A^{-1}Q_2\|+\|(A^{-1}Q_1)^2-(A^{-1}Q_2)^2\|+\dots+\|(A^{-1}Q_1)^n-(A^{-1}Q_2)^n\|+$$

$$+\dots.$$

The mean value theorem yields

$$\|(A^{-1}Q_1)^i-(A^{-1}Q_2)^i\| \leqq \|A^{-1}\|^i\|Q_1^i-Q_2^i\| \leqq \tag{13}$$

$$\leqq \|A^{-1}\|^i \sup\{\|i[(1-t)Q_1+tQ_2]^{i-1}\| : 0 \leqq t \leqq 1\}\|Q_1-Q_2\| \leqq i\|A^{-1}\|^i\|Q_1-Q_2\|.$$

From (11), (12) and (13) it follows that

$$\|\mathcal{F}_\epsilon(Q_1)-\mathcal{F}_\epsilon(Q_2)\| \leqq \epsilon S_1S_2\|Q_1-Q_2\| \tag{14}$$

for all $Q_1, Q_2 \in V$, where

$$S_1 = \|R_1\|\|A^{-1}\|+\|R_2\|\|A^{-1}\|^2\frac{2}{1-\nu}+\dots+\|R_n\|\|A^{-1}\|^n\frac{n}{(1-\nu)^{n-1}}+\dots,$$

$$S_2 = \|A^{-1}\|+2\|A^{-1}\|^2+\dots+n\|A^{-1}\|^n+\dots,$$

i. e.

$$S_1 = (1-\nu)\sum_{n=1}^{\infty} n\left(\sqrt[n]{\|R_n\|}\,\frac{\|A^{-1}\|}{1-\nu}\right)^n,$$

$$S_2 = \|A^{-1}\|\sum_{n=1}^{\infty} n\|A^{-1}\|^{n-1}.$$

One can show using the condition (4) and the D'Alambert convergence criterion that the series S_1 is convergent. Since $\|A^{-1}\| < 1$ the series S_2 is also convergent. Therefore the inequality (14) implies that if $\epsilon \in (0, \frac{1}{S_1S_2})$ then the mapping $\mathcal{F}_\epsilon|V$ is contractive and thus there exists a unique fixed point of $\mathcal{F}_\epsilon$ in V , i. e. the assertion (a) is proved. It remains to prove the assertion (b). We shall prove that for any $y_o \in Y$ there exists a unique solution $x = \{x_n\}_{n=-\infty}^{\infty}$ of (3) satisfying the condition $x_0 = y_0$ and $\sup\{|x_n| : -\infty < n \leqq 0\} < \infty$.

Define the space

$$B = \{x = \{x_n\}_{n=-\infty}^{0} : x_n \in Y, \sup\{|x_n| : -\infty < n \leqq 0\} < \infty\}$$

which is a Banach space with the norm $\|x\| = \sup\{|x_n| : -\infty < n \leqq 0\}$.

From the equation (3) it follows that

$$x_{-1} = A^{-1}x_0 - \epsilon A^{-1}(R_0x_{-1}+R_1x_{-2}+\dots+R_kx_{-(1+k)}+\dots),$$

$$x_{-2} = A^{-1}x_{-1} - \epsilon A^{-1}(R_0x_{-2} + R_1x_{-3} + \dots R_kx_{-(2+k)} + \dots) =$$
$$= A^{-2}x_0 - \epsilon A^2(R_0x_{-2} + R_1x_{-3} + \dots R_kx_{-(2+k)} + \dots) -$$
$$-\epsilon A^{-1}(R_0x_{-2} + R_1x_{-3} + \dots + R_{-(2+k)} + \dots),$$

etc.

$$x_{-p} = A^{-p}x_0 - \epsilon A^{-p}(R_0x_{-1} + R_1x_{-2} + \dots + R_kx_{-(1+k)} + \dots) -$$
$$-\epsilon A^{-p+1}(R_0x_{-2} + R_1x_{-3} + \dots + R_kx_{-(2+k)} + \dots) - \dots -$$
$$-\epsilon A^{-1}(R_0x_{-p} + R_1x_{-(p+1)} + \dots + R_kx_{-(p+k)} + \dots).$$

Therefore we define the mapping

$$G_\epsilon : B \to \{x : x = \{x_n\}_{n=-\infty}^{0}, x_n \in Y\},$$

$$(G_\epsilon x)_{-p} = -\epsilon A^{-p}(R_0x_{-1} + R_1x_{-2} + \dots + R_kx_{-1-k} + \dots) - \dots -$$
$$-\epsilon A^{-1}(R_0x_{-p} + R_1x_{-(p+1)} + \dots + R_kx_{-(p+1)} + \dots)$$

for all $p \in N, p > 0$.

If $x = \{x_n\}_{n=-\infty}^{0} \in B$ and $\nu = ||A^{-1}||$ then

$$|(G_\epsilon x)_{-p}| \leqq \epsilon||x||\{||A^{-1}||^p(||R_0|| + ||R_1|| + \dots + ||R_k|| + \dots) +$$
$$+||A^{-1}||^{p-1}(||R_0|| + ||R_1|| + \dots ||R_k|| + \dots) + \dots +$$
$$+||A^{-1}||(||R_0|| + ||R_1|| + \dots + ||R_k|| + \dots) \leqq$$
$$\leqq \frac{\epsilon||x||}{1-\nu}[1 + \gamma^{-1} + \dots \frac{\gamma^{-k}}{k^{1-\alpha}} + \dots] \leqq$$
$$\leqq \frac{\epsilon||x||}{1-\nu}\int_0^\infty (\exp\{-t\})t^{\alpha-1}dt = \frac{\epsilon||x||}{1-\nu}\Gamma(\alpha).$$

This means that $|(G_\epsilon x)_{-p}| \leqq \frac{\epsilon||x||}{1-\nu}\Gamma(\alpha)$ for all $p \in N, p > 0$, i. e. $G_\epsilon x \in B$. Since G_ϵ is linear, we have

$$||G_\epsilon x_1 - G_\epsilon x_2|| = ||G_\epsilon(x_1 - x_2)|| \leqq \epsilon\frac{\Gamma(\alpha)}{1-\nu}||x_1 - x_2||$$

for all $x_1, x_2 \in B$. This implies that if $\epsilon \in (0, \frac{1-\nu}{\Gamma(\alpha)})$ then the mapping G_ϵ is contractive and thus it has a unique fixed point $z \in B$. Since G_ϵ is linear, we conclude that $z = 0$.

Let $\phi_1 = \{x_n\}_{n=-\infty}^{\infty}, \phi_2 = \{y_n\}_{n=-\infty}^{\infty}$ be two solutions of (3) satisfying the condition $x_0 = y_0, \sup\{|x_n| : -\infty < n \leqq 0\} < \infty, \sup\{|y_n| : -\infty < n \leqq 0\} < \infty$ and let $\phi = \phi_1 - \phi_2 = \{x_n - y_n\}_{n=-\infty}^{\infty}$. Then $||\phi|| = \sup\{|x_n - y_n| : -\infty < n \leqq 0\} < \infty$. The sequence $\Phi = \{x_n - y_n\}_{n=-\infty}^{0}$ is the fixed point of G_ϵ and therefore $\Phi = 0$. Thus if there exists a solution $x = \{x_n\}_{n=-\infty}^{\infty}$ of (3) such that $\{x_n\}_{n=-\infty}^{0} \in B$ then it is uniquely defined for all $n \leqq 0$. We shall prove that such a solution exists and it is uniquely defined also for all $n \geqq 0$.

The sequence $\Psi = \{\Psi_n\}_{n=-\infty}^{\infty} = \{(D(\epsilon))^n x_0\}_{n=-\infty}^{\infty}$ is a solution of (3) satisfying the condition $\Psi_0 = x_0$. From the condition (5) and the assertion (a) it follows that there exists an $\epsilon_1 > 0$ such that $||D(\epsilon)^{-p}|| \leqq ||D(\epsilon)^{-1}||^p < 1$ for all $p \in N, p > 0, \epsilon \in (0, \epsilon_1)$ and this implies that $\sup\{|\psi_n| : -\infty < n \leqq 0\} \leqq |x_o| < \infty$, i. e. $\{\Psi_n\}_{n=-\infty}^{0} \in B$. It suffices to prove the uniqueness of solutions of (3) for $n \geqq 0$.

Let $\phi_1 = \{x_n\}_{n=-\infty}^{\infty}, \phi_2 = \{y_n\}_{n=-\infty}^{\infty}$ be two solutions of (3) such that $\{x_n\}_{n=-\infty}^{0}, \{y_n\}_{n=-\infty}^{0} \in B$ and $x_0 = y_0$.

Since for $n \geqq 0$ we have

$$|x_n - y_n| = \epsilon|\sum_{i=0}^{n-1} A^{n-i-1}[R_0(x_i - y_i) + R_1(x_{i-1} - y_{i-1}) + \dots + R_k(x_{i-k} - y_{i-k}) + \dots]|$$

and $x_k = y_k$ for all $k \in N, k \leqq 0$, what we have proved above, we obtain

$$|x_n - y_n| \leqq \epsilon\{||A||^{n-1}|x_0 - y_0| + ||A||^{n-2}[|x_1 - y_1| + \gamma^{-1}|x_0 - y_0|] +$$

$$\dots + [|x_{n-1} - y_{n-1}| + \gamma^{-1}|x_{n-2} - y_{n-2}| + \dots + \frac{\gamma^{-(n-2)}}{(n-2)^{1-\alpha}}|x_0 - y_0|]\}.$$

From this inequality we obtain

$$|x_1 - y_1| \leqq \epsilon|x_0 - y_0| = 0,$$

$$|x_2 - y_2| \leqq \epsilon\{||A||[|x_0 - y_0| + |x_1 - y_1|]\} = 0,$$

etc. By induction one can show that $x_n = y_n$ for all $n \in N$, $n \geqq 0$. From the assertion (a) it follows that if $x_n(\epsilon) = D(\epsilon)^n x_0$ then

$$\sup\{|x_n(\epsilon) - A^n x_0| : -L \leqq n \leqq L\} = \sup\{|[D(\epsilon)^n - A^n]x_0| : -L \leqq n \leqq L\} \leqq$$

$$\leqq \sup\{||D(\epsilon) - A||^n |x_0| : -L \leqq n \leqq L\} \to 0 \text{ as } \epsilon \to 0 \text{ for any } L > 0.$$

References

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